AMSTERDAM 2008

European Congress of Mathematics
Amsterdam, 14 – 18 July, 2008

André Ran
Herman te Riele
Jan Wiegerinck
Editors

European Mathematical Society

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2010 Mathematics Subject Classification: 00Bxx

ISBN 978-3-03719-077-7

The Swiss National Library lists this publication in The Swiss Book, the Swiss national bibliography, and the detailed bibliographic data are available on the Internet at http://www.helveticat.ch.

European Mathematical Society Publishing House
Seminar for Applied Mathematics
ETH-Zentrum FLI C4
CH-8092 Zürich
Switzerland

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Email: info@ems-ph.org
Homepage: www.ems-ph.org

Background picture of Amsterdam courtesy www.amsterdam.info, travel guide to Amsterdam featuring information on hotels, apartments, sights and entertainment in Amsterdam and throughout The Netherlands.

Typeset using the authors' TEX files: I. Zimmermann, Freiburg

Printed in Germany

9 8 7 6 5 4 3 2 1

Preface

The Fifth European Congress of Mathematics (5ECM) took place from July 14–18, 2008 in the RAI Convention Center Amsterdam. It was organized by the Centrum Wiskunde en Informatica Amsterdam, the University of Amsterdam, and the VU University Amsterdam, under auspices of the European Mathematical Society. Included in this congress was the 44th Nederlands Mathematisch Congres, the yearly congress of the Royal Dutch Mathematical Society (KWG). 5ECM stood under the special patronage of the KWG. Previous European Congresses of Mathematics were held in Paris (1992), Budapest (1996), Barcelona (2000), and in Stockholm (2004).

About 1000 mathematicians from 68 different countries attended the congress.

The first of ten plenary lectures, to get the congress started, was delivered by Richard Taylor. As all the other plenary lecturers, he did a wonderful job of explaining his work to a general mathematical audience. Another major item on the programme were three science lectures. These lectures outlined applications of mathematics in other sciences. Mathematical modeling plays a crucial role in predicting climate change, as was stressed by Tim Palmer (European Centre for Medium Range Weather Forecasts). He also outlined what would be necessary to improve on the current state of affairs in the mathematical modeling to obtain predictions on a finer scale than is possible at the moment. Ignacio Cirac (Max Planck Institute für Quantenoptik) discussed quantum information theory, and the challenges in this area. The third science lecture was given by Jonathan Sherratt (Heriot-Watt University) who talked about the latest developments in mathematical modeling for population dynamics. Thirty-three invited lectures were presented in sessions of four or five parallel talks.

As in the four preceding EMS congresses, ten EMS prizes were given to young researchers, not older than 35 years, who had been selected by a Prize Committee appointed by the EMS. In addition, the Felix Klein Prize was awarded for the second time, jointly by the EMS and the Institute for Industrial Mathematics in Kaiserslautern, for an application of mathematics to a concrete and difficult industrial problem.

There were twenty-two minisymposia, spread over the whole mathematical area. These minisymposia played a role in attracting people to the ECM meeting that would otherwise perhaps not have come to such a broad mathematics congress. The organizers are grateful to the organizers of the minisymposia for their valuable help.

Two Round Table meetings were organized: one on Industrial Mathematics and one on Mathematics and Developing Countries.

As part of the 44th Nederlands Mathematisch Congres, the so-called Brouwer lecture was given, by Phillip Griffiths of IAS Princeton. The Brouwer lecture is organized every three years by the KWG. The Brouwer lecturer receives a gold medal commemorating the Dutch mathematician L. J. Brouwer. The Brouwer lecture with the Brouwer medal is The Netherlands' most prestigious award in mathematics. Information about Brouwer was given by Dirk van Dalen in an invited historical lecture during the congress. Other parts of NMC44 were the 9th Beeger lecture by Dan Bernstein of the University of Illinois at Chicago (organized once every two years to commemorate the Dutch number theorist N. G. W. H. Beeger and sponsored by CWI Amsterdam) and the third Philips PhD prize lectures for Dutch PhD students (sponsored by Philips Eindhoven and this time won by Erik Jan van Leeuwen of CWI Amsterdam).

These proceedings present extended versions of nineteen of the invited talks which were delivered during 5ECM. We are grateful to the authors for their contributions and to the following referees: Keith Ball, Henk Broer, Arjeh Cohen, Gerard van der Geer, Robbert Dijkgraaf, Klaas Landsman, Eduard Looijenga, Terry Lyons, Yvan Martel, Andrzej Pelczar, Nicolai Reshetikhin, David Riley, Benjamin Rossman, Marta Sanz-Solé, Floris Takens, Constantin Teleman, Rob Tijdeman, Bruno Vallette, and Don Zagier.

A congress of this size is impossible to organize without generous financial support from the government, businesses and industry, and the local mathematical community. The full list of subsidy-providers and sponsors is given in the section Sponsors of these Proceedings. Although all subsidy-providers and sponsors are equally appreciated, we like to single out the most important ones here. The largest single subsidy was provided by NWO, the Netherlands Organization for Scientific Research. Biggest sponsors were Foundation Compositio Mathematica and ING Corporation.

The editors

André Ran
Herman te Riele
Jan Wiegerinck

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Jan Schipper and Jos van der Werf (CWI) (editing and printing of numerous 5ECM documents)

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Opening Ceremony

The 5ECM attendants were first welcomed by André Ran, chairman of the 5ECM Organizing Committee. Following this, a spectacular *tableau vivante* of Rembrandt's most famous painting "The Nightwatch" from 1642 was built up on the stage, accompanied by rolls of drum from the drummers in the painting. Next, Robbert Dijkgraaf, president of the Royal Netherlands Academy of Arts and Sciences, Amsterdam, stepped out of this *tableau*, dressed in mediaeval costume, to give the official opening address to the congress. Finally, Ari Laptev, president of the European Mathematical Society, welcomed the attendants on behalf of the EMS.

Welcome Address by André Ran, Chairman of the Organizing Committee of 5ECM

Ladies and gentlemen, it is my great pleasure to welcome you all to Amsterdam to enjoy this special mathematical event. For The Netherlands it was an honour to be selected for the fifth European Congress of Mathematics, and many people within the Dutch mathematical community have worked hard to prepare for you a special congress. As a matter of fact, preparations for this congress started as early as 2001, so we have been at it for almost seven years. We hope the week will be a successful event for all of you, and that all of you will come away from this congress with the feeling that you have learned something new. Above all, we hope you will have an enjoyable week in Amsterdam.

We know that the mathematical events here at the RAI will have to compete with a wonderful city and its surrounding countryside. However, since you are all mathematicians we hope that the math will win from interesting sites like the museums like a boat tour on the canals, or a walk in the city streets.

I would like to take this opportunity to thank all those who supported us financially, via subsidies, via sponsoring, or via gifts. There are too many to name them all. However, I want to name just a few and then show the complete list, which you can also find in your programme on page 2.

The first organization I want to single out is the most important one for us. The Netherlands Organization for Scientific Research provided us with a very substantial subsidy. As a matter of fact, about 1/5 of the total budget was covered from this subsidy. Without their support the congress would not have been possible in Amsterdam.

NWO is investing in mathematics in terms of money, but also in terms of ensuring that the knowledge generated through its initiatives is disseminated and utilised. Since science knows no borders, NWO also is looking across the national border, mainly aiming to increase cooperation in Europe. The development of the European Research

Area is seen as a key element in this. Apart from this NWO cooperates and maintains contacts with Russia, Asia, Africa and America. Representatives of NWO will be around during the congress, and are more then willing to discuss with you. I would like to ask you all to show your appreciation to them in a round of applause.

Other subsidies were donated by many organizations, a full list can be found in your programme book.

There were also many companies who sponsored us with substantial amounts. For most of those you will find an item in your congress bag, or an advertisement in the program, or just read the label on the water bottles that will be distributed during the conference. Their logos can also be found on our website, and just clicking on those will get you to the website of the company.

Other companies, like ING Bank, even have a stand, and have representatives here at the congress. Again, may I ask you to show your appreciation by a round of applause?

Finally, we received gifts from several companies, organizations and private individuals, and important guarantee subsidies from several organizations. These too are of course highly appreciated.

Important support in terms of money, manpower, moral support, and help in the organization was received from the Dutch mathematical community and the Koninklijk Wiskundig Genootschap. Mathematicians from all over the country helped us by being chairs of committees, organizing mini-symposia or helping out with many trivial matters. People from almost all universities in the Netherlands were involved in the organization in one way or another. Without them it would have been impossible to organize this event.

Before I give the floor to the President of the Royal Netherlands Academy of Arts and Sciences, Professor Robbert Dijkgraaf, for the official opening of this congress, I would like to draw your attention to the screen: to save you a trip to the Rijksmuseum, we have already given you a taste of what is undoubtedly the most famous 17th century Dutch painting: the "Nightwatch" by Rembrandt.

Opening Address by Robbert Dijkgraaf, President of the Royal Netherlands Academy of Arts and Sciences, Amsterdam

Distinguished Guests, Fellow Mathematicians, Ladies and Gentlemen,

It is a great privilege and a real pleasure to welcome you to Amsterdam and the fifth European Congress of Mathematics on behalf of the Dutch scientific community in general and the Royal Netherlands Academy of Arts and Sciences in particular.

Given this embarrassing grand entrance, I feel it is appropriate to say a few words on the relation between mathematics and the arts from a historical perspective.

This year the Academy celebrates its bicentenary. It is among the oldest royal institutions of the Netherlands, even antedating the Kingdom of the Netherlands by

The Nightwatch, 1642
Rembrandt van Rijn (1606–1669)
Rijksmuseum Amsterdam

seven years. This mathematical paradox is resolved by the fact that the Academy was founded during the French occupation by a different King of Holland, Louis Napoléon Bonaparte, brother of the French emperor. Louis Napoléon was a somewhat tragic figure, rather unhappy in gloomy and cold Amsterdam, mostly vacationing in the sunny south of France. Yet he bravely tried to master the Dutch language, making himself rather infamous by pronouncing his title of King of Holland consistently as something that translates into English as 'rabbit of Olland'.

The Academy was founded along the French model as the Royal Institute of Science, Letters and Fine Arts. Painters, writers and composers were elected among its first members. Ludwig van Beethoven was a foreign member, as was the French painter Jacques-Louis David. The magnificent 17th century lodgings of the Academy, the Trippenhuis in the historic centre of Amsterdam also known as the small Royal Palace, was for a long time home to the Rijksmuseum. Rembrandt's masterpiece 'de Nachtwacht', that has just materialized before your eyes, was for many years exhibited in our Great Hall. But both the artists and the paintings left the Academy in 1851 when our government in the spirit of that time judged them of little practical use for science.

The Rembrandt Nightwatch tableau with Robbert Dijkgraaf addressing the audience

In the early nineteenth century the bond between science and art was thought to be much more self-evident, although one can wonder for the right reasons. In these days it was argued that scientists should be foremost eloquent and artists should be learned and academic. That attitude was demonstrated by one of the founding fathers of the Royal Academy, the mathematician Jan Hendrik van Swinden, professor at the University of Amsterdam and chair of the international committee defining the metric units, in which capacity he addressed the National Congress in Paris on the 4th of July 1799. He was eloquent in both old and new languages, always speaking with the required 'genteel appearance and civilized posture' and 'without a single letter in writing before him.'

That not all mathematicians could rise to the standard of Van Swinden, now and two hundred years ago, becomes clear from a contemporary description of a mathematician as in general 'stiff, dull, pale and drawn, also in domestic circles, where even the gentle words of a loving spouse or the flattering of precious children could not bring life into his frigid countenance.' 'The mathematician views a lovely landscape with the cold eyes of a land surveyor.'

Ladies and Gentlemen, the last time that the Netherlands saw such a distinguished gathering of mathematicians was the year 1954 when Amsterdam hosted the International Congress of Mathematicians. I am happy to see quite a few distinguished guests present today who attended that Congress as students or young and upcoming professors. This reminds us of the strong historical bonds in mathematics, where only a few degrees of separation connect us all to the grand historical figures. For example, Sir Michael Atiyah recently told me how the ICM in Amsterdam was his first visit abroad as a graduate student and how at that time he was looking forward with great expectations to hear the lectures of Hermann Weyl and John von Neumann. His fellow student at that time, Sir Roger Penrose attended, as many other participants, a reception in the Stedelijk Museum where a small exhibition was organized around an unknown Dutch graphic artist by the name of M. C. Escher. As one says: the rest is history. Penrose subsequently wrote a paper with his father, the distinguished geneticist Lionel Penrose, introducing Escher's impossible figures to the world. One of the remarkable side-effects of the ICM has been the interactions between Escher and mathematicians from all over the world leading to many new pieces of art and many new mathematical ideas.

Ladies and Gentlemen, the ECM is a wonderful initiative of the European Mathematical Society that brings the best of mathematics together. Viewed from a historical perspective this Congress is a remarkable illustration of the rapid growth of mathematics, the diverse spectrum of interactions touching more and more fields in science and applications in society. At the same time it is a testament to the unity of mathematics. The Dutch mathematical community has to be praised for their efforts to host this prestigious congress and for carrying a large share of the financial burden. The unity of mathematics is further enhanced by the incorporation of the Dutch Mathematical Congress within the ECM, this despite the earlier Dutch rejection of the European constitution. Indeed, this congress reminds us of the special role of the Netherlands and Amsterdam through the centuries as a place of scientific diplomacy and cooperation, both within Europe and the world.

With these thoughts I am happy to give the word to the President of the European Mathematical Society, Professor Ari Laptev.

Thank you and enjoy these days in Amsterdam.

Welcome Address by Ari Laptev, President of the European Mathematical Society

Ladies and gentlemen,

It is a great pleasure for me to welcome you all to the Fifth European Congress of Mathematics here in this culturally rich city of Amsterdam and I am delighted to see so many participants.

The Programme Committee has worked hard to provide us with an exciting week of distinguished lectures that I am sure we will find stimulating, challenging and enjoyable.

The 5th European Congress of Mathematics is, without doubt, the main mathematical event of the year 2008. It enables mathematicians from all over the world to meet their fellow colleagues, some of whom they might otherwise never have the chance to meet. Such Congresses also provide opportunities for interaction between different areas of mathematics which often leads to new exciting development of sometimes completely new areas of mathematics.

Mathematics is a very dynamic subject that has a growing number of applications in both traditional and new areas such as environmental science, biology, medicine, finances and telecommunication. The present impressive technological development is unthinkable without the new discoveries in Mathematics made during the last decades.

The European Mathematical Society, founded in 1990, together with the National European Mathematical Societies play an increasingly important role in promoting Mathematical Science in Europe.

In a few moments the names of the ten EMS Prize Winners will be announced. These brilliant young European mathematicians have already made substantial contributions in different areas of our beautiful subject and we thank the Prize Committee for their excellent choices.

Finally on behalf of all the Congress participants I would like to congratulate the 5ECM organizers on their committed work and for making this event possible.

And lastly I would like to express our gratitude to all the sponsors without whom this Congress would not be possible. We are very grateful to all funding agencies supporting the Congress, in particular, the Royal Dutch Mathematical Society for their involvement in organizing this event.

Prize Ceremony

Ten *EMS prizes* were awarded during 5ECM by the European Mathematical Society in recognition of distinguished contributions in Mathematics by young researchers not older than 35 years. The EMS prizes are presented every four years at the European Congress of Mathematics.

The Prize Committee was appointed by the EMS and consisted of fifteen recognized mathematicians from a wide variety of fields (listed in the section Committees). The prizes were first awarded in Paris in 1992, followed by Budapest in 1996, Barcelona in 2000, and Stockholm in 2004. During 5ECM in Amsterdam, the prizes were awarded after the Opening Ceremony on July 14, 2008. Each prize winner received 5,000 Euro.

The prize money for the EMS Prizes was generously made available by the Dutch Foundation Compositio Mathematica.

The *Felix Klein Prize* has been established by the European Mathematical Society and the endowing organization: the Institute for Industrial Mathematics in Kaiserslautern. It is awarded to a young scientist or a small group of young scientists (normally under the age of 38) for using sophisticated methods to give an outstanding solution to a concrete and difficult industrial problem, which meets with the complete satisfaction of industry. The Prize is presented every four years at the European Congress of Mathematics. The prize committee consisted of six members appointed by agreement of the EMS and the Institute for Industrial Mathematics in Kaiserslautern (listed in the section Committees). The first prize was presented at 3ECM in Barcelona to David C. Dobson. During 4ECM in Stockholm, no Felix Klein Prize was awarded.

The eleven Prize Winners are listed below. Two of the EMS Prize winners, Artur Avila and Laure Saint-Raymond, were invited by the 5ECM Scientific Committee to present an Invited Lecture, before they were selected as Prize Winner by the Prize Committee.

The Prize Ceremony during 5ECM was chaired by the Chair of the EMS prize Committee Rob Tijdeman.

The Prize Winners

Artur Avila

Full name: Artur Avila Cordeiro de Melo, born: June 29, 1979; citizenship: Brazilian; Ph.D.: IMPA Rio de Janeiro, Brazil; presently: Clay Mathematics Institute, Paris 6, France and IMPA, Rio de Janeiro, Brazil.

Artur Avila has obtained many important results in dynamical systems, especially in the theory of iterated rational maps and the Teichmüller geodesic flow. Several of them provide the final solution to longstanding and major problems, for example: his proof with Lyubich that there are infinitely renormalizable Julia sets in the quadratic family $f(z) = z^2 + c$ with Hausdorff dimension strictly less than 2, his proof with Jitomirskaya of the "ten Martini Conjecture" of B. Simon, his proof with Viana of the Kontsevich–Zorich conjecture on simplicity of the Lyapunov spectrum for the Teichmüller geodesic flow, his proof with Forni that almost every interval exchange which does not have the combinatorics of a rotation is weakly mixing and his proof with Gouëzel and Yoccoz of exponential mixing for the Teichmüller flow. He is internationally recognized as a leader of research in these areas.

Alexei Borodin

Born: June 25, 1975; citizenship Russian; Ph.D.: University of Pennsylvania, U.S.A. 2001; presently: CalTech, Pasadena, U.S.A.

Alexei Borodin has made substantial contributions to the representation theory of "big" groups, to combinatorics, interacting particle systems and random matrix theory. A key observation of Borodin and Olshanski in the representation theory of big groups is that the irreducible characters for the group are associated with stochastic point processes. Borodin found a determinantal formula for the correlation functions of the so-called generalized regular representation of the infinite symmetric group and, with Olshanski, also of the unitary group. A stunning consequence of his work is one of the first proofs of a conjecture of Baik, Deift and Johansson in Combinatorics. In later work Borodin analyzed the irreducible character associated with the generalized regular representation. Borodin and his collaborators also developed a radical new approach for analyzing totally antisymmetric simple exclusion processes. Equally remarkable is his work on isomonodromy transformations of linear systems of difference equations and his solution of a problem of Widom on the spectrum of some matrix. Borodin is a brilliant mathematician.

Ben Green

Full name: Ben Joseph Green, born: February 27, 1977; citizenship: British; Ph.D.: University of Cambridge, 2002; presently University of Cambridge, England.

Ben Green is best known for his celebrated result with Terence Tao that there exist arbitrarily long arithmetic progressions of primes. Some basic ideas for the proof can already be found in the earlier work of Green. Therein he proved that every relative dense subset of the primes contains an arithmetic progression of length 3. In another paper he improved a result of Bourgain on the sumset of two dense subsets of an interval. Where Bourgain obtained a lower bound 1/3 in the exponent and Ruzsa an upper bound 2/3, Green got a lower bound 1/2. One of the essential steps in the proof of the famous result with Tao is the discovery by Green that the work of Goldston and Yildirim on short intervals between primes provided precisely the "random-like" superset of the primes that they needed. After their proof Green and Tao have continued their investigations. This has allowed them to give an asymptotic for how many progressions of length 4 there are in the primes up to N. By now Green has a string of highly impressive results.

Olga Holtz

Name: Olga V. Holtz; born: August 19, 1973; citizenship: Russian; Ph.D.: University of Wisconsin-Madison, 2000; presently: Technische Universität Berlin, Germany, and University of California-Berkeley, U.S.A.

Olga Holtz has made substantial contributions to several mathematical areas including algebra, numerical linear algebra, approximation theory, theoretical computer science and numerical analysis. Some of these are spectacular results such as the proof of the

Newton inequalities for M-matrices, the fundamental work on accurately evaluating polynomials in finite arithmetic and the proof that all group theory based fast matrix multiplication methods are numerically stable. These are not only very strong results in theoretical computer science that may have a fundamental impact on computational methods of the coming years, but they also required very deep mathematical theory in the context of finite group theory. Her new work on zonotopal algebra is a substantial contribution to combinatorial commutative algebra. Olga Holtz is a mathematician who truly transcends the traditional boundaries of applied versus pure mathematics.

Bo'az Klartag

Born: April 25, 1978; citizenship: Israeli; Ph.D.: Tel-Aviv University, 2004; presently: Clay Mathematics Institute, Princeton University, U.S.A.

Bo'az Klartag's main achievements are in Asymptotic Geometric Analysis. He has solved a number of long standing problems in this field. He broke the record on the minimum number of symmetrization steps of convex bodies required to transform them into near balls, thereby solving problems posed by Hadwiger and Bourgain–Lindenstrauss–Milman. He solved the isomorphic version of a slicing problem posed by Bourgain 20 years ago, exhibiting novel ideological and technical ideas. This work has a strong impact on Functional Analysis. He proved a central limit theorem for convex bodies, a beautiful result bringing, in a novel way ideas of Convex Geometry into Probability Theory. With Feffermann he solved a fundamental problem on optimal extrapolation of smooth functions. Bo'az Klartag is a surprisingly productive young mathematician who has succeeded, in a very short time, to make breakthroughs in a number of different directions of major significance in modern analysis.

Alexander Kuznetsov

Born: November 1, 1973; citizenship: Russian; Ph.D.: Moscow State University, 1998; presently: Steklov Mathematical Institute, Moscow, Russia.

Kuznetsov has made fundamental contributions to birational projective geometry, representation theory, mathematical physics, homological algebra, and non-commutative geometry. A trademark of his work is the blend of his ground-breaking ideas and technical sophistication. His work on birational projective geometry includes theories of homological Lefschetz decompositions, homological projective duality and categorical resolutions of singularities. Kuznetsov boldly and innovatively combines several ideas ranging from very classical algebraic geometry such as Mori's Minimal Model Program to such hot topics as Kontsevich's Homological Mirror Symmetry Program. His techniques can be used in situations where the conventional constructions do not apply and thus extend the range of birational projective geometry considerably. Kuznetsov's work is a great source of inspiration.

Assaf Naor

Born: May 7, 1975; citizenship: Czech/Israeli; Ph.D.: Hebrew University, Jerusalem, Israel; presently Courant Institute, New York, U.S.A.

Assaf Naor has made ground-breaking contributions to three mathematical fields: functional analysis, the theory of algorithms and combinatorics. Naor is the leading architect of the modern theory of non-linear functional analysis: a theory that has taken off in recent years and has become an essential tool in mathematical computer science. Among other things, Naor and a variety of collaborators discovered an unpredicted threshold phenomenon in the non-linear Dvoretzky Theorem, found a non-linear analogue of the cotype invariant and proved a sophisticated non-linear analogue of the celebrated Maurey–Pisier Theorem. Naor's work has led to essentially optimal embeddings of finite subsets of L_1 into Hilbert space and thence, the best available polynomial time approximation algorithm to compute the sparsest cut in a network with several commodities. Assaf Naor's versatility, originality and technical power are overwhelming and his work has a profound influence on functional analysis and mathematical computer science.

Laure Saint-Raymond

Born: August 4, 1975, citizenship: French; Ph.D.: Paris VII, France, 2000; presently: ENS Paris, France.

Laure Saint-Raymond is well known for her outstanding results on nonlinear partial differential equations in the dynamics of gases and plasmas and also in fluid dynamics. Her most striking work concerns the study of the hydrodynamic limits of the equation of Boltzmann in the kinetic theory of gases, where she answered a question posed by Riemann within the framework of his 6th problem. Recently, in collaboration with I. Gallagher, she aims at understanding the equations of rotating fluids within the limit where the number of Rossby tends to 0. They have already obtained surprising results in this direction. At 32 years, Laure Saint-Raymond is at the origin of several outstanding and difficult results in the field of nonlinear partial differential equations of mathematical physics. She is one of the most brilliant young mathematicians in her generation.

Agata Smoktunowicz

Born: October 12, 1973; citizenship: Polish; Ph.D.: PAN, Warsaw, Poland; presently: University of Edinburgh, Scotland and Institute of Mathematics of the Polish Academy of Sciences.

Agata Smoktunowicz has solved a number of outstanding problems in noncommutative algebra. She has made the first significant progress for decades on some fundamental problems concerning nil rings. The most spectacular of these results is the construction, over any countable field, of a simple nil algebra. This solves a famous problem of Levitsky, Jacobson and later Kaplansky from around 1970. This work is a technical tour-de-force. Other outstanding problems she has solved include an answer to a problem about polynomial rings over nil rings first asked by Amitsur in 1971, the proof of the Artin–Stafford Gap Theorem for graded domains, and the first examples of finitely generated nil, but not nilpotent algebras with polynomially bounded growth. In all her work, Smoktunowicz has introduced novel techniques and constructions and she displays a great ability to deal with long, difficult and technically demanding calculations.

Cédric Villani

Born: October 5, 1973; citizenship: French; Ph.D.: ENS, Paris, France, 1998; presently: ENS Lyon, France.

Cédric Villani has contributed to the theory of non-equilibrium statistical mechanics, in particular in connection with the Boltzmann equation and the Landau equation in plasma physics. He proved the Cercignani conjecture and obtained with Desvillettes the first convergence result to a global gaussian equilibrium for the Boltzmann equation without any smallness assumption. A second component of Villani's work is at a crosspoint between probability, functional analysis, partial differential equations, differential and Riemannian geometries. With Otto he studied the link between diffusion equations, Talagrand inequalities and logarithmic Sobolev inequalities. More recently, Lott and Villani obtained a new characterization of Riemannian manifolds with bounded Ricci curvature from below, in terms of convexity of the Boltzmann entropy with respect to optimal transportation (Monge–Kantorovich–Wasserstein) metrics. By his way of looking at problems Villani has inspired many.

Josselin Garnier, the Felix Klein Prize Winner

Born: June 18, 1971; citizenship: French; Ph.D.: École Polytechnique, 1996; presently: Université Paris 7.

Josselin Garnier was appointed associate Professor in Mathematics in Toulouse at the (remarkably young) age of 30, and he joined the Université Paris Diderot (Paris 7) in 2005, where he became a full professor in 2007. He is affiliated to the Laboratoire de Probabilités et Modèles Aléatoires and the Laboratoire Jacques Louis Lions. He is also a scientific consultant at the Nuclear Energy Agency (CEA), he has a number of research contracts with many teams of CEA, with the French Electric Company (EDF), and with the European Aeronautic Defence and Space company (EADS). In 2006, he has been one of the organizers (with Guillaume Bal and Didier Lucor) of the CERMRACS summer activity of SMAI that aims at promoting the collaboration between academic and industrial mathematicians on dedicated problems.

His research is at the interface of stochastics and applied analysis, and the fields of applications are mainly in optics, wave propagation and plasma physics. He is a leading scientist dealing with probabilistic aspects in the framework of partial differential equations and he has shown his ability to apply powerful theoretical tools to deal with real industrial problems.

Josselin Garnier has both an impressive academic curriculum (wave propagation in random medium where a recent breakthrough is the analysis of time reversal of the wave when the medium is randomly layered, first proof of the existence of solitons in random media with qualitative and quantitative information, analysis of Bose–Einstein condensates...) where he has published numerous high level publications in international scientific journals both in the mathematical area and in applied physics area but he is also deeply involved in real applications (new techniques in imaging for the detection of buried objects, telecommunication for comparison of signal-to-noise ratio and signal-to-interference ratio for various protocols in wireless communication, design of the target in the Laser Mega Joule experimental device in the framework of Inertial Confinement Fusion, problems in aeronautics where for acoustic problems, electromagnetic compatibility analysis, design of antennas.... the industrial conception has to incorporate now Random modeling and uncertainty management). Finally he knows very well the state-of-the-art about most of the numerical methods in Computational Fluid Dynamics and he can provide very useful orientations for robust simulations of these problems.

Congress Programme

Plenary Lectures

Luigi Ambrosio	Optimal transportation and evolution problems in spaces of probability measures
Christine Bernardi	From a posteriori analysis to automatic modelling
Jean Bourgain	New developments in arithmetic combinatorics
Jean-François Le Gall	The continuous limit of large random planar maps
François Loeser	The geometry behind non-archimedean integrals
László Lovász	Very large graphs
Matilde Marcolli	Renormalization, Galois symmetries and motives
Felix Otto	Pattern formation and partial differential equations
Nicolai Reshetikhin	Topological quantum field theory: 20 years later
Richard Taylor	The Sato–Tate conjecture

Science Lectures

J. Ignacio Cirac	Quantum information theory: applications and challenges
Tim Palmer	Climate change and the trillion-dollar millenium mathematics
Jonathan Sherratt	Periodic travelling waves in field vole populations

Invited Lectures

Nalini Anantharaman	Entropy and localization of eigenfunctions
Christoph Böhm	Ricci flow in higher dimensions
Annalisa Buffa	On the discretization of differential forms
José Antonio Carrillo	The Patlak–Keller–Segel model: free energies, geometric inequalities
Nils Dencker	The solvability of differential equations
Bas Edixhoven	On the computation of the coefficients of modular forms
Manfred Einsiedler	Spectral gap and effective equidistribution

László Erdös	Derivation of the Gross–Pitaevskii equation for the dynamics of the Bose–Einstein condensate
Nicola Fusco	The sharp Sobolev inequality in quantitative form
Søren Galatius	Homotopy theory and automorphism groups
Dmitry Kaledin	Motivic structures in non-commutative geometry
Nikita Karpenko	Essential dimension of finite p-groups
Arno Kuijlaars	Critical phenomena in random matrix theory
Miklós Laczkovich	Whitney constants, twisted sums, and the difference property
Michel Ledoux	Markov operators, classical orthogonal polynomial, ensembles and random matrices
Wolfgang Lück	Topological rigidity of aspherical manifolds
Yvan Martel	Inelastic collision of two solitons for nonintegrable gKdV equations
Sergei Merkulov	Wheeled pro(p)file of Batalin–Vilkovisky formalism and BF theory of unimodular Poisson structures
Ralf Meyer	Equivariant non-commutative topology
Oleg Musin	Positive definite functions in distance geometry
Nikolai Nadirashvili	Singular solutions to fully nonlinear elliptic equations
Jaroslav Nešetřil	From sparse to nowhere dense structures: dualities and first order properties
Yuval Peres	Internal aggregation with multiple sources
Christoph Schweigert	Bundle gerbes and surface holonomy
H. Mete Soner	Nonlinear parabolic PDEs and pricing intervals
Balázs Szegedy	Non-standard methods, regularity and the completion of hyper-graphs
Constantin Teleman	Topological field theories in 2 dimensions
Ana Vargas	Bilinear restriction theorems and applications to dispersive equations
Frank Wagner	Geometric model theory
Reinhard Werner	Locality and unitarity in the structure of quantum cellular automata
Andreas Winter	High dimensional geometry in quantum information
Ragnar Winther	Finite element exterior calculus – a link between algebraic topology and numerical analysis
Stanislaw Woronowicz	The trace formula for Haar weight on locally compact quantum groups

Prize Winner Lectures

Artur Avila	Dynamics of quasiperiodic cocycles and the spectrum of the almost Mathieu operator
Alexei Borodin	Random surfaces in dimensions two, three, and four
Ben Green	Patterns of primes
Olga Holtz	Complexity and stability of linear problems
Bo'az Klartag	High-dimensional distributions with convexity properties
Alexander Kuznetsov	Derived categories and rationality of cubic fourfolds
Assaf Naor	The story of the sparsest cut problem
Laure Saint-Raymond	Some results about the sixth problem of Hilbert
Agata Smoktunowicz	On some open questions in noncommutative ring theory
Cédric Villani	Optimal transport and Riemannian geometry: Monge meets Riemann
Josselin Garnier	Passive sensor imaging using cross correlations of noisy signals

Round Table on Industrial Mathematics

Wim Mulder	The seismic inverse problem
Valtteri Niemi	Mathematics in mobile communications
Wil Schilders	Mathematics in the electronics industry, and in industry as a whole
Gerrit T. Timmer	Applied mathematics at work: lessons learned in 25 years at ORTEC

Round Table on Mathematics and Developing Countries

(moderators: Andreas Griewank, Tsou Sheung Tsun)

This round table discussion was organised as a follow up to one on "Developing Mathematics in the Developing World" held at ICIAM07. While the previous event had a global scope, this one has focused on developing mathematics in Africa. Apart from the moderators the panel included Wandera Ogana (Kenya), Laura Pauline Fotso (Cameroon), Gareth Whitten (South Africa), Leif Abrahamsson (Uppsala University), Paulus Gerdes (Mocambique), Mohamed Jaoua (Nice) and Bernard Philippe

(Rennes). They are all actively involved in several development activities and organizations.

Each of the panelists spoke for about 10 minutes, followed by a discussion among and between themselves and the audience on the following topics: Status quo of mathematics in statistical terms; Challenges with the development of advanced Centres of Excellence; Barriers: political, economical, and cultural; Remedies: "Twinning" of departments from developing countries with departments from the developed countries; Strategies to persuade African governments to support the development of mathematics in their countries.

Special Lectures

Dan Bernstein	Edwards curves (Beeger Lecture)
Dirk van Dalen	Brouwer's revolution – a century later
Phillip Griffiths	Complex algebraic geometry (Brouwer Lecture)

Philips PhD Prize Lectures

Stefanie Donauer	Infinitely many unobservable data – asymptotics in deconvolution problems
Willemien Ekkelkamp	Predicting the sieving effort for the number field sieve factorization method
Robbert de Haan	More efficient cryptography from error correcting codes
Erik Jan van Leeuwen	Geometric optimization for wireless networks and computational biology
Arjen Stolk	An algebraic approach to discrete tomography
Yana Volkovich	Probabilistic analysis of web ranking

Minisymposia

Advances in Variational Evolution (org.: Alexander Mielke, Ulisse Stefanelli)

Yann Brenier	A non-convex gradient flow structure for mass transport, convection and magnetic relaxation
Nassif Ghoussoub	Navier–Stokes evolutions as self-dual variational problems
Giuseppe Savaré	Gradient flows and diffusion in metric spaces under lower curvature bounds

Ulisse Stefanelli	The weighted-energy-dissipation functional

Algebra and Optimization (org.: Jan Draisma, Monique Laurant)

Harm Derksen	G-invariant tensors
Marie-Françoise Roy	Certificates of positivity in the Bernstein's basis
Markus Schweighofer	Which sets can be described by linear matrix inequalities?
Frank Vallentin	Semidefinite programming bounds

Applications of Noncommutative Geometry
(org.: Gunther Cornelissen, Klaas Landsman)

Caterina Consani	Noncommutative geometry and motives
Yuri Manin	An update on real multiplication
Pedro Resende	Noncommutative geometry and Bohr's doctrine of classical concepts
Walter van Suijlekom	On the geometry of noncommutative gauge fields

Applied Algebraic Topology (org.: Michael Farber)

Yuliy Baryshnikov	Enumeration in sensor networks and integrals with respect to Euler characteristics
Gunnar Carlsson	Persistent topology and data
Konstantin Mischaikow	Databases for global nonlinear dynamics
Marian Mrozek	Reduction homology algorithms
Shmuel Weinberger	A topological view of unsupervised learning from noisy data

Combinatorics of Hard Problems (org.: Josep Diaz, Oriol Serra, Jaroslav Nešetřil)

Jiří Matoušek	Low-distortion embeddings in $\mathbb{R}^d$
Colin McDiarmid	Random graphs from a minor-closed class
Marc Noy	Enumeration of planar graphs and related families of graphs
Vera T. Sós	Convergence of dense graph sequences

Coupled Cell Networks (org.: Peter Ashwin, Ana Dias, Jeroen Lamb)

Konstantinos Efstathiou	Unstable attractors and heteroclinic cycles in pulse coupled networks with delay

Michael Field	Global dynamics and heteroclinic cycles in coupled cell systems
Hiroshi Kori	Synchronization engineering via global delayed nonlinear feedback
Oleksandr V. Popovych	Decoupling of oscillatory ensembles by mixed nonlinear delayed feedback
Eric Shea-Brown	Reliable and unreliable dynamics in driven oscillator networks

Discrete Structures in Geometry and Topology (org.: Dmitry Feichtner-Kozlov)

Corrado De Concini	Hyperplane arrangements, polytopes and box-splines
Peter Littelmann	Equations defining symmetric varieties and affine grassmannians
Frank Sottile	Bounds for real solutions to equations from geometry
Sergey Yuzvinsky	Completely reducible fibers of a pencil of curves and combinatorics

Galois Theory and Explicit Methods (org.: Bart de Smit)

Anna Morra	Counting cubic extensions with given quadratic resolvent
Samir Siksek	A multi-Frey approach to Diophantine equations
William Stein	Computing with abelian varieties of GL_2-type using Sage
Jan Tuitman	A generalized sparse effective nullstellensatz

Global Attractors in Hyperbolic Hamiltonian Systems
(org.: Andrew Comech, Alexander Komech)

Vladimir Buslaev	Generic scenario of the scattering for nonlinear wave equations
Andrew Comech	Global attraction to solitary waves in models based on the Klein–Gordon equation
Scipio Cuccagna	On asymptotic stability of standing waves of nonlinear Schrödinger equations
Elena Kopylova	Scattering of solitons for the Schrödinger equation coupled to a particle
Markus Kunze	Radiation in classical particle systems
A. E. Merzon	On scattering states in the nonlinear Lamb system
David Stuart	Vortices in a Chern–Simons–Schrödinger system

Graphs and Matroids (org.: Bert Gerards, Hein van der Holst, Rudi Pendavingh)

Jim Geelen	Matroid minors
Paul Seymour	Perfect matchings in planar cubic graphs
Robin Thomas	K_t minors in large t-connected graphs
Carsten Thomassen	Graph decomposition

Hypoellipticity, Analysis on Groups and Functional Inequalities
(org.: Waldemar Hebisch, Boguslaw Zegarlinski)

Jean-Philippe Anker	Evolution equations on homogeneous spaces
Dominique Bakry	Gradient bounds for hypoelliptic heat equations
Martin Hairer	Slow energy dissipation in anharmonic chains
Krzysztof Oleszkiewicz	Noise stability of functions with low influences

Mathematical Challenges in Cellular Systems
(org.: Frank Bruggeman, Mark Peletier)

Pauline Hogeweg	Spatial pattern formation and multilevel evolutionary dynamics
Johan Paulsson	Fundamental limits on the suppression of noise
David Rand	Global sensitivity and summation laws for cellular network dynamics
Jens Timmer	Data-based identifiability analysis of non-linear dynamical models

Mathematical Finance (org.: Hans Schumacher, Peter Spreij)

Hans Föllmer	Probabilistic quantification of financial uncertainty
Philip Protter	Modelling financial bubbles
Walter Schachermayer	The fundamental theorem of asset pricing for continuous processes under small transaction costs
Thaleia Zariphopoulou	SPDE and portfolio choice

Mathematical Logic (org.: Peter Koepke, Benedikt Löwe, Jaap van Oosten)

Mirna Džamonja	Combinatorics of trees
Sy-David Friedman	Consistency completeness
Mai Gehrke	Duality theory as a Rosetta Stone for relational semantics

Erik Palmgren	Point-free topology versus topology according to Brouwer and Bishop
Giovanni Sambin	Minimalist foundation and pluralism in mathematics: computation and structure in topology
Katrin Tent	Simplicity of certain automorphism groups
Boban Velickovic	PCF structures of height less than ω_3

Mathematics of Cryptology (org.: Ronald Cramer)

Steven Galbraith	Elliptic curves, pairings and public key cryptography
Oded Goldreich	The bright side of hardness – relating computational complexity and cryptography
Eyal Kushilevitz	The private information retrieval problem
Renato Renner	Induction and quantum cryptography

Random and Quasi-periodic Operators
(org.: Frédéric Klopp, François Germinet)

Alexander Fedotov	On the behavior at infinity of solutions of an almost periodic equation
Abel Klein	Recent results concerning localization in continuum random Schrödinger operators
Wolfgang König	A two-cities theorem for the parabolic Anderson model
Raphael Krikorian	KAM-Liouville theory and an extension of a theorem by Dinaburg and Sinai
Armen Shirikyan	Control and mixing for nonlinear PDE's
Simone Warzel	On the joint distribution of energy levels of random Schrödinger operators

Representation Theoretical Methods and Quantization
(org.: Stefaan Caenepeel, Jürgen Fuchs, Alexander Stolin, Christoph Schweigert, Freddy van Oystaeyen)

Giovanni Felder	Riemann–Roch–Hirzebruch formulae for traces of differential operators
Gilles Halbout	Universal deformations and propic methods in quantum groups
Bernhard Keller	Mutations of quivers with potentials and derived equivalences
Wendy Lowen	Stacks and dg categories in deformation theory
Ingo Runkel	Conformal field theory, vertex operators, and Frobenius algebras

Rough Path Theory (org.: Peter K. Friz)

Sandy M. Davie	Discrete approximation to solutions of rough path equations
Peter Friz	On some properties of SDEs and first order SPDEs with Gaussian noise
Massimiliano Gubinelli	Some infinite dimensional rough-paths
Christian Litterer	Numerical analysis on Wiener space: cubature methods

Singular Structures in Variational PDE's (org.: Matthias Roeger, Mark Peletier)

Piotr Gwiazda	Flat metric and structural stability of a nonlinear population model
Matthias Kurzke	Vortices in Chern–Simons Higgs theories away from self-duality
Xiaofeng Ren	On the resonance condition of a singular limit problem from the Ohta–Kawasaki and the Gierer–Meinhardt theories
Marc Oliver Rieger	Two applications of transport theory: gradient flows of Young measures and portfolio optimization
Didier Smets	Stability of the kink for the NLS flow

Spectral Problems and Hilbert Spaces of Entire Functions
(org.: Joaquim Bruna, Hakan Hedenmalm, Kristian Seip, Mikhail Sodin)

Aharon Atzmon	Weighted Hardy spaces and the uncertainty principle for Fourier transforms
Alexander Borichev	Approximation on the line: weight's perturbations
Jean-François Burnol	The Fourier transform as a spectral problem
Nikolai Makarov	Linear complex analysis and de Branges spaces
Alexander Ulanovskii	Universal sampling and interpolation of band-limited signals

Spectral Theory
(org.: E. Brian Davies, Timo Weidl, Frédéric Klopp, Thomas Hoffmann-Ostenhof)

Maria Esteban	Self-adjoint extensions via Hardy-like inequalities
Rupert L. Frank	Lieb–Thirring and Hardy–Sobolev inequalities
Gian Michele Graf	Quantization of charge transport: equivalence of scattering and Chern number approaches
Yoram Last	On the structure of Hofstadter's butterfly

Alexander V. Sobolev	Some aspects of perturbation theory for the periodic Schrödinger operators

Weak Approximations of Stochastic Differential Equations (org.: Dan Crisan)

Mireille Bossy	Discretization of non-linear Langevin SDEs
Emmanuel Gobet	Closed pricing formula via weak approximation of financial models
Peter Kloeden	Convergence in stochastic numerics: some new developments
Terry Lyons	Resampling and cubature on Wiener space

Invited Lectures

5ECM Amsterdam 2008

Uniqueness of bounded solutions to aggregation equations by optimal transport methods

José A. Carrillo and Jesús Rosado

Abstract. We show how to extend the method used in [22] to prove uniqueness of solutions to a family of several nonlocal equations containing aggregation terms and aggregation/diffusion competition. They contain several mathematical biology models proposed in macroscopic descriptions of swarming and chemotaxis for the evolution of mass densities of individuals or cells. Uniqueness is shown for bounded nonnegative mass-preserving weak solutions without diffusion. In diffusive cases, we use a coupling method [16], [33], and thus we need a stochastic representation of the solution to hold. In summary, our results show, modulo certain technical hypotheses, that nonnegative mass-preserving solutions remain unique as long as their L^∞-norm is controlled in time.

Mathematics Subject Classification (2000). 35A02.

Keywords. Optimal transport, uniqueness, aggregation equation.

1. Introduction

We aim to study the uniqueness of solutions to continuity equations evolving a nonnegative density $\rho(t,x)$ at position $x \in \mathbb{R}^N$ and time $t > 0$ by the equation

$$\begin{cases} \dfrac{\partial \rho}{\partial t}(t,x) + \operatorname{div}\left[\rho(t,x)u(t,x)\right] = 0, & t > 0,\ x \in \mathbb{R}^N, \\ u(t,x) := -\nabla K * \rho(t,x), & t > 0,\ x \in \mathbb{R}^N, \\ \rho(0,x) = \rho_0(x) \geq 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u(t,x) := -\nabla K * \rho(t,x)$ is the velocity field. We will also deal with this uniqueness issue for the associated equations in which a linear diffusion term is added, i.e.,

$$\begin{cases} \dfrac{\partial \rho}{\partial t}(t,x) + \operatorname{div}\left[\rho(t,x)u(t,x)\right] = \Delta\rho, & t > 0,\ x \in \mathbb{R}^N, \\ \rho(0,x) = \rho_0(x) \geq 0, & x \in \mathbb{R}^N. \end{cases} \tag{1.2}$$

The initial data is assumed to have total finite mass, $\rho_0 \in L^1(\mathbb{R}^N)$. Moreover, since solutions of (1.1) formally preserves the total mass of the system

$$\int_{\mathbb{R}^N} \rho(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^N} \rho_0(y)\,\mathrm{d}y := M, \tag{1.3}$$

we can assume, without loss of generality, that we work with probability measures, i.e., $M = 1$, by suitable scalings of the equation. A further assumption that will be made through this work is the boundedness of the initial data, i.e., $\rho_0 \in L^\infty(\mathbb{R}^N)$.

This kind of nonlocal interaction equations have been proposed as models for velocity distributions of inelastic colliding particles [3], [4], [31], [14], [15]. Here, typical interaction kernels $K(x)$ are convex and increasing algebraically at infinity. Convexity gives rates of expansion/contraction of distances between solutions, see also [1], and thus uniqueness.

Another source of these models is in the field of collective animal behavior. One of the mathematical problems arising there is the analysis of the long time behavior of a collection of self-interacting individuals via pairwise potentials leading to patterns such as flocks, schools or swarms formed by insects, fishes and birds. The simplest models based on ODE/SDEs systems, for instance [10], [27], led to continuum descriptions [13], [12], [29], [30] for the evolution of densities of individuals. Here, one of the typical potentials used is the Morse potential, which is radial $K(x) = k(|x|)$ and given by

$$k(r) = -C_a e^{-r/\ell_a} + C_r e^{-r/\ell_r}, \tag{1.4}$$

with C_a, C_r attractive and repulsive strengths and ℓ_a, ℓ_r their respective length scales. Typically, these interaction potentials are not convex, and they are composed of an attraction part usually in a certain annular region and a repulsive region closed to the origin while the interaction gets asymptotically zero for large distances, see [12]. Global existence and uniqueness of weak solutions in Sobolev spaces when the potential is well-behaved and smooth, say $K \in C^2(\mathbb{R}^N)$ with bounded second derivatives, were established in [29], [21]. Uniqueness results in the smooth potential case also follow from the general theory developed in [1] as used in [13].

One of the interesting mathematical difficulties in these problems relates to the case of only attractive potentials with a Lipschitz point at the origin as the Morse potential with $C_r = 0$. In this particular case, finite time blow-up for $L^1 - L^\infty$ solutions have been proved for compactly supported initial data, see [21], [6], [5], [7] for a series of results in this direction. In this particular case, a result of uniqueness of $L^1 - L^\infty$ solutions under some additional technical hypotheses was obtained in [5] inspired by ideas from 2D-incompressible Euler equations in fluid mechanics [34].

Finally, another source of problems of this form is the so-called Patlak–Keller–Segel (PKS) model [28], [20] for chemotaxis in the parabolic–elliptic approximation. This equation corresponds to the case in which the potential is the fundamental solution of the operator $-\Delta$ in any dimension. Originally, this model was written in two dimensions with linear diffusion, see [18], [9], [8] for a state of the art in two dimensions and [17] in larger dimensions. Therefore, in the rest we will refer to as "PKS equation without diffusion" and the "PKS equation", respectively. In the case without diffusion, it is known that bounded solutions will exist locally in time and that smooth fast-decaying solutions cannot exist globally. In the classical PKS system in 2D dimensions, the mass is a critical quantity and thus there are global solutions below a critical mass and local in time solutions that may blow-up in time for mass values larger than the critical one. In more dimensions, this dichotomy is not so well known and there are criteria for both situations.

Here, we will essentially work with the three type of interaction potentials above: bounded second derivatives, pointy potentials and Poisson kernels, to show uniqueness of bounded weak solutions on a given time interval $[0,T]$. The idea is based on G. Loeper's work [22] who showed the uniqueness of bounded weak solutions for the Vlasov–Poisson system and the 2D-incompressible Euler equations using as "distance" an estimate on the Euclidean optimal transport distance between probability measures. We handle the case without diffusion in the next section. Finally, in Section 3 we present the adaptation of this idea using a coupling method ([16], [33]) to the case with diffusion by assuming that we have an stochastic representation formula.

2. Uniqueness for aggregation equations

2.1. Notion of solution. Let us start by working with the continuity equation (1.1) with a given velocity field $u\colon [0,T]\times\mathbb{R}^N\to\mathbb{R}^N$. The continuity equation comes from the assumption that the mass density of individuals in a set is preserved by the flow map or characteristics associated to the ODE system determined by

$$\begin{cases} \dfrac{dX(t,\alpha)}{dt} = u(t,X(t,\alpha)), & t\geq 0,\\ X(0,\alpha)=\alpha, & \alpha\in\mathbb{R}^N. \end{cases}$$

Let us assume that the given velocity field u is such that the solutions to the ODE system are globally defined in $[0,T]$ and unique. Moreover, let us assume that the flow map $X(t)\colon\mathbb{R}^N\to\mathbb{R}^N$ for all $t\geq 0$ associated to the velocity field $u(t,x)$, $X(t)(\alpha):=X(t,\alpha)$ for all $\alpha\in\mathbb{R}^N$, is a family of homeomorphisms from $\mathbb{R}^N$ onto $\mathbb{R}^N$. Typically in our cases, $u\in C([0,T]\times\mathbb{R}^N;\mathbb{R}^N)$ and is either Lipschitz or Log-Lipschitz in space, which implies the above statements on the ODE system, see for instance [23], [24].

Given $\rho\in C_w([0,T],L^1_+(\mathbb{R}^N))$, we will say that it is a distributional solution to the continuity equation (1.1) with the given velocity field u and initial data $\rho_0\in$

$L^1_+(\mathbb{R}^N)$, if it verifies

$$\int_0^T \int_{\mathbb{R}^N} \left(\frac{\partial \varphi}{\partial t}(t,x) + u(t,x) \cdot \nabla \varphi(t,x) \right) \rho(t,x) \, \mathrm{d}x \, \mathrm{d}t = \int_{\mathbb{R}^N} \varphi(0,x) \rho_0(x) \, \mathrm{d}x$$

for all $\varphi \in C_0^\infty([0,T) \times \mathbb{R}^N))$. Here, the symbol C_w means continuity with the weak-* topology of measures. Let us point out that under the above hypotheses the term $(u \cdot \nabla \varphi)\rho$ makes perfect sense as duality $L^1 - L^\infty$.

In fact, the distributional solution of the continuity equation with initial data $\rho_0 \in L^1_+(\mathbb{R}^N)$ is uniquely characterized by

$$\int_B \rho(t,x) \, \mathrm{d}x = \int_{X(t)^{-1}(B)} \rho_0(x) \, \mathrm{d}x$$

for any measurable set $B \subset \mathbb{R}^N$, see [1]. In the optimal transport terminology, this is equivalent to saying that $X(t)$ *transports* the measure ρ_0 onto $\rho(t)$ and we denote it by $\rho(t) = X(t) \# \rho_0$ defined by

$$\int_{\mathbb{R}^N} \zeta(x) \rho(t,x) \, \mathrm{d}x = \int_{\mathbb{R}^N} \zeta(X(t,x)) \rho_0(x) \, \mathrm{d}x \quad \text{for all } \zeta \in \mathcal{C}_b^0(\mathbb{R}^N) \, . \tag{2.1}$$

With these ingredients, we can define the notion of solution for which we will prove its uniqueness.

Definition 2.1. A function ρ is a bounded weak solution of (1.1) on [0,T] for a nonnegative initial data $\rho_0 \in L^1(\mathbb{R}^N)$ if it satisfies the following conditions.

(1) $\rho \in C_w([0,T], L^1_+(\mathbb{R}^N))$.

(2) The solutions of the ODE system $X'(t,\alpha) = u(t, X(t,\alpha))$ with the velocity field $u(t,x) := -\nabla K * \rho(t,x)$ are uniquely defined in $[0,T]$ for any initial data $\alpha \in \mathbb{R}^N$.

(3) $\rho(t) = X(t) \# \rho_0$ is the unique distributional solution to the continuity equation with given velocity field u.

(4) $\rho \in L^\infty(0,T; L^\infty(\mathbb{R}^N))$.

Remark 2.2. (1) Let us point out that even if the solution to the continuity equation with given velocity field u is unique, the uniqueness issue for (1.1) is not settled due to the nonlinear coupling through $u = -\nabla K * \rho$.

(2) In order to show that bounded weak solutions exist, one usually needs more assumptions on the initial data depending on the particular choices of the kernel K. Typically for initial data $\rho_0 \in L^1_+ \cap L^\infty(\mathbb{R}^N)$, we will have a time $T > 0$ possibly depending on the initial data and a bounded weak solution on the time interval $[0,T]$.

2.2. Ingredients about optimal transport. Given two probability measures ρ_1 and ρ_2 with bounded second moment, the Euclidean Wasserstein Distance is defined as

$$W_2(\rho_1, \rho_2) = \inf_{\Pi \in \Gamma} \left\{ \iint_{\mathbb{R}^N \times \mathbb{R}^N} |x - y|^2 \, \mathrm{d}\Pi(x, y) \right\}^{1/2} \tag{2.2}$$

where Π runs over the set of transference plans Γ, that is, the set of joint probability measures on $\mathbb{R}^N \times \mathbb{R}^N$ with marginals ρ_1 and ρ_2, i.e.,

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \varphi(x) \, \mathrm{d}\Pi(x, y) = \int_{\mathbb{R}^N} \varphi(x) \, \rho_1(x) \, \mathrm{d}x$$

and

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \varphi(y) \, \mathrm{d}\Pi(x, y) = \int_{\mathbb{R}^N} \varphi(y) \, \rho_2(y) \, \mathrm{d}y$$

for all $\varphi \in C_b(\mathbb{R}^N)$, the set of continuous and bounded functions on $\mathbb{R}^N$.

Remark 2.3. As it will be used below, given two maps $X_1, X_2 \colon \mathbb{R}^N \to \mathbb{R}^N$ and a given probability measure with bounded second moment ρ_0, then

$$W_2^2(X_1 \# \rho_0, X_2 \# \rho_0) \le \int_{\mathbb{R}^N} |X_1(x) - X_2(x)|^2 \mathrm{d}\rho_0(x)$$

just by using $\Pi = (X_1 \times X_2) \# \rho_0$ in the definition of the distance W_2.

First we recall a result, already used in [22], for estimating the displacement interpolation between two absolutely continuous measures ρ_1 and ρ_2 with respect to Lebesgue. Let us define the displacement interpolation between these measures as

$$\rho_\theta = ((\theta - 1)T + (2 - \theta)\mathbb{I}_{\mathbb{R}^N})_{\#}\rho_1 \tag{2.3}$$

for $\theta \in [1, 2]$, where T is the optimal transport map between ρ_1 and ρ_2 due to Brenier's theorem [11] and $\mathbb{I}_{\mathbb{R}^N}$ is the identity map.

Theorem 2.4 ([2], [25], [19], [1]). *Let ρ_1 and ρ_2 be two probability measures on $\mathbb{R}^N$, such that they are absolutely continuous with respect to the Lebesgue measure and $W_2(\rho_1, \rho_2) < \infty$. Then there exists a vector field $v_\theta \in L^2(\mathbb{R}^N, \rho_\theta \, dx)$ such that*

i. $\dfrac{d}{d\theta}\rho_\theta + \operatorname{div}(\rho_\theta v_\theta) = 0$ *for all* $\theta \in [1, 2]$.

ii. $\displaystyle\int_{\mathbb{R}^N} \rho_\theta |v_\theta|^2 \, \mathrm{d}x = W_2^2(\rho_1, \rho_2)$ *for all* $\theta \in [1, 2]$.

iii. *We have the L^∞-interpolation estimate*

$$\|\rho_\theta\|_{L^\infty(\mathbb{R}^N)} \le \max\left\{ \|\rho_1\|_{L^\infty(\mathbb{R}^N)}, \|\rho_2\|_{L^\infty(\mathbb{R}^N)} \right\}$$

for all $\theta \in [1, 2]$.

One of the ingredients in the proof of Loeper for the Vlasov–Poisson and the 2D-incompressible Euler equations is an interpolation estimate between the associated Newtonian potentials.

Proposition 2.5 ([22]). *Let ρ_1 and ρ_2 be two probability measures on $\mathbb{R}^N$ with L^∞ densities with respect to the Lebesgue measure. Let c_i be the solution of the Poisson equation $-\Delta c_i = \rho_i$ in $\mathbb{R}^N$ given by $c_i = \Gamma_N * \rho_i$ with Γ_N the fundamental solution of $-\Delta$ in $\mathbb{R}^N$. Then*

$$\|\nabla c_1 - \nabla c_2\|_{L^2(\mathbb{R}^N)} \le \max(\|\rho_1\|_{L^\infty(\mathbb{R}^N)}, \|\rho_2\|_{L^\infty(\mathbb{R}^N)})^{1/2}\, W_2(\rho_1, \rho_2).$$

We will proceed similarly to the proof of the previous proposition to get the following interpolation in smoother situations where the kernel has integrable Hessian.

Proposition 2.6. *Let ρ_1 and ρ_2 be two probability measures on $\mathbb{R}^N$ with L^∞ densities with respect to the Lebesgue measure. Let $c_i = K * \rho_i$ with $\nabla K \in L^2(\mathbb{R}^N)$ and $|D^2 K| \in L^1(\mathbb{R}^N)$. Then*

$$\begin{aligned}
&\|\nabla c_1 - \nabla c_2\|_{L^2(\mathbb{R}^N)} \\
&\qquad \le \||D^2 K|\|_{L^1(\mathbb{R}^N)} \max(\|\rho_1\|_{L^\infty(\mathbb{R}^N)}, \|\rho_2\|_{L^\infty(\mathbb{R}^N)})^{1/2}\, W_2(\rho_1, \rho_2).
\end{aligned}$$

Proof. Let us first point out that $\nabla c_i \in L^2(\mathbb{R}^N)$ due to the assumption $\nabla K \in L^2(\mathbb{R}^N)$. By using Theorem 2.4, we can write

$$\begin{aligned}
\|\nabla c_1 - \nabla c_2\|^2_{L^2(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |\nabla K * (\rho_1 - \rho_2)(x)|^2 \, \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left| \left(\nabla K * \int_1^2 \frac{\partial}{\partial\theta} \rho_\theta \, \mathrm{d}\theta \right)(x) \right|^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left| \left(\nabla K * \int_1^2 \operatorname{div}(\rho_\theta v_\theta)\, \mathrm{d}\theta \right)(x) \right|^2 \mathrm{d}x.
\end{aligned}$$

Applying Jensen's inequality, integrating-by-parts and estimating the modulus and the L^2-norm of the convolution, we get

$$\begin{aligned}
\|\nabla c_1 - \nabla c_2\|^2_{L^2(\mathbb{R}^N)} &\le \int_1^2 \int_{\mathbb{R}^N} \big| |D^2 K| * (\rho_\theta |v_\theta|) \big|^2 \mathrm{d}x \, \mathrm{d}\theta \\
&\le \|\rho_\theta\|_{L^\infty(\mathbb{R}^N)} \||D^2 K|\|^2_{L^1(\mathbb{R}^N)} \int_1^2 \int_{\mathbb{R}^N} \rho_\theta |v_\theta|^2 \mathrm{d}x \, \mathrm{d}\theta,
\end{aligned}$$

from which the desired inequality follows using the other information from Theorem 2.4. $\square$

2.3. Uniqueness for bounded weak solutions. Let us set some notation for the rest of this section. Let us assume that ρ_1 and ρ_2 are two bounded weak solutions to (1.1). We look at the two characteristics flow maps, X_1 and X_2, such that $\rho_i = X_i \# \rho_0$, $i = 1, 2$, and provide a bound for the distance between them at time t in terms of its distance at time $t = 0$.

In the following, we will address the uniqueness with $u = -\nabla K * \rho$, first providing the details of the computation for a regular smooth kernel $K \in C^2(\mathbb{R}^N)$ and with L^∞-bounded Hessian and then modify it in order to include a more general family of kernels with possibly Lipschitz point at the origin, namely, for kernels with the Hessian bounded in $L^1(\mathbb{R}^N)$. Let us remark that the potential $K(x) = e^{-|x|}$ belongs to this class for $N \geq 2$. Finally, we look at the Keller–Segel model without diffusion, i.e., taking $u = \nabla c = -\nabla \Gamma_N * \rho$. The main theorem is summarized as follows.

Theorem 2.7. *Let ρ_1, ρ_2 be two bounded weak solutions of equation* (1.1) *in the interval $[0, T]$ with initial data $\rho_0 \in L^1_+(\mathbb{R}^N)$ and assume that either*

- *u is given by $u = -\nabla K * \rho$, with K such that $K \in C^2(\mathbb{R}^N)$ and $|D^2 K| \in L^\infty(\mathbb{R}^N)$; or*
- *u is given by $u = -\nabla K * \rho$, with K such that $\nabla K \in L^2(\mathbb{R}^N)$ and $|D^2 K| \in L^1(\mathbb{R}^N)$; or*
- *$u = -\nabla \Gamma_N * \rho$.*

Then $\rho_1(t) = \rho_2(t)$ for all $0 \leq t \leq T$.

Idea of the proof. Given the two bounded weak solutions to (1.1), let us define the quantity

$$Q(t) := \frac{1}{2} \int_{\mathbb{R}^N} |X_1(t) - X_2(t)|^2 \rho_0(x)\, \mathrm{d}x, \tag{2.4}$$

with X_i the flow map associated to each solution, $\rho_i(t) = X_i \# \rho_0$, $i = 1, 2$. Taking into account the remarks to the definition of the W_2-distance, we may conclude that $W_2^2(\rho_1(t), \rho_2(t)) \leq 2Q(t)$. It is then clear that $Q(t) \equiv 0$ would imply that $\rho_1 = \rho_2$.

(1) *Regular kernel case.* In this case, the velocity field is continuous and Lipschitz in space, therefore the characteristics are globally defined and unique. Now, by taking the derivative of Q w.r.t. time, we get

$$\begin{aligned} \frac{\partial Q}{\partial t} &= \int_{\mathbb{R}^N} \langle X_1 - X_2, u_1(x_1) - u_2(x_2) \rangle \rho_0(x) \mathrm{d}x \\ &= \int_{\mathbb{R}^N} \langle X_1 - X_2, u_1(x_1) - u_1(x_2) \rangle \rho_0(x) \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} \langle X_1 - X_2, u_1(x_2) - u_2(x_2) \rangle \rho_0(x) \mathrm{d}x \end{aligned} \tag{2.5}$$

where the dependence on the time variable has been omitted for clarity. Now, taking into account the Lipschitz properties of u into the first integral and using Hölder inequality in the second one, we can write

$$\begin{aligned}\frac{\partial Q}{\partial t} &\le CQ(t) + Q(t)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N} |u_1\left(X_2(t,x)\right) - u_2\left(X_2(t,x)\right)|^2 \rho_0(x)\mathrm{d}x\right)^{\frac{1}{2}} \\ &= CQ(t) + Q(t)^{\frac{1}{2}} I(t)^{\frac{1}{2}}. \end{aligned} \tag{2.6}$$

Now, let us work in the term $I(t)$. By using that the solutions are constructed transporting the initial data through their flow maps, we deduce

$$\begin{aligned} I(t) &= \int_{\mathbb{R}^N} |\nabla K * (\rho_1 - \rho_2)\left[X_2(t,x)\right]|^2 \rho_0(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \Bigg| \int_{\mathbb{R}^N} \nabla K(X_2(x) - y)\rho_1(y)\,\mathrm{d}y \\ &\qquad\qquad - \int_{\mathbb{R}^N} \nabla K(X_2(x) - y)\rho_2(y)\,\mathrm{d}y \Bigg|^2 \rho_0(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \Bigg| \int_{\mathbb{R}^N} \big[\nabla K(X_2(x) - X_1(y)) \\ &\qquad\qquad - \nabla K(X_2(x) - X_2(y))\big]\rho_0(y)\,\mathrm{d}y \Bigg|^2 \rho_0(x)\,\mathrm{d}x \\ &\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \big|\nabla K(X_2(x) - X_1(y)) \\ &\qquad\qquad - \nabla K(X_2(x) - X_2(y))\big|^2 \rho_0(y)\rho_0(x)\,\mathrm{d}y\,\mathrm{d}x, \end{aligned}$$

the last step holding due to Jensen's inequality. Using Taylor's theorem, since K is twice differentiable, we deduce

$$\begin{aligned}\nabla K(A) = \nabla K(B) + \int_0^1 (D^2K)\big[X_2(x) - X_1(y) \\ + \zeta(X_1(y) - X_2(y))\big](X_1(y) - X_2(y))\,\mathrm{d}\zeta\end{aligned}$$

with $A = X_2(x) - X_2(y)$ and $B = X_2(x) - X_1(y)$, and thus $|\nabla K(A) - \nabla K(B)| \le C|X_1(y) - X_2(y)|$ since $|D^2K| \in L^\infty(\mathbb{R}^N)$. This finally gives that $I(t) \le CQ(t)$. Going back to (2.6), we recover $\frac{\partial Q}{\partial t} \le CQ(t)$, and hence we can conclude that if $Q(0) = 0$ then $Q(t) \equiv 0$, implying $\rho_1 = \rho_2$.

(2) *Kernels allowing Lipschitz singularity.* Under the assumptions on the kernel K and the properties of bounded weak solutions, it was shown in [5, Lemma 4.2] that the velocity field u is Lipschitz continuous in space and time. Therefore,

we can recover exactly the relation (2.6) again. Now, in order to estimate $I(t)$, we write it as

$$I(t) = \int_{\mathbb{R}^N} |\nabla K * (\rho_1 - \rho_2)\,[X_2(t,x)]\,|^2 \rho_0(x)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^N} |\nabla K * (\rho_1 - \rho_2)\,(x)|^2 \rho_2(x)\,\mathrm{d}x.$$

Using Proposition 2.6, we deduce that

$$I(t)$$
$$\leq \|\rho_2\|_{L^\infty(\mathbb{R}^N)} \| |D^2 K| \|_{\mathrm{L}^1(\mathbb{R}^N)} \max(\|\rho_1\|_{L^\infty(\mathbb{R}^N)}, \|\rho_2\|_{L^\infty(\mathbb{R}^N)})\, W_2^2(\rho_1,\rho_2)$$
$$\leq CQ(t),$$

and we can conclude similarly as in the previous case.

(3) *PKS model without diffusion.* Under the assumptions on bounded weak solutions, the velocity field in our case is Log-Lipschitz in space. This is a classical result used in 2D-incompressible Euler equations and easily generalized to any dimension [23], [24], [22]. More precisely, $u \in L^\infty((0,T)\times\mathbb{R}^N;\mathbb{R}^N)$ and there exists a constant C depending on the L^1 and L^∞ norms of $\rho(t)$ such that

$$|u(t,x) - u(t,y)| \leq C|x-y|\log\frac{1}{|x-y|} \quad \text{when } |x-y| \leq \frac{1}{2}$$

for any $t \in [0,T]$. The flow map under these conditions can be uniquely defined and is a Hölder homeomorphism.

The uniqueness proof follows estimating the second term in (2.5) as in the previous case. More precisely, we use Proposition 2.5 to infer that

$$I(t) \leq \|\rho_2\|_{L^\infty(\mathbb{R}^N)} \max(\|\rho_1\|_{L^\infty(\mathbb{R}^N)}, \|\rho_2\|_{L^\infty(\mathbb{R}^N)})\, W_2^2(\rho_1,\rho_2) \leq CQ(t),$$

implying

$$\int_{\mathbb{R}^N} \langle X_1 - X_2, u_1(x_2) - u_2(x_2)\rangle \rho_0(x)\mathrm{d}x \leq Q(t)^{\frac{1}{2}} I(t)^{\frac{1}{2}} \leq CQ(t).$$

Now, let us concentrate in the first term of (2.5), we just repeat the standard arguments in [22] to get that by taking T small enough then

$$\int_{\mathbb{R}^N} \langle X_1 - X_2, u_1(x_1) - u_1(x_2)\rangle \rho_0(x)\mathrm{d}x \leq CQ(t)\log^2(2Q(t))$$

where the log-Lipschitz property of u was used. This finally gives the differential inequality

$$\frac{d}{dt}Q(t) \leq CQ(t)\Big(1 + \log\frac{1}{Q(t)}\Big),$$

for $0 \leq t \leq T$ with T small enough. Standard Gronwall-like arguments as in [23] imply $Q(t) = 0$, and thus, the uniqueness. □

3. Uniqueness for aggregation/diffusion competition

We start by defining the concept of solution we will work with.

Definition 3.1. A function ρ is a bounded weak solution of (1.2) on [0,T] for a nonnegative initial data $\rho_0 \in L^1(\mathbb{R}^N)$, if it satisfies the following conditions.

(1) $\rho \in C_w([0,T], L^1_+(\mathbb{R}^N))$.

(2) The SDE system
$$dX(t) = u(t, X(t))\, dt + \sqrt{2}\, dW_t$$
with the velocity field $u(t,x) := -\nabla K * \rho(t,x)$ and initial data $X(0)$ with law $\rho_0(x)$ has a solution given by a Markov process $X(t)$ of law $\rho(t,x)$. Here W_t is the standard Wiener process.

(3) $\rho(t)$ is a distributional solution to (1.2).

(4) $\rho \in L^\infty(0,T; L^\infty(\mathbb{R}^N))$.

We point out that again more additional assumptions on the kernel K and the initial data ρ_0 are needed to prove the existence of such solutions. Solutions of this form have been obtained for particular cases of K in [26], [16]. Moreover, stability estimates, leading in particular to uniqueness of solutions, are obtained under convexity assumptions on the kernel K in [16]. Here, we will assume the existence of bounded weak solutions for the three models introduced in the previous section with diffusion. The existence theory seems a challenging problem to be tackled in the PKS system. The main theorem for these models with diffusion can be summarized as follows.

Theorem 3.2. *Let ρ_1, ρ_2 be two bounded weak solutions of equation (1.2) in the interval $[0,T]$ with initial data $\rho_0 \in L^1_+(\mathbb{R}^N)$ and assume that either*

- *u is given by $u = -\nabla K * \rho$, with K such that $K \in C^2(\mathbb{R}^N)$ and $|D^2K| \in L^\infty(\mathbb{R}^N)$; or*
- *u is given by $u = -\nabla K * \rho$, with K such that $\nabla K \in L^2(\mathbb{R}^N)$ and $|D^2K| \in L^1(\mathbb{R}^N)$; or*
- *$u = -\nabla \Gamma_N * \rho$.*

Then $\rho_1(t) = \rho_2(t)$ for all $0 \le t \le T$.

Proof. Given two bounded weak solutions to (1.2), let us consider that the solutions of the SDE systems
$$dX_i(t) = u(t, X_i(t))\, dt + \sqrt{2} dW_t$$

with initial data $X_1(0) = X_2(0)$ a random variable with law ρ_0, are constructed based upon the same Wiener process as in [16], see also Chapter 2 in [33]. Then, the stochastic process $X_1(t) - X_2(t)$ follows a deterministic equation:

$$\frac{d}{dt}(X_1(t) - X_2(t)) = u(t, X_1(t)) - u(t, X_2(t)).$$

Therefore, the quantity used in this case will be

$$Q(t) := \frac{1}{2}\,\mathbb{E}\left[|X_1(t) - X_2(t)|^2\right]. \tag{3.1}$$

It is also easy to check that $W_2^2(\rho_1(t), \rho_2(t)) \leq 2Q(t)$ by defining an admissible plan π transporting $\rho_1(t)$ to $\rho_2(t)$ by

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \varphi(x, y)\mathrm{d}\pi(x, y) = \mathbb{E}\left[\varphi(X_1(t), X_2(t))\right]$$

for all $\varphi \in C_b(\mathbb{R}^N)$. This plan has the right marginals since the law of $X_i(t)$ is given by $\rho_i(t, x)$ meaning that

$$\int_{\mathbb{R}^N} \varphi(x)\rho_i(t, x)\,\mathrm{d}x = \mathbb{E}\left[\varphi(X_i(t))\right].$$

It is clear then that $Q(t) \equiv 0$ would imply that $X_1(t) = X_2(t)$, and thus their laws $\rho_1 = \rho_2$. With this new quantity the proof now follows exactly the same steps as in Theorem 2.7. We make a quick summary of the new ingredients to consider. We first compute the time derivative of $Q(t)$ as

$$\frac{dQ}{dt} = \mathbb{E}\left[\langle X_1 - X_2, u_1(X_1) - u_1(X_2)\rangle\right] + \mathbb{E}\left[\langle X_1 - X_2, u_1(X_2) - u_2(X_2)\rangle\right],$$

with abuse of notation since an integrated in time version of it would give full rigor. Now, the proof of the smooth case can be really copied directly to this case by replacing integration with respect to the measure ρ_0 by expectations. The second and third cases can be also adapted by using the following ingredients:

(1) The interpolation results in Propositions 2.6 and 2.5 can be used by realizing that

$$I(t) = \mathbb{E}\left[|\nabla K * (\rho_1 - \rho_2)\left[X_2(t)\right]|^2\right] = \int_{\mathbb{R}^N} |\nabla K * (\rho_1 - \rho_2)\,(x)|^2\rho_2(x)\,\mathrm{d}x$$

since $\rho_2(t, x)$ is the law of $X_2(t)$.

(2) In the case of the PKS system, one of the ingredients used by G. Loeper in his proof in [22] was the continuity in time of the solutions of the ODE system for

small time. This step was not detailed in the previous section since the proof coincides with the one in [22]. This needed continuity can be also proved in the present case since being the two SDE systems solved with the same Brownian motion, we have

$$\frac{d}{dt}(X_1(t) - X_2(t)) = u_1(t, X_1(t)) - u_2(t, X_2(t)).$$

Using that under the assumptions of bounded densities the velocity fields are bounded and Log-Lipschitz, and since the initial data is the same, we deduce $|X_1(t) - X_2(t)| \le Ct$ for all $0 \le t \le T$ a.e. in the probability space.

All the rest of the details are left to the interested reader. □

Acknowledgements. We acknowledge support from the project MTM2008-06349-C03-03 DGI-MICINN (Spain), 2009-SGR-345 from AGAUR-Generalitat de Catalunya and IPAM–UCLA where part of this work was done.

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5ECM Amsterdam 2008

On the computation of the coefficients of modular forms

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Abstract. An overview for a non-specialised audience is given of joint work with Jean-Marc Couveignes, Robin de Jong, Franz Merkl, and Johan Bosman. This joint work concerns fast computation of coefficients of modular forms, via the computation of associated Galois representations. For example, for p prime, Ramanujan's $\tau(p)$ can be computed in time polynomial in $\log p$. The overview focuses on the main results and ideas. Developments since 2006 are included: more examples by Johan Bosman, generalisation to forms of level one and of arbitrary weight, and an application to theta functions of lattices. Some future developments are mentioned.

Mathematics Subject Classification (2000). Primary 11Y40; Secondary 11F80, 11G18, 14G35, 14G40.

Keywords. Computation, modular form, coefficient, Galois representations.

1. Introduction

The aim of this article is to give an overview of joint work with Jean-Marc Couveignes, Robin de Jong, Franz Merkl, and Johan Bosman. Details of this joint work can be found in [9], in [7] and in [3] and [2]. This joint work will appear as a book in the series "Annals of Mathematics Studies" of Princeton University Press. The book version will contain deterministic variants of the probabilistic algorithms given in [9]. In this overview, by *algorithm* we mean deterministic algorithm. We will focus on the main results and ideas, skipping technical details, and we will also mention some future developments. In comparison to the previous overview [10] of this joint work, this text discusses the developments since 2006: more examples by Johan Bosman, generalisation to forms of level one and of arbitrary weight, and application to theta functions of lattices; it says much less about the method by which Galois representations are computed.

An important example of our main results can be formulated easily. Ramanujan's τ-function $\tau\colon \mathbb{N}_{>0} \to \mathbb{Z}$ is defined by the equality of formal power series with integer coefficients:

$$q\prod_{n\geq 1}(1-q^n)^{24}=\sum_{n\geq 1}\tau(n)q^n=q-24q^2+252q^3+\cdots \quad \text{in } \mathbb{Z}[[q]]. \tag{1.1}$$

Hecke already showed that $|\tau(n)| = O(n^6)$ as n tends to infinity. One can then ask how fast $\tau(n)$ can be computed, as a function of n. More precisely, one can ask if there is an algorithm that on input $n \in \mathbb{N}_{>0}$ computes $\tau(n)$ in time polynomial in $\log n$, i.e., in a running time that is bounded by a fixed power of $\log n$. As a partial answer to this question we have the following result.

1.2 Theorem. *There exists an algorithm that on input a prime number p gives $\tau(p)$, in running time polynomial in* $\log p$.

Let us first indicate why this is fast. For $n \in \mathbb{N}$, a straightforward way to compute $\tau(n)$ is to compute the product in (1.1) up to order q^n, i.e., to do the necessary multiplications in the ring $\mathbb{Z}[[q]]/(q^{n+1})$. Clearly, this takes time at least linear in n, hence exponential in $\log n$. A faster algorithm for computing $\tau(n)$, based on computation of class numbers, is given in [5]; but, even assuming the generalised Riemann hypothesis (GRH), that algorithm has running time approximately $O(n^{1/2})$, which is still exponential in $\log n$.

Let us emphasise that in Theorem 1.2 the integer p must be a prime number. This condition is not there for some artificial reason. For $n \in \mathbb{N}_{>0}$, the computation of $\tau(n)$ is reduced to the computation of the $\tau(p)$ for p dividing n via well-known properties of the τ-function. These are summarised in the identity of (formal) Dirichlet series

$$\sum_{n\geq 1}\tau(n)n^{-s}=\prod_{p}(1-\tau(p)\cdot p^{-s}+p^{11}\cdot p^{-2s})^{-1}, \tag{1.3}$$

where the index p of the Euler product on the right ranges over the set of prime numbers. On the other hand, if p and q are distinct prime numbers and $n = pq$, then one can easily compute $\tau(p)^2/p^{11}$ and $\tau(q)^2/q^{11}$ from $\tau(n)$ and $\tau(n^2)$, which shows that factoring n is equivalent to computing $\tau(n)$ and $\tau(n^2)$, provided $\tau(n) \neq 0$. See [1] for details.

The importance of the series $\sum_{n\geq 1}\tau(n)q^n$ in (1.1) comes from the fact that the complex analytic function $\Delta\colon \mathbb{H} \to \mathbb{C}$ on the complex upper half plane defined by

$$\Delta\colon \mathbb{H}\to\mathbb{C}, \quad z\mapsto \sum_{n\geq 1}\tau(n)e^{2\pi i n z} \tag{1.4}$$

is a modular form of level 1 and weight 12, the so-called discriminant modular form. This means that for all $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z})$ and all $z \in \mathbb{H}$ one has

$$\Delta\left(\frac{az+b}{cz+d}\right)=(cz+d)^{12}\Delta(z). \tag{1.5}$$

Behind our proof of Theorem 1.2 is the existence of Galois representations associated to modular forms. This will be explained in some detail in the next section.

In our opinion, the fact that such Galois representations are accessible to computation is of much interest. We get congruences for $\tau(p)$ modulo all primes ℓ. The classical congruences only involve the primes 2, 3, 5, 7, 23, and 691. Whereas the classical congruences are given by explicit formulas, these other congruences are "encoded" by number fields K_ℓ, and can now, in theory, be made explicit. More generally, one can hope that non-solvable global field extensions whose existence is guaranteed by the Langlands program can be made accessible to computation. Our result gives an example of the computation of higher degree étale cohomology with $\mathbb{F}_\ell$-coefficients together with its Galois action. It provides some evidence towards the existence of polynomial time algorithms for computing the number of solutions in $\mathbb{F}_p$ of a fixed system of polynomial equations over $\mathbb{Z}$, when p varies.

To end this introduction, let us note that as a consequence of Theorem 1.2, for $m \in \mathbb{N}$ given together with its factorisation into primes, the number of elements x of the Leech lattice with $\|x\|^2 = 2m$ can be computed in time polynomial in $\log m$. This will be explained in Section 5.

2. Galois representations

Our proof of Theorem 1.2 uses that Δ is an eigenform for certain operators named after Hecke, that such an eigenform implies the existence of certain Galois representations (Deligne), and that, for p prime, $\tau(p)$ is the trace of the Frobenius conjugacy class. We make this more explicit.

Deligne has shown in 1969 (see [8]) that for each prime number ℓ there exists a number field K_ℓ (i.e., finite extension of $\mathbb{Q}$), Galois over $\mathbb{Q}$, together with a faithful representation

$$\rho_\ell\colon\ \mathrm{Gal}(K_\ell/\mathbb{Q}) \hookrightarrow \mathrm{GL}_2(\mathbb{F}_\ell), \tag{2.1}$$

uniquely determined by the modular form Δ by the following conditions. First of all, the representation ρ_ℓ is semisimple (i.e., irreducible, or the direct sum of two 1-dimensional representations). Secondly, the extension $\mathbb{Q} \to K_\ell$ is unramified at all primes $p \neq \ell$. Lastly, for all $p \neq \ell$ the characteristic polynomial of the Frobenius element $\rho_\ell(\mathrm{Frob}_p)$ is given by

$$\det(1 - x \cdot \rho_\ell(\mathrm{Frob}_p)) = 1 - \tau(p)x + p^{11}x^2. \tag{2.2}$$

The notions "unramified" and "Frobenius element" will be made explicit in a moment. What is important now is that we have a description of $\tau(p)$ mod ℓ:

$$\text{for all } \ell \neq p, \quad \mathrm{trace}(\rho_\ell(\mathrm{Frob}_p)) = \tau(p) \quad \text{in } \mathbb{F}_\ell. \tag{2.3}$$

The fields K_ℓ, which encode non-explicit congruences mod ℓ for $\tau(p)$, for all $p \neq \ell$, can be thought of as an analog in the GL_2 context of the fields $\mathbb{Q}(\zeta_\ell)$ generated by the roots of unity of order ℓ. Serre and Swinnerton-Dyer have shown that for ℓ not in $\{2, 3, 5, 7, 23, 691\}$ we have $\mathrm{im}(\rho_\ell) \supset \mathrm{SL}_2(\mathbb{F}_\ell)$. Hence, for these ℓ, called non-exceptional, the extension $\mathbb{Q} \to K_\ell$ is not solvable. Nevertheless, these K_ℓ can now be computed efficiently.

2.4 Theorem. *There exists an algorithm that on input ℓ computes K_ℓ and ρ_ℓ in time polynomial in ℓ. More precisely, it gives*

- *the extension $\mathbb{Q} \to K_\ell$, given in terms of the "structure constants" $a_{i,j,k} \in \mathbb{Q}$ with respect to a $\mathbb{Q}$-basis e: $e_i e_j = \sum_k a_{i,j,k} e_k$;*
- *a list of the elements σ of $\mathrm{Gal}(K_\ell/\mathbb{Q})$, where each σ is given as its matrix with respect to e;*
- *the injective morphism $\rho_\ell\colon \mathrm{Gal}(K_\ell/\mathbb{Q}) \hookrightarrow \mathrm{GL}_2(\mathbb{F}_\ell)$.*

Before discussing the proof of this result, let us describe how it implies Theorem 1.2, via standard methods from computational number theory. So, let p be a prime number. The strategy is then to compute $\tau(p)$ mod ℓ for all ℓ up to some sufficiently large number $x(p)$. One knows that $|\tau(p)| < 2p^{11/2}$ by Deligne, and that $\prod_{\ell<x(p)} \ell \approx e^{x(p)}$. So, in order to deduce $\tau(p)$ from the congruences modulo all $\ell < x(p)$, it is sufficient to take $x(p)$ a suitable constant times $\log p$. Hence, for proving Theorem 1.2, it suffices to show that for primes p and ℓ, one can compute $\tau(p)$ mod ℓ in time polynomial in $\ell \cdot \log p$. Theorem 2.4 gives us ρ_ℓ in time polynomial in ℓ. Then one computes a $\mathbb{Q}$-basis e' of K_ℓ such that the denominators of the structure constants $a'_{i,j,k}$ with respect to e' are not divisible by p and such that the $\mathbb{F}_p$-algebra obtained by reduction mod p of the $a'_{i,j,k}$ is a product of fields (unramifiedness at p means that such a basis e' exists); here one uses an algorithm of Buchmann and Lenstra (see [4]). The group $\mathrm{Gal}(K_\ell/\mathbb{Q})$ then permutes these fields, and for each of them, there is a unique element in $\mathrm{Gal}(K_\ell/\mathbb{Q})$ that induces the p-power automorphism on it. This gives, up to conjugation, an element Frob_p in $\mathrm{Gal}(K_\ell/\mathbb{Q})$. Then one has $\tau(p) = \mathrm{trace}(\rho_\ell(\mathrm{Frob}_p))$ in $\mathbb{F}_l$.

Let us now discuss how one proves Theorem 2.4. As this will become somewhat technical, some readers may want to skip it from some point on and continue with the next section.

So, let ℓ be a prime number. We may and do assume that the image of ρ_ℓ contains $\mathrm{SL}_2(\mathbb{F}_\ell)$. According to [8], ρ_ℓ is realised on a 2-dimensional sub-$\mathbb{F}_\ell$-vector space V_ℓ of the dual of the étale cohomology group $H^{11}(E^{10}_{\overline{\mathbb{Q}},\mathrm{et}}, \mathbb{F}_\ell)$, where E^{10} is the 10-fold self-product of the "universal elliptic curve". In particular, E^{10} is an 11-dimensional algebraic variety, defined over $\mathbb{Q}$, and independent of ℓ. At this point, the reader is not required to know what all this is; we just want to convince him/her that this realisation of ρ_ℓ is not easily accessible for computation in a direct way.

Via some standard methods in étale cohomology (the Leray spectral sequence, and passing to a finite cover to trivialise a locally constant sheaf of finite dimensional $\mathbb{F}_\ell$-vector spaces), or from the theory of congruences between modular forms, it is well known that V_ℓ also occurs in the ℓ-torsion $J_\ell(\overline{\mathbb{Q}})[\ell]$ of the Jacobian variety J_ℓ of some modular curve X_ℓ defined over $\mathbb{Q}$. The field K_ℓ is then the field generated by suitable "coordinates" of the points $x \in V_\ell \subset J_\ell(\overline{\mathbb{Q}})[\ell]$. The Riemann surface $X_\ell(\mathbb{C})$ of complex points of X_ℓ can be described as

$$X_\ell(\mathbb{C}) = \Gamma_1(\ell)\backslash(\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})), \tag{2.5}$$

where $\Gamma_1(\ell) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv 1 \pmod{\ell} \text{ and } c \equiv 0 \pmod{\ell} \right\}$.

We are now in the more familiar situation of torsion points on abelian varieties. But the price that we have paid for this is that the abelian variety J_ℓ depends on ℓ, and that its dimension, equal to the genus of X_ℓ, i.e., equal to $(\ell-5)(\ell-7)/24$, grows quadratically with ℓ. This makes it impossible to directly compute the $x \in V_\ell$ using computer algebra: known algorithms for solving systems of non-linear polynomial equations take time exponential in the dimension.

At this point, Couveignes suggested to use approximations and height bounds. This is an important idea. In its simplest form, it works as follows. Suppose that x is a rational number, $x = a/b$, with a and b in $\mathbb{Z}$ coprime. Suppose that we have an upper bound M for $\max(|a|, |b|)$. Then x is determined by any approximation $y \in \mathbb{R}$ of x such that $|y - x| < 1/2M^2$, simply because for all $x' \neq x$ with $x' = a'/b'$, where a' and b' in $\mathbb{Z}$ satisfy $\max(|a'|, |b'|) < M$, we have $|x' - x| = |(a'b - ab')/bb'| \geq 1/M^2$.

For the computation of K_ℓ, we consider the minimal polynomial P_ℓ in $\mathbb{Q}[T]$ of a carefully theoretically constructed generator α of K_ℓ. We use approximations of all Galois conjugates of α, i.e., of all roots of P_ℓ. Instead of working directly with torsion points of J_ℓ, we work with divisors on the curve X_ℓ. Using this strategy, the problem of showing that P_ℓ can be computed in time polynomial in ℓ is divided into two different tasks. Firstly, to show that the number of digits necessary for a good enough approximation of P_ℓ is bounded by a fixed power of ℓ. Secondly, to show that, given ℓ and n, the coefficients of P_ℓ can be approximated with a precision of n digits in time polynomial in $n \cdot \ell$. The first problem was solved by Bas Edixhoven and Robin de Jong, with some help by Franz Merkl, using Arakelov geometry. The second problem was solved by Jean-Marc Couveignes, in two ways: complex approximations (numerical analysis), and approximations in the sense of reductions modulo many small primes, using exact computations in Jacobians of modular curves over finite fields. We emphasise that the solutions to each of these problems required much work, which occupies most of the pages of [9] and of [7].

3. Johan Bosman's examples

Using the Magma system to do computer computations over $\mathbb{C}$, Johan Bosman has found, for all $\ell \leq 23$ and for every normalised cuspidal eigenform f_k of level one and weight $k \leq 22$, a polynomial $P_{k,\ell}$ of degree $\ell + 1$ that gives the projective Galois representation over $\mathbb{F}_\ell$ associated to f_k:

$$\bar{\rho}_{f_k,\ell}\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_\ell) \to \mathrm{PGL}_2(\mathbb{F}_\ell).$$

More precisely, the action of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the roots of $P_{k,\ell}$ corresponds to the action of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via $\rho_{f_k,\ell}\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_\ell)$ on the set of 1-dimensional sub-$\mathbb{F}_\ell$-vector spaces of $\mathbb{F}_\ell^2$. We refer to [3] for these examples, where the 13 cases with non-solvable image are listed in a table at the end. Three of the examples come from ℓ-torsion of elliptic curves. For the 10 other cases, one must really work with the Jacobian J_ℓ, which is of dimension 12 for $\ell = 23$.

In order to find the polynomials $P_{k,\ell}$, Bosman computed, with a high precision, approximations of them, which allowed him to *guess* the $P_{k,\ell}$. The theoretically proved sufficient precision is not really made explicit in [9], and even if it was, it would not be practical. The $P_{k,\ell}$ thus obtained do have the property that their splitting field is unramified outside ℓ, and that it has the right Galois group. To really *prove* that his $P_{k,\ell}$ are correct, he then uses the recent progress by Khare, Wintenberger, Kisin (see [11] and [12]) on Serre's conjecture on modularity of 2-dimensional Galois representations over finite fields. The projective representations to $\mathrm{PGL}_2(\mathbb{F}_\ell)$ coming from the $P_{k,\ell}$ can be lifted to $\mathrm{GL}_2(\mathbb{F}_\ell)$, still being unramified outside ℓ, and thus come from a modular form of level one and of minimal weight, which is then shown to be f_k.

We list some of Bosman's examples. The polynomials given here are not the approximated ones, but have been obtained by taking suitable elements in the ring of integers of the field given by the approximated polynomials.

$$\begin{aligned}
P_{12,17} = x^{18} &- 9x^{17} + 51x^{16} - 170x^{15} + 374x^{14} - 578x^{13} + 493x^{12} - 901x^{11}\\
&+ 578x^{10} - 51x^9 + 986x^8 + 1105x^7 + 476x^6 + 510x^5 + 119x^4\\
&+ 68x^3 + 306x^2 + 273x + 76;
\end{aligned}$$

$$\begin{aligned}
P_{12,19} = x^{20} &- 7x^{19} + 76x^{17} - 38x^{16} - 380x^{15} + 114x^{14} + 1121x^{13} - 798x^{12}\\
&- 1425x^{11} + 6517x^{10} + 152x^9 - 19266x^8 - 11096x^7 + 16340x^6\\
&+ 37240x^5 + 30020x^4 - 17841x^3 - 47443x^2 - 31323x - 8055;
\end{aligned}$$

$$\begin{aligned}
P_{22,23} = x^{24} &- 11x^{23} + 46x^{22} - 1127x^{20} + 6555x^{19} - 7222x^{18} - 140737x^{17}\\
&+ 1170700x^{16} - 2490371x^{15} - 16380692x^{14} + 99341324x^{13}\\
&+ 109304533x^{12} - 2612466661x^{11} + 4265317961x^{10}
\end{aligned}$$

$$
\begin{aligned}
&+ 48774919226x^9 - 244688866763x^8 - 88695572727x^7\\
&+ 4199550444457x^6 - 10606348053144x^5 - 25203414653024x^4\\
&+ 185843346182048x^3 - 228822955123883x^2\\
&- 1021047515459130x + 2786655204876088.
\end{aligned}
$$

As an application of his computation of the $P_{12,\ell}$ for ℓ in $\{13, 17, 19\}$, Bosman has verified Lehmer's conjecture that for all $n \in \mathbb{Z}_{\geq 1}$, $\tau(n) \neq 0$ up to a higher bound than what was done before. More precisely, he has shown that for all $n < 22798241520242687999 \approx 2 \cdot 10^{19}$ one has $\tau(n) \neq 0$. The previous bound was $22689242781695999 \approx 2 \cdot 10^{16}$.

Using the same methods, Johan Bosman could also produce a polynomial that gives an $\mathrm{SL}_2(\mathbb{F}_{16})$ extension of $\mathbb{Q}$, corresponding to a weight 2 modular form on $\Gamma_0(137)$ (genus 11). Such an example was still missing in tables of Jürgen Klüners. See [2].

4. Modular forms of level 1 and arbitrary weight

In this section we present the generalisation of Theorem 1.2 on fast computation of $\tau(p)$ to forms of level 1 and arbitrary weight.

For $k \in \mathbb{Z}$, a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form of level 1 and weight k if it satisfies the following two conditions. The first condition is that for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z})$ and for all z in $\mathbb{H}$ we have $f((az+b)/(cz+d)) = (cz+d)^k f(z)$. This implies that for all $z \in \mathbb{H}$ we have $f(z+1) = f(z)$, hence that f has a q-expansion $f = \sum_{n\in\mathbb{Z}} a_n(f) q^n$ (recall that $q(z) = \exp(2\pi i z)$). The second condition is that f is "holomorphic at the cusp", i.e., that for all $n < 0$ we have $a_n(f) = 0$.

For $k \in \mathbb{Z}$, we let M_k denote the $\mathbb{C}$-vector space of modular forms of level 1 and weight k. The subspace S_k consisting of the f with $a_0(f) = 0$ is called the space of cuspforms. The direct sum M of all M_k is a graded $\mathbb{C}$-algebra, and it is well known to be generated by the Eisenstein series of weights 4 and 6, together with Δ, satisfying one relation:

$$M = \mathbb{C}[E_4, \Delta] \oplus E_6 \cdot \mathbb{C}[E_4, \Delta],$$

where

$$E_4 = 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n, \quad E_6 = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n, \quad \text{and} \quad \Delta = \frac{E_4^3 - E_6^2}{1728},$$

and where, for m and n in $\mathbb{N}$, $\sigma_m(n) = \sum_{0<d|n} d^m$. The space M_k is zero if $k < 0$, and, for $k \geq 0$ its dimension grows linearly with k: $\dim M_k - k/12$ is bounded. For each k in $\mathbb{Z}$ we have $S_k = \Delta M_{k-12}$. For $k \geq 4$, S_k has codimension one in M_k.

For each $k \in \mathbb{N}$, the space M_k is equipped with Hecke operators, coming from the action of $\mathrm{GL}_2(\mathbb{Q})^+$ on $\mathbb{H}$. These operators preserve S_k. For each $i \in \mathbb{N}_{>0}$ one has an operator T_i on M_k (we do not include k in the notation, and we denote the restriction of T_i to S_k still by T_i). The $\mathbb{Z}$-algebra in $\mathrm{End}_{\mathbb{C}}(S_k)$ generated by the T_i is called the Hecke algebra $\mathbb{T}_k$ acting on cuspforms of level 1 and weight k. It is commutative, generated by the T_p with p prime. For p prime, the action of T_p on M_k is as follows:

$$\text{for } f \text{ in } M_k \text{ and } p \text{ prime,} \quad T_p f = \sum_{n\geq 0} a_{np}(f)q^n + \sum_{n\geq 0} p^{k-1}a_n(f)q^{np}.$$

The Hecke algebra $\mathbb{T}_k$, together with its elements T_i, $i \in \mathbb{N}_{>0}$, gives us another interpretation of S_k: the pairing $S_k \times \mathbb{T}_k \to \mathbb{C}$, $(f,t) \mapsto a_1(t(f))$, identifies S_k with the space of $\mathbb{Z}$-linear maps from $\mathbb{T}_k$ to $\mathbb{C}$. The subset of morphisms of $\mathbb{Z}$-algebras corresponds to the set of normalised cuspidal eigenforms: the f in S_k such that $a_1(f) = 1$ and $T_i(f) = a_i(f) \cdot f$ for all i. Each S_k has a natural inner product, for which the T_i are self-adjoint, hence S_k has a basis of eigenforms, and all eigenvalues are real.

The structure of M given above implies that as a $\mathbb{Z}$-module, $\mathbb{T}_k$ is generated by the T_i with $i \leq k/12$, and that it is free of rank $\dim_{\mathbb{C}} S_k$. Therefore, an element f of S_k is determined by its values on the T_i with $i \leq k/12$. The following theorem is the generalisation of Theorem 1.2 to arbitrary weights: it says that the coefficients $a_p(f)$ can be computed quickly, if the $a_m(f)$ for $m \leq k/12$ are given.

4.1 Theorem. *Assume the generalised Riemann hypothesis for number fields, or, in the following, assume that k bounded. There is an algorithm that on input $k \in \mathbb{N}$ and p prime gives the element T_p of $\mathbb{T}_k$ as a $\mathbb{Z}$-linear combination of the T_i with $i \leq k/12$, in time polynomial in $k \log p$.*

The principle of the proof of Theorem 4.1 is simply to compute the image of T_p in sufficiently many quotients $\mathbb{T}_k/m$ of $\mathbb{T}_k$ by maximal ideals. We only consider maximal ideals m of $\mathbb{T}_k$ with $\mathbb{T}_k/m$ a prime field, and with $\#(\mathbb{T}_k/m) \leq x$ for a suitable bound x to be specified later. We let $P(k,x)$ be the set of these m. We will use the LLL-algorithm (see [13]) to compute T_p from all these congruences, replacing the Chinese remainder theorem that we used in the case $k = 12$, where $\mathbb{T}_{12} = \mathbb{Z}$.

The $\mathbb{Z}$-algebra $\mathbb{T}_k$ can be computed, in the form of a $\mathbb{Z}$-basis and a multiplication table, in time polynomial in k, using algorithms for computing with modular symbols (see [15]).

For each maximal ideal m of $\mathbb{T}_k$ there is a unique semi-simple Galois representation ρ_m from $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $\mathrm{GL}_2(\mathbb{T}_k/m)$ that is unramified at all primes q not equal to the characteristic of $\mathbb{T}_k/m$, and such that for all such q the Frobenius element $\rho_m(\mathrm{Frob}_q)$ has trace T_q and determinant q^{k-1} in $\mathbb{T}_k/m$. Just as in Theorem 2.4, for m

with $\mathbb{T}/m$ a prime field, ρ_m can be computed in time polynomial in $k \cdot \log(\#(\mathbb{T}_k/m))$. Hence we can compute the image of T_p in all the $\mathbb{T}_k/m$ with m in $P(k,x)$ in time polynomial in kx. We let $\mathbb{I}_{k,x}$ be the intersection in $\mathbb{T}_k$ of all m in $P(k,x)$. Then we have the exact sequence

$$0 \to \mathbb{I}_{k,x} \to \mathbb{T}_k \to \prod_{m \in P(k,x)} \mathbb{T}_k/m \to 0,$$

and we can compute the image $\overline{T_p}$ of T_p in $\mathbb{T}_k/\mathbb{I}_{k,x}$, as well as a pre-image T'_p in $\mathbb{T}_k$ of $\overline{T_p}$ in time polynomial in kx. We will now address the problem of how to choose x so that we can efficiently compute T_p from T'_p.

In the case where k is not fixed, we will use the assumption of GRH to show that there are sufficiently many m's, in the form of the following effective prime number theorem for number fields (see [16]).

4.2 Theorem (Weinberger). *Assume GRH for number fields. For K a number field and x in $\mathbb{R}$ let $\pi_1(x, K)$ denote the number of maximal ideals m of the ring of integers O_K of K with O_K/m a prime field and with $|O_K/m| \le x$. For $x > 2$ in $\mathbb{R}$ let* $\mathrm{li}(x) = \int_2^x (1/\log y)dy$. *Then there exists c_1 in $\mathbb{R}$ such that for every number field K and for every $x > 2$ one has*

$$|\pi_1(x, K) - \mathrm{li}(x)| \le c_1 \sqrt{x} \log\left(|\operatorname{discr}(O_K) x^{\dim_{\mathbb{Q}} K}|\right),$$

where $\operatorname{discr}(O_K) \in \mathbb{Z}$ *denotes the discriminant of O_K.*

As all eigenvalues of all T_i on all S_k are real, we have, for each k, an isomorphism of $\mathbb{R}$-algebras $\mathbb{R} \otimes \mathbb{T}_k \to \mathbb{R}^{\dim S_k}$, unique up to permutation of the factors. The standard inner product on $\mathbb{R}^{\dim S_k}$ is the trace form of this $\mathbb{R}$-algebra, hence is obtained by extension of scalars from $\mathbb{Z}$ to $\mathbb{R}$ of the trace form of $\mathbb{T}_k$. We will view each $\mathbb{T}_k$ as a lattice in $\mathbb{R} \otimes \mathbb{T}_k$, and we equip each $\mathbb{R} \otimes \mathbb{T}_k$ with the standard volume form, i.e., the one for which the unit cube has volume one. From the Ramanujan bound on the eigenvalues of the T_i, proved by Deligne, one easily derives that

$$\log \operatorname{Vol}(\mathbb{R} \otimes \mathbb{T}_k/\mathbb{T}_k) = \frac{1}{2} \log \operatorname{discr} |\mathbb{T}_k| \le \frac{k^2}{24} \log k.$$

Let us now explain how we choose x as a function of k and p, assuming GRH for number fields. Let n_k be the rank of $\mathbb{T}_k$, i.e., $n_k = \dim_{\mathbb{C}} S_k$. We can and do assume that $n_k > 0$. The norm of the element T_p that we want to compute from a congruence modulo $\mathbb{I}_{k,x}$ is bounded, by Deligne, as follows:

$$\|T_p\| \le 2 n_k^{1/2} p^{(k-1)/2}.$$

Applying Theorem 4.2 to the number fields of which $\mathbb{Q}\otimes\mathbb{T}_k$ is a product, one proves that for x a suitable constant times fixed powers of k and $\log p$, one has the following lower bound for the length of a shortest non-zero element of $\mathbb{I}_{k,x}$:

$$\mu_1(\mathbb{I}_{k,x}) > 2^{(n_k+1)/2}\cdot\|T_p\|, \quad \text{where } \mu_1(\mathbb{I}_{k,x}) = \min\{\|t\| \mid t\in\mathbb{I}_{k,x}-\{0\}\}.$$

Under these conditions, the standard approach for using the LLL-algorithm for the "closest vector problem" shows that T_p can be computed from our element T_p' in $T_p+\mathbb{I}_{k,x}$, in time polynomial in $k\log p$, as follows.

Let b denote our inner product on $\mathbb{T}_k$, i.e., the trace form. Let $e=(e_1,\ldots,e_n)$ be an "LLL-reduced basis" of $\mathbb{I}_{k,x}$: if $e^*=(e_1^*,\ldots,e_n^*)$ denotes the orthogonal $\mathbb{R}$-basis of $\mathbb{R}\otimes\mathbb{T}_k$ obtained from e by letting e_i^* be the orthogonal projection of e_i to the orthogonal complement of the subspace of $\mathbb{R}\otimes\mathbb{T}_k$ generated by $\{e_j \mid j<i\}$ (i.e., by the Gram–Schmidt orthogonalisation process), and $\mu_{i,j} := b(e_i,e_j^*)/b(e_j^*,e_j^*)$, then we have

$$|\mu_{i,j}| \le \frac{1}{2} \quad \text{for } 1\le j<i\le n,$$

and

$$\|e_i^*\|^2 \ge \left(\frac{3}{4}-\mu_{i,i-1}^2\right)\|e_{i-1}^*\|^2 \quad \text{for } 1<i\le n.$$

Then T_p can be recovered from T_p' as follows:

- put $x_n := T_p'$;
- for i going down from n to 1 let $x_{i-1} := x_i - [b(x_i,e_i^*)/b(e_i^*,e_i^*)]e_i$, where, for y in $\mathbb{Q}$, $[y]$ denotes the largest of the (one or two) integers nearest to y;
- then $T_p = x_0$.

In the case where k is fixed, in Theorem 4.1, the $\mathbb{Z}$-algebra $\mathbb{T}_k$ is fixed, and the ordinary asymptotic prime number theorem for each of the factors of $\mathbb{Q}\otimes\mathbb{T}_k$ suffices for what we do.

4.3 Theorem. *Assume GRH for number fields, or, in the following, assume that k is bounded. There exists an algorithm that on input positive integers k and n, together with the factorisation of n into prime factors, computes the element T_n of $\mathbb{T}_k$ as $\mathbb{Z}$-linear combination of the T_i with $i\le k/12$, in time polynomial in $k\log n$.*

Theorem 4.3 follows from Theorem 4.1 by using the standard way to express T_n in the T_p for the prime numbers p dividing n:

$$T_m = \prod_{p\mid m} T_{p^{v_p(m)}}, \quad T_{p^r} = T_pT_{p^{r-1}} - p^{k-1}T_{p^{r-2}}.$$

5. Lattices

Theorem 4.3 has an interesting application to certain modular forms that come from lattices: theta functions of even unimodular lattices.

Let us consider a free $\mathbb{Z}$-module L of finite rank n_L, equipped with a positive definite symmetric bilinear form $b\colon L \times L \to \mathbb{Z}$. Then $L_{\mathbb{R}} := \mathbb{R} \otimes L$ is an $\mathbb{R}$-vector space of dimension n_L on which b gives an inner product, and hence L is a lattice in the euclidean space $L_{\mathbb{R}}$. For m in $\mathbb{Z}$ we define

$$r_L(m) := \#\{x \in L \mid b(x,x) = m\}. \tag{5.1}$$

In this situation, one considers the so-called *theta-function* $\theta_L\colon \mathbb{H} \to \mathbb{C}$ associated to (L,b):

$$\theta_L = \sum_{x \in L} q^{b(x,x)/2} = \sum_{m \geq 0} r_L(m) q^{m/2}, \quad \text{where } q\colon z \mapsto \exp(2\pi i z). \tag{5.2}$$

The form b is called *even* if $b(x,x)$ is even for all x in L. Equivalently, b is even if and only if the matrix of b with respect to a basis of L has only even numbers on the diagonal. The form b is called *unimodular* if the map $x \mapsto (y \mapsto b(x,y))$ from L to its dual $L^\vee$ is an isomorphism of $\mathbb{Z}$-modules. Equivalently, b is unimodular if and only if the matrix of b with respect to a basis of L has determinant 1. If (L,b) is even and unimodular, then n_L is even, and θ_L is a modular form of level 1 and weight $n_L/2$ (see [14, VII,§6]). This explains that Theorem 4.3 has the following consequence.

5.3 Theorem. *Assume GRH, or, in the following, consider only (L,b) whose ranks n_L are bounded. There is an algorithm that, on input the rank n_L and the integers $r_L(i)$ for $1 \leq i \leq n_L/24$ of an even unimodular lattice (L,b), and an integer $m > 0$ together with its factorisation into primes, computes $r_L(m)$ in time polynomial in $n_L \log m$.*

To prove this theorem, one writes θ_L as a rational multiple of the Eisenstein series $E_{n_L/2}$ of weight $n_L/2$ plus a cuspform f with rational coefficients. The coefficient $a_m(E_{n_L/2})$ can be computed easily, because m is given with its factorisation. For $a_m(f)$ one first computes the $a_i(f)$ for $i \leq n_L/24$, using the $r_L(i)$. Then, viewing f as a $\mathbb{Z}$-linear map $\mathbb{T}_{n_L/2} \to \mathbb{Q}$, one has $a_m(f) = f(T_m)$ and one applies Theorem 4.3.

We give an example. Let L be the *Leech lattice*. It is the unimodular lattice of rank 24 that, according to Henry Cohn and Abhinav Kumar [6], gives the densest lattice sphere packing in dimension 24. As M_{12} is two-dimensional, generated by E_{12} and Δ, θ_L is a linear combination of these two. Comparing the coefficients of q^m for $m = 0$ and $m = 1$ gives

$$\theta_L = E_{12} - \frac{65520}{691}\Delta, \quad \text{with } E_{12} = 1 + \frac{65520}{691}\sum_{n \geq 1} \sigma_{11}(n) q^n.$$

Theorem 5.3 says that if $m > 0$ is given, with its factorisation into primes, then $r_L(m)$ can be computed in time polynomial in $\log m$.

We can also consider direct sums of copies of L. For $n \in \mathbb{N}$, we have

$$\theta_{L^n} = \sum_{x_1,\ldots,x_n \in L} q^{(b(x_1,x_1)+\cdots+b(x_n,x_n))/2} = \Big(\sum_{x\in L} q^{b(x,x)/2}\Big)^n = \theta_L^n.$$

This means that we can compute the $r_{L^n}(i)$ for $1 \leq i \leq n$ in time polynomial in n by doing our multiplications in $\mathbb{Z}[[q]]/(q^{n+1})$. Hence, assuming GRH, if $m > 0$ is given, with its factorisation into primes, then $r_{L^n}(m)$ can be computed in time polynomial in $n \log m$.

It is interesting to note that theta functions are usually considered as modular forms whose coefficients are easy to compute. As such, they can be used to compute coefficients of cuspforms. But for coefficients a_m with m large, one now concludes that the situation is reversed.

6. Perspectives

It is to be expected that Theorem 4.3 will be generalised to the spaces of cuspforms of varying level and weight, with running time for computing T_n polynomial in $\log n$, the level and the weight. A PhD-student, Peter Bruin, is working on this.

Hence, it is also to be expected that, assuming GRH, there is an algorithm that on input positive integers n and m, together with the factorisation of m into primes, computes the number

$$r_{\mathbb{Z}^{2n}}(m) = \#\{x \in \mathbb{Z}^{2n} \mid x_1^2 + \cdots + x_{2n}^2 = m\}$$

in time polynomial in n and $\log m$. Hence, even in the absence of explicit simple formulas for the $r_{\mathbb{Z}^{2n}}(m)$ as one has for $n \leq 5$, there will be an algorithm that computes the $r_{\mathbb{Z}^{2n}}(m)$ as fast as if one had such formulas.

Another consequence of the expected generalisation of Theorem 4.3 mentioned above is that, again assuming GRH, there is an algorithm that on input a positive number n and a finite field $\mathbb{F}_q$ computes the number $\#X_1(n)(\mathbb{F}_q)$ in time polynomial in n and $\log q$. Indeed, this is a matter of computing the element T_p (where p is the prime dividing q) in the Hecke algebra acting on the space $S_2(\Gamma_1(n))$ of modular forms of weight 2 on $\Gamma_1(n)$. At this moment, there is no algorithm known for point counting on curves C over $\mathbb{F}_q$ that has running time polynomial in $\log q$ and the genus of C, if both the genus and the characteristic of $\mathbb{F}_q$ are not bounded. The case of modular curves is interesting, but does not indicate how to solve this for general curves.

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5ECM Amsterdam 2008

Effective equidistribution and spectral gap

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Abstract. In these notes we discuss some equidistribution problems with the aim to give reasonable error rates, i.e., we are interested in effective statements. We motivate some arguments by studying a concrete problem on a two-torus, and then describe recent results on the equidistribution of semisimple orbits obtained in joint work with G. Margulis and A. Venkatesh. We end by studying the relationship between equidistribution of closed orbits and mixing properties. This leads to a way of transporting spectral gap from one group – via an effective equidistribution result on a quotient by an irreducible lattice – to another group. The latter topic is ongoing joint work with G. Margulis and A. Venkatesh.

Mathematics Subject Classification (2000). 37A17, 37C85, 11F99.

Keywords. Homogeneous dynamics, equidistribution, spectral gap.

1. Purpose

These notes are the combination of a few lectures given on an effective equidistribution theorem and related material. The main theorem that we discuss e.g. describes how dense closed orbits xH of $H = \mathrm{SO}(2,1)(\mathbb{R})^\circ$ on $\mathrm{SL}(3,\mathbb{Z})\backslash \mathrm{SL}(3,\mathbb{R})$ with big volume have to be. A more general version of this was obtained in joint work [7] with G. Margulis and A. Venkatesh and will be described in §6. A crucial input to the method that we used in [7] was spectral gap – in §4 we state what is used in general and prove the statement in the special case of $\mathrm{SL}(3,\mathbb{R})$ being the acting group. We motivate these questions and give a brief historical discussion in §2–§3. In §7 we outline the idea of ongoing joint work with G. Margulis and A.Venkatesh. Most of the material herein is well known to experts, but we think that assembling the material in these notes is worthwhile as it may help someone reading [7].

The author would like to thank his coauthors and students for many discussions on these topics. This research has been supported by the NSF-FRG collaborative grant 0554373.

2. Model cases of equidistribution problems

2.1. (Too) general setup. Let us start with the following kind of equidistribution problems (which we specialize later to a more concrete setup). Suppose $T\colon X \to X$ is a continuous map on a compact metrizable space and $x \in X$. Then one can ask about the distribution properties of the finite sequence of points

$$x, T(x), T^2(x), \ldots, T^{n-1}(x) \in X.$$

We can specify the question more concretely by defining the measure

$$\int f \, d\delta_{x,n} = \frac{1}{n} \sum_{\ell=0}^{n-1} f(T^\ell(x)) \text{ for all } f \in C(X),$$

and asking about the behavior of $\delta_{x,n}$ for large n and a given x. If $\delta_{x,n}$ converges for $n \to \infty$ in the weak* topology to some measure μ, then we say that the orbit of x *equidistributes* (w.r.t. μ). If we have a reasonable error for the expression $\left| \int f \, d\delta_{x,n} - \int f \, d\mu \right|$ for smooth functions, then we speak of *effective equidistribution*.

If T is ergodic with respect to an invariant probability measure μ, one knows that $\delta_{x,n}$ converges to μ in the weak* topology as $n \to \infty$ for μ-a.e. $x \in X$ by Birkhoff's pointwise ergodic theorem, i.e., a.e. orbit equidistributes w.r.t. μ. This is an interesting statement and can be quite useful in applications, but it does by no means provide a complete answer to the problem. For instance, it does not say anything about orbits of points $x \in X$ that are not typical for μ. Moreover, if we want to work with large but fixed n then again this provides no information about $\delta_{x,n}$ as the general ergodic theorem does not provide an effective error rate.

2.2. Rotation on the circle $\mathbb{T}$. In the generality discussed above, one cannot hope to say anything about a given point $x \in X$ and also not anything – even if x is typical for an ergodic measure μ – about the speed of approximation. However, there are cases where both can be achieved.

Still in the same generality as above, if T has only one invariant probability measure on X, say μ, then more is true: For every point x one has that $\delta_{x,n}$ converges in the weak* topology to μ – and does so uniformly. It is easy to give an example for this, as e.g. the circle rotation defined by $T(x) = x + \alpha$ for $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and some fixed irrational $\alpha \in \mathbb{R}$ (with addition being understood modulo $\mathbb{Z}$) preserves only the Lebesuge measure $m_\mathbb{T}$ and $\delta_{x,n}$ converges to $m_\mathbb{T}$ for any $x \in \mathbb{T}$. Moreover, one can also answer the refined question for an error rate but this requires[1] additional

[1] If one asks for a weaker form of an effective error rate, then one can do any irrational α. We refer to the work of Green and Tao [11] for what one can say without the Liouville-assumption.

assumptions on α: If α is not a Liouville number and $f \in C^\infty(X)$, then

$$\left|\frac{1}{n}\sum_{\ell=0}^{n-1} f(x+\ell\alpha) - \int_{\mathbb{T}} f\,d\lambda\right| < \frac{1}{n}S(f),$$

where $S(f)$ depends on the function f (respectively on the sizes of the first few derivatives $f, f', f'', \dots, f^{(L)}$ with L dependent on α). This is quite well known and can be proven directly for characters $e_k(x) = \exp(2\pi i k x)$ (using the geometric series and the assumption on α), and then can be boot-strapped to any smooth function f by an application of the Cauchy–Schwarz inequality. Instead of proving this, we give a proof of a different effective equidistribution statement on $\mathbb{T}^2$ below.

2.3. Polynomial curves on $\mathbb{T}^2$. Let us study now two[2] polynomials $p_1(t), p_2(t)$ for $t \in [0,1]$ of degree $\leq D$ and how the corresponding curve[3] $\{(p_1(t), p_2(t)) : t \in [0,1]\}$ behaves modulo $\mathbb{Z}^2$ as a subset of $\mathbb{T}^2$. I.e., we wish to estimate

$$\left|\int_0^1 f(p_1(t), p_2(t))\,dt - \int_{\mathbb{T}^2} f(\boldsymbol{x})\,d\boldsymbol{x}\right| \tag{1}$$

for a given smooth f defined on $\mathbb{T}^2$. Clearly, if e.g. $p_2 = 0$ then there will not be a reasonable estimate for (1) as the curve in questions stays in a subtorus, more generally the same holds if $(p_1(t), p_2(t))$ is close to a rational line for all $t \in [0,1]$. To avoid this problem, let us assume that there exists some L and T such that for any integer $\boldsymbol{n} \in \mathbb{Z}^2$ we have that the polynomial

$$p_{\boldsymbol{n}}(t) = (n_1 p_1 + n_2 p_2)(t) = c_0 + c_1 t + \dots + c_D t^D$$

is nonconstant and moreover that[4]

$$\max_{j=1,\dots,D} |c_j| \geq T \quad \text{for all } \boldsymbol{n} \in \mathbb{Z}^2 \text{ with } \|\boldsymbol{n}\|^L \leq T. \tag{2}$$

We fix L and think of T being a very large number (but without actually taking the limit $T \to \infty$ as one often would do in ergodic theory). We wish to estimate (1) by a negative power of T (which will depend on D) times a constant that depends on the sizes of the first few partial derivatives of f (where the number of derivatives used will depend on L). As this section's main purpose is to introduce some ideas in a

[2] The only reason for restricting the dimension to 2 is just to restrict the number of parameters in this discussion.

[3] The continuous setting is in some aspects easier than the discrete one considered before, but is also more relevant to the following discussion.

[4] Clearly, making a restriction on the $\boldsymbol{n}$ for which we require (2), will lead to a stronger result. In particular, with this restriction one can apply the result (5) to the case of a flow $(p_1(t), p_2(t)) = T(t, \alpha t)$ whenever α is not a Liouville number.

very concrete setup, we make no claims regarding the optimality[5] of the estimates. We refer to [2] for the case of expanded images of a fixed curve where the Van der Corput lemma (see e.g. [10, pg. 146]) is used to prove a sharper estimate. Instead of using Van der Corput we will combine harmonic with more geometric arguments as this generalizes more easily to the context considered later.

2.3.1. Characters first. We start the calculation for the desired estimate by the case where $f(\boldsymbol{x}) = e_{\boldsymbol{n}}(\boldsymbol{x}) = \exp(2\pi i(n_1x_1 + n_2x_2))$ is a character. The following argument is relatively simple but requires some games with exponents and hence a few constants that we will optimize at the end.

By definition of $p_{\boldsymbol{n}}$ we have $e_{\boldsymbol{n}}((p_1(t), p_2(t)) = \exp(2\pi i p_{\boldsymbol{n}}(t))$. We assume first that $\|\boldsymbol{n}\|^L \le T$. By our assumption (2) and the equivalence of norms on the space of polynomials of degree $\le D-1$ we have[6]

$$S = \max_{t\in[0,1]} |p'_{\boldsymbol{n}}(t)| \gg T$$

and similarly

$$\max_{t\in[0,1]} |p''_{\boldsymbol{n}}(t)| \ll S.$$

As $p'_{\boldsymbol{n}}(t)$ is a polynomial of degree $D-1$ we can easily convince ourselves that the Lebesgue measure of the points $t \in [0,1]$ where $|p'_{\boldsymbol{n}}(t)|$ is much smaller than S must in fact be small. In fact, for the polynomial $\frac{1}{S}p'_{\boldsymbol{n}}$ on $[0,1]$ of supremum norm about equal to one, the value of this polynomial can only be small, say smaller than $S^{\alpha-1}$, roughly speaking, close to the roots. Here the worst case happens if all the $D-1$ roots are equal, in which case $\frac{1}{S}p'_{\boldsymbol{n}}(t)$ is smaller than $S^{\alpha-1}$ on an interval of size $S^{\frac{\alpha-1}{D-1}}$. More formally, this estimate follows from the interpolation formula for polynomials, see for instance [12, Proposition 3.2], and gives that

$$m_{\mathbb{R}}\big(\{t \in [0,1] : |p'_{\boldsymbol{n}}(t)| < S^{\alpha}\}\big) \ll S^{\frac{\alpha-1}{D}}, \tag{3}$$

where $\alpha \in (0,1)$ is to be determined later. Vaguely speaking, for any α we will be able to ignore those t with $|p'_{\boldsymbol{n}}(t)| < S^{\alpha}$ as we are aiming to obtain an estimate involving a negative power of T.

Next we fix some $\beta \in (0,1)$, again to be determined later, and divide $[0,1]$ into subintervals of size $S^{-\beta}$ and one interval that may be shorter than that. Let I be one such interval of length $\le S^{-\beta}$ and assume that for some $t_0 \in I$ we have $|p'_{\boldsymbol{n}}(t_0)| \ge S^{\alpha}$. Then as the second derivative is bounded by $\ll S$ on $[0,1]$ we have for any $t \in [0,1]$ that

$$p_{\boldsymbol{n}}(t) = p_{\boldsymbol{n}}(t_0) + (t-t_0)p'_{\boldsymbol{n}}(t_0) + O((t-t_0)^2 S).$$

[5] This is partly but not only because we will be wasteful at places in the estimates if this helps to keep the expressions tidy.

[6] Implicit constants in the $\ll$-notation we allow to depend on D.

By choosing $t \in I$ and $\beta > \frac{1}{2}$ we can make the error term here of the form

$$|(t - t_0)^2| S \leq S^{-2\beta+1}.$$

As the derivative of $\exp(2\pi i \cdot)$ is of absolute value 2π we have with these choice that

$$\left| e_{\boldsymbol{n}}\big((p_1(t), p_2(t)\big) - \exp\big(2\pi i \|\boldsymbol{n}\| (p_{\boldsymbol{n}}(t_0) + (t - t_0) p'_{\boldsymbol{n}}(t_0))\big) \right| \ll S^{-2\beta+1}.$$

We make that approximation because it is trivial to integrate an exponential function, which leads to

$$\left| \int_I \exp\big(2\pi i (p_{\boldsymbol{n}}(t_0) + (t - t_0) p'_{\boldsymbol{n}}(t_0))\big)\, dt \right| \ll \left| p'_{\boldsymbol{n}}(t_0) \right|^{-1} \leq S^{-\alpha}.$$

Together we get

$$\left| \int_I e_{\boldsymbol{n}}\big((p_1(t), p_2(t)\big)\, dt \right| \ll S^{-\alpha} + S^{-2\beta+1} m_{\mathbb{R}}(I).$$

We are summing this estimate over all intervals I that contain some t_0 with $|p'_{\boldsymbol{n}}(t_0)| \geq S^{\alpha}$, of which there are at most $S^{\beta} + 1 \ll S^{\beta}$, and add the integral of the trivial estimate $\|e_{\boldsymbol{n}}\|_\infty = 1$ over the remaining intervals. As the union of the latter intervals is contained in the set in (3), we obtain

$$\left| \int_0^1 e_{\boldsymbol{n}}\big((p_1(t), p_2(t)\big)\, dt \right| \ll S^{-\alpha} S^{\beta} + S^{-2\beta+1} + S^{\frac{\alpha-1}{D}}.$$

It is clear that if we choose e.g. $\beta = \frac{3}{5}$ and $\alpha = \frac{4}{5}$, then all of the exponents are negative. A slightly better negative exponent is achieved by setting all the exponents equal and solving for α and β, which then turns the right-hand side into $\ll S^{-\frac{1}{2D+3}}$. Using in addition $S \gg T$ gives for all $\boldsymbol{n} \in \mathbb{Z}^2 \setminus \{0\}$ that

$$\left| \int_0^1 e_{\boldsymbol{n}}\big((p_1(t), p_2(t)\big)\, dt \right| \ll T^{-\frac{1}{2D+3}} \|\boldsymbol{n}\|^{L}. \tag{4}$$

2.3.2. Bootstrapping to any smooth function. Using Cauchy–Schwarz and the relation between smoothness and Fourier-coefficients we can now generalize (4) to an estimate for (1). In fact, we claim that

$$\left| \int_0^1 f(p_1(t), p_2(t))\, dt - \int_{\mathbb{T}^2} f(\boldsymbol{x})\, d\boldsymbol{x} \right| \ll T^{-\frac{1}{2D+3}} \mathcal{S}_{L+2}(f) \tag{5}$$

whenever p_1, p_2 are polynomials of degree $\leq D$ satisfying (2) and $f \in C^\infty(\mathbb{T}^2)$. We note that the error is independent of the starting point of the curve $(p_1(0), p_2(0))$

and of the particular polynomial as long as it satisfies our assumptions. Here S_{L+2} is the L^2-Sobolev norm of f of degree $L+2$ defined by

$$S_{L+2}(f)^2 = \int |f(\boldsymbol{x})|^2 d\boldsymbol{x} + \int \left|\left(\frac{\partial}{\partial x_1}\right)^{L+2} f(\boldsymbol{x})\right|^2 d\boldsymbol{x} + \int \left|\left(\frac{\partial}{\partial x_2}\right)^{L+2} f(\boldsymbol{x})\right|^2 d\boldsymbol{x}$$
$$= \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \left(1 + (2\pi)^{L+2}\|\boldsymbol{n}\|^{2(L+2)}\right)|\hat{f}(\boldsymbol{n})|^2.$$

To obtain (5) recall also that the Fourier series $f = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \hat{f}(\boldsymbol{n}) e_{\boldsymbol{n}}$ converges uniformly for $f \in C^\infty(\mathbb{T}^2)$. Hence we may sum (4) multiplied by $\hat{f}(\boldsymbol{n})$ over all $\boldsymbol{n} \in \mathbb{Z}^2 \setminus \{0\}$ to obtain

$$\left| \int_0^1 f(p_1(t), p_2(t))\, dt - \int_{\mathbb{T}^2} f(\boldsymbol{x})\, d\boldsymbol{x} \right| \ll T^{-\frac{1}{2D+3}} \sum_{\boldsymbol{n}\in\mathbb{Z}^2\setminus\{0\}} |\hat{f}(\boldsymbol{n})| \|\boldsymbol{n}\|^L.$$

Here the sum on the right-hand side can be estimated via Cauchy–Schwarz:

$$\sum_{\boldsymbol{n}\in\mathbb{Z}^2\setminus\{0\}} |\hat{f}(\boldsymbol{n})| \|\boldsymbol{n}\|^L = \sum_{\boldsymbol{n}\in\mathbb{Z}^2\setminus\{0\}} |\hat{f}(\boldsymbol{n})| \|\boldsymbol{n}\|^{L+2} \frac{1}{\|\boldsymbol{n}\|^2}$$
$$\ll \Bigg(\sum_{\boldsymbol{n}\in\mathbb{Z}^2\setminus\{0\}} |\hat{f}(\boldsymbol{n})|^2 \|\boldsymbol{n}\|^{2(L+2)} \Bigg)^{\frac{1}{2}},$$

where we used that $\left(\frac{1}{\|\boldsymbol{n}\|^2}\right)_{\boldsymbol{n}\in\mathbb{Z}^2\setminus\{0\}}$ belongs to ℓ^2. As the last expression is $\leq S_{L+2}(f)$ this finishes the proof of (5).

3. Equidistribution of unipotent and closed orbits on homogeneous spaces

3.1. Unipotent orbits. We replace the setup of a single transformation on a compact space discussed in §2.1 by a one-parameter flow, i.e., an action of $\mathbb{R}$, on a homogeneous space. Let $X = \Gamma\backslash G$ be a quotient of a linear group G by a lattice Γ, and let $U = \{u_t = \exp(tw) : t \in \mathbb{R}\} < G$ be a one-parameter unipotent subgroup – here w is a nilpotent element of the Lie algebra of G. Then instead of the above we consider the pieces of orbits $xu_{[0,T]} = \{xu_t : 0 \leq t \leq T\}$ for points $x \in X$. For this Ratner [15] has shown that the normalized image of the Lebesgue measure on $xu_{[0,T]}$ converges to a natural measure on X – as before we say the orbit equidistributes with respect to this measure. This natural invariant measure on X is for many points[7] the Haar measure on X, but the theorem applies to any point as follows. For a given x Ratner proves [15] that the orbit closure $\overline{xU} \subset X$ is of the form xL for some closed

[7] Unlike the abstract ergodic theorem Ratner's theorem establishes precisely for which points this is true.

connected subgroup $L < G$ and that this orbit supports an L-invariant probability measure, the Haar measure m_{xL}, this is known as Raghunathan's conjecture. Then the measure on $xu_{[0,T]}$ converges in the weak* topology to m_{xL}. However, the problem of estimating the error in this theorem and in this generality is wide open.

A special case of the above setup is given by $U = \left\{u_t = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) : t \in \mathbb{R}\right\}$ acting on $X = \Gamma \backslash \mathrm{SL}(2,\mathbb{R})$, i.e., the horocycle flow on the unit tangent bundle of a hyperbolic surface. If $X = \Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ is compact, then the equidistribution of orbits has been established by Furstenberg [9] already much earlier. Moreover, in this case error rates are known:

$$\left|\frac{1}{T}\int f(xu_t)\,dt - \int f\,dm_X\right| \ll S(f)T^{-\delta},$$

where m_X is the Haar measure on X, $S(f)$ is a Sobolev norm of the function f and $\delta > 0$ is a constant which depends on the spectral properties of X. In particular, this error is independent of the starting point x. We refer the reader to [3], Section 9.3.1 in [17], and [8] for more details. If X is noncompact with finite volume, e.g. $X = \mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R})$, then the above error is again more delicate. This is because in X there are periodic orbits for the action of U. Even assuming that x is not periodic, x could in fact be very close to a periodic orbit for U which makes it impossible to give an error that is independent of x.

3.2. Equidistribution of closed orbits. A problem related to the distribution of pieces of the orbit is the distribution of closed orbits. Here a toy problem is the effective distribution properties of rational lines in the two-dimensional torus (which is a special case of the discussion in §2.3).

3.2.1. Periodic horocycles. A more interesting case concerns the distribution properties of closed horocyle orbits on noncompact quotients. Here an error rate has been established by Sarnak [16]. We now describe this result for $\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R})$ and outline the argument from [17, Section 9] which establishes a slightly weaker form of the effective equidistribution.

We start by recalling that periodic orbits of U are easily visualized using the unit tangent bundle of the hyperbolic plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Here the fundamental domain for $\mathrm{SL}(2,\mathbb{Z})$ is the triangle bounded by $\mathrm{Re}(z) = \pm\frac{1}{2}$ and the unit circle. In this picture the horocycle transports vectors along the horocycle normal to the vector, and horocycles are horizontal lines and circles touching the real axis. In particular, we can visualize periodic orbits for the horocycle flow as horizontal line segments cutting through the fundamental domain with the arrows pointing up. Let y be the y-coordinate of the points in the orbit and write P_y for the periodic orbit. Going up inside the fundamental domain (i.e., for $y \to \infty$) the length of the periodic orbit, which equals $\frac{1}{y}$ goes to zero and the orbit escapes to infinity.

However, as $y \to 0$ we can still draw periodic orbits as horizontal lines outside the standard fundamental domain. If we draw P_y for small y inside the fundamental domain (applying the appropriate isometries from $\mathrm{SL}(2,\mathbb{Z})$) the orbit will look much more complicated, but will be periodic of large length $\frac{1}{y}$. In fact, the orbit P_y becomes equidistributed in X as $y \to 0$. Moreover, as Sarnak showed (in greater generality and with more information regarding δ) this can be made effective, i.e.,

$$\left| \int_{P_y} f - \int_X f \, dm_X \right| \ll y^\delta S(f)$$

for any $f \in C_c^\infty(X)$. Here $\int_{P_y} f$ denotes the normalized integral over the periodic orbit P_y.

3.2.2. Outline of a proof. The geodesic flow is hyperbolic, i.e., inside the 3-dimensional space X there are three special directions:

(o) the orbit direction of the geodesic flow,

(s) the horocycle direction (corresponding to U) which is contracted by the geodesic flow in forward time (the stable direction), and

(u) the opposite horocycle direction (corresponding to $\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$) which is expanded (the unstable direction).

Now let B be a small *box* (using the above three directions as "directions for the sides") around the periodic orbit P_1 (the periodic orbit for U going through i), then

$$\langle f, g_t \cdot \chi_B \rangle \to m_X(B) \int_X f \, dm_X \quad \text{as } t \to \pm\infty$$

by the Howe–Moore theorem on vanishing of matrix coefficients or (equivalently) the mixing property of the geodesic flow. Here $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(X)$, χ_B is the characteristic function, $g_t \cdot$ denotes the unitary action of the geodesic flow on $L^2(X)$, and m_X denotes the Haar measure on X. However, $g_t \cdot \chi_B$ equals the characteristic function of $B_t = B \begin{pmatrix} e^{-\frac{t}{2}} & \\ & e^{\frac{t}{2}} \end{pmatrix}$, which we should think of as a box around the periodic orbit $P_{e^{-t}}$. The distance of points in this new box to the new periodic orbit in the direction of geodesic flow is unchanged, and in the direction of the opposite horocycle flow has decreased exponentially. Hence for $f \in C_c(X)$ and large enough t we have (by uniform continuity and the careful construction of a thin enough box)

$$\int_{P_{e^{-t}}} f \approx \frac{1}{m_X(B)} \int_{B_t} f \, dm_X = \frac{1}{m_X(B)} \langle f, g_t \cdot \chi_B \rangle \approx \int_X f,$$

which can be made more precise to give a proof of the (noneffective) claim.

Using $f \in C_c^\infty$ one can use the same argument as above (the method in [16] is different and gives a better constant for δ), replacing the box with a smooth box-like function, and replacing the Howe–Moore theorem with the *effective decay of matrix coefficients* as discussed in §4.

3.2.3. More general closed orbits. More generally, one may ask about the distribution properties of closed, finite volume orbits xH of closed subgroups $H \subset G$ on quotients $X = \Gamma\backslash G$. If H is generated by unipotent subgroups, a theorem of Mozes and Shah [13] describes limits of such finite volume orbits – the limit measure is again a Haar measure m_{xL} just as in Ratner's theorem. However, also just as in Ratner's equidistribution theorem for individual orbits, the problem of establishing an error rate in this generality is wide open. We will discuss in §6 a special case where an error rate has been obtained in joint work with Margulis and Venkatesh [7].

4. Effective decay of matrix coefficients and spectral gap

We assume G is a closed linear semisimple group. We say we have *effective decay of matrix coefficients* for $X = \Gamma\backslash G$ if there exists some $\delta > 0$ such that

$$\left|\langle g \cdot f_1 - \textstyle\int f_1 \, dm_X, f_2 - \int f_2 \, dm_X \rangle\right| \ll \|g\|^{-\delta} S(f_1) S(f_2), \tag{6}$$

where $g \in G$, $f_1, f_2 \in C_c^\infty(X)$, and $\|g\|$ denotes the maximum of the matrix entries of g. As before $S(f)$ denotes a Sobolev norm of f.

Also we say that the action of G on X has a *spectral gap* if there exists some nonnegative $\chi \in C_c(G)$ with $\int_G \chi(g) dm_G(g) = 1$ such that for any $f \in L^2(X)$ with $\int_X f dm_X = 0$ we have $\|\chi * f\|_2 \leq \theta \|f\|_2$ for some fixed $\theta < 1$. Here

$$\chi * f(x) = \int \chi(g) f(xg) \quad \text{for } x \in X$$

may be thought of as the average of the images $g \cdot f$ of f under the unitary transformation induced by right multiplication by g on X with respect to the weights $\chi(g)$. As $\chi * 1 = 1$ the assumption that $\theta < 1$ amounts to having a gap in the spectrum of the operator $\chi *$.

Both of the above notions generalize to more general unitary representations of G, in both notions we restrict ourself to representations without fixed vectors (or equivalently the orthogonal complement of the space of vectors fixed under G). The existence of a spectral gap $\theta < 1$ that is independent of the unitary representation is the well-known property (T) of the group G. We recall that $\mathrm{SL}(3, \mathbb{R})$ has property (T), but that $\mathrm{SL}(2, \mathbb{R})$ does not have property (T).

From representation theory one knows that (6) is equivalent to spectral gap for the G-action on $L^2(X)$ (where δ and the gap $1 - \theta$ are related). We refer to [7, Section 6] and the references there for a discussion of this equivalence.

4.1. Effective decay for SL(3, ℝ). Spectral gap, in the form of effective decay of matrix coefficients, is an essential input for establishing effective equidistribution for homogeneous spaces, and so we would like to discuss where it comes from. However, instead of describing the general argument for establishing spectral gap and effective decay of matrix coefficients on "congruence" quotients, which would be quite hard in these short notes, we will give a direct proof of the effective decay of matrix coefficients for unitary representations of $\mathrm{SL}(3,\mathbb{R})$. I.e., we will prove (6) by showing

$$\left|\langle g\cdot v, w\rangle\right| \ll \|g\|^{-\frac{3}{4}} S(v)S(w), \tag{7}$$

whenever v, w are smooth vectors belonging to a Hilbert space $\mathcal{H}$ which has a unitary action of $\mathrm{SL}(3,\mathbb{R})$ on it and does not contain any $\mathrm{SL}(3,\mathbb{R})$-fixed vectors. E.g. this will apply to the subspace of $L^2(\Gamma\backslash\mathrm{SL}(3,\mathbb{R}))$ of functions of integral zero for any lattice Γ. Again we will use a Sobolev norm $S(v)$ for smooth vectors $v\in\mathcal{H}$ which we define below. The argument we present is an effectivization of the proof that $\mathrm{SL}(3,\mathbb{R})$ has property (T) and is likely well known to experts of the field.

4.1.1. Smooth vectors and the Sobolev norm. Let π be a unitary representation of $\mathrm{SL}(3,\mathbb{R})$ on a Hilbert space $\mathcal{H}$, for which we will also write $\pi(g)v = g\cdot v$ for $g\in\mathrm{SL}(3,\mathbb{R})$ and $v\in\mathcal{H}$. A vector $v\in\mathcal{H}$ is called *smooth* if all partial derivates of $g\mapsto\pi(g)v$ as a map from G to $\mathcal{H}$ exist. Taking a basis $e_1,\ldots,e_8$ of the Lie algebra $\mathfrak{sl}_3$ of $\mathrm{SL}(3,\mathbb{R})$ we can define the *Sobolev norm* (of degree one) by

$$S(v)^2 = \|v\|^2 + \sum_{j=1}^{8}\left\|\left(\frac{\partial}{\partial t}\exp(te_j)\cdot v\right)\Big|_{t=0}\right\|^2,$$

where the sum is over all partial derivatives corresponding to the basis elements.

4.2. Spectral measures. We will also be needing some basic properties of the spectral measures which we recall next. Let π be a unitary representation of $\mathbb{R}^2$. Then $\boldsymbol{t}\to\langle\pi(\boldsymbol{t})v,v\rangle$ is a positive definite function and so equals $\int_{\mathbb{R}^2}\exp(2\pi i\,\boldsymbol{t}\cdot\boldsymbol{s})\,d\mu_{v,v}(\boldsymbol{s})$ for some finite measure $\mu_{v,v}$ on $\mathbb{R}^2$ by Bochner's theorem. We will refer to $\mu_{v,v}$ as the *spectral measure* of v. These are used in the theory of unitary representations of $\mathbb{R}^2$ to define the projection-valued measure E_B on $\mathcal{H}$ for any Borel subset $B\subset\mathbb{R}^2$ which have the property that the spectral measure μ_{E_Bv,E_Bv} is the restriction of $\mu_{v,v}$ to B. The map E_B is an orthogonal projection commuting with $\pi(\boldsymbol{t})$ for all $\boldsymbol{t}\in\mathbb{R}^2$, satisfies that $E_{\mathbb{R}^2}$ is the identity and that $E_{B_1\cup B_2} = E_{B_1}+E_{B_2}$ whenever $B_1, B_2\subset\mathbb{R}^2$ are disjoint. In particular, if $\mathcal{H}$ does not contain any vectors fixed under $\mathbb{R}^2$, then $\mu_{v,v}(\{0\}) = 0$. Finally, we note that if $v,w\in\mathcal{H}$ have singular spectral measures, then v and w are orthogonal.

We will be using these tools for the restriction of the unitary representation of $\mathrm{SL}(3,\mathbb{R})$ to the subgroup

$$U = \left\{ \begin{pmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} : \boldsymbol{t} \in \mathbb{R}^2 \right\}.$$

Note that by the Mautner phenomenon we have no $\mathbb{R}^2$-fixed vectors in $\mathcal{H}$ as we assumed that there are no $\mathrm{SL}(3,\mathbb{R})$-invariant vectors.

We note that the subgroup $\mathrm{SL}(2,\mathbb{R})$ embedded into the upper left corner of $\mathrm{SL}(3,\mathbb{R})$ normalizes the subgroup U. This leads to a relationship of the spectral measure of v and of $g \cdot v$ for $g \in \mathrm{SL}(2,\mathbb{R})$. In fact, we claim that $\mu_{g\cdot v, g\cdot v} = (g^{-1})^T_* \mu_{v,v}$. This follows from uniqueness of the measure in Bochner's theorem and the equation

$$\begin{aligned} \langle \pi(\boldsymbol{t}) g \cdot v, g \cdot v \rangle = \langle \pi(g^{-1}\boldsymbol{t}) v, v \rangle &= \int_{\mathbb{R}^2} \exp(2\pi i (g^{-1}\boldsymbol{t}) \cdot \boldsymbol{s}) \, d\mu_{v,v}(\boldsymbol{s}) \\ &= \int_{\mathbb{R}^2} \exp(2\pi i \boldsymbol{t} \cdot ((g^{-1})^T \boldsymbol{s}) \, d\mu_{v,v}(\boldsymbol{s}). \end{aligned}$$

Here we used that $\left(\begin{smallmatrix} g^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$ belongs to U and is the element corresponding to $g^{-1}\boldsymbol{t}$.

4.3. Eigenfunctions of SO(2) first. We assume first that $v, w \in \mathcal{H}$ are eigenfunctions of $\mathrm{SO}(2)$, i.e., that for the matrix $k_\theta \in \mathrm{SO}(2)$ corresponding to a rotation by angle θ we have $k_\theta \cdot v = \exp(i\theta n) v$ and $k_\theta \cdot w = \exp(i\theta m) v$ for some $n, m \in \mathbb{Z}$. In this case we have $\langle \pi(\boldsymbol{t}) k_\theta \cdot v, k_\theta \cdot v \rangle = \langle \pi(\boldsymbol{t}) v, v \rangle$ which shows that the spectral measures of v and $k_\theta \cdot v$ are the same. This implies that the spectral measure of v, and similarly for w, is invariant under $\mathrm{SO}(2)$.

We claim that for such eigenfunctions v, w we have

$$|\langle a_r \cdot v, w \rangle| \ll e^{-|r|} \|v\| \|w\| \quad \text{with } a_r = \begin{pmatrix} e^{-r} & 0 & 0 \\ 0 & e^r & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{8}$$

where as before we assume $\mathcal{H}$ does not contain any $\mathrm{SL}(3,\mathbb{R})$-invariant vectors. We assume that $r > 0$, the argument for the other case is similar. We show this by splitting both v and w into two components $v = v_{\text{main}} + v_{\text{vertical}}$ and $w = w_{\text{main}} + w_{\text{horizontal}}$. Here v_{vertical} is defined as the image of v under the orthogonal projection defined by the set $\left\{(t_2, t_1) : \left|\frac{t_2}{t_1}\right| \geq e^r\right\}$ which is a sector shaped neighborhoods of the t_2-axis of angle $\ll e^{-r}$. Hence by invariance of the spectral measure under $\mathrm{SO}(2)$ we get $\|v_{\text{vertical}}\| \ll e^{-r}\|v\|$. Similarly, $w_{\text{horizontal}}$ is defined as the image of w under the orthogonal projection defined by $\left\{(t_2, t_1) : \left|\frac{t_2}{t_1}\right| \leq e^{-r}\right\}$ which also has

$\|w_{\mathrm{horizontal}}\| \ll e^{-r}\|w\|$. The other two vectors v_{main} and w_{main} are defined as the projections w.r.t. the complements of these sets. Recall that the spectral measure of v_{main} is supported on $\left\{(t_2,t_1) : \left|\frac{t_2}{t_1}\right| < e^r\right\}$ and that the spectral measure of $a_r \cdot v_{\mathrm{main}}$ is the push forward of the spectral measure of v_{main} under $(a_r^{-1})^T = a_r^{-1}$. This shows that the spectral measure of $a_r \cdot v_{\mathrm{main}}$ is supported on $\left\{(t_2,t_1) : \left|\frac{t_2}{t_1}\right| < e^{-r}\right\}$ and so $a_r \cdot v_{\mathrm{main}}$ is orthogonal to w_{main} as their spectral measures are supported on disjoint sets. Applying this to

$$\begin{aligned}|\langle a_r \cdot v, w\rangle| \le{}& |\langle a_r \cdot v_{\mathrm{main}}, w_{\mathrm{main}}\rangle| + |\langle a_r \cdot v_{\mathrm{main}}, w_{\mathrm{horizontal}}\rangle| \\ &+ |\langle a_r \cdot v_{\mathrm{vertical}}, w_{\mathrm{main}}\rangle| + |\langle a_r \cdot v_{\mathrm{vertical}}, w_{\mathrm{horizontal}}\rangle|,\end{aligned}$$

we get that the first term is zero, and the other are bounded by $\ll e^{-r}\|v\|\|w\|$ which gives (8).

4.4. Bootstrapping to general vectors and general group elements. We first extend (8) to any diagonal matrix

$$a = \begin{pmatrix} e^{r_1} & 0 & 0 \\ 0 & e^{r_2} & 0 \\ 0 & 0 & e^{r_3} \end{pmatrix}$$

with $r_1 + r_2 + r_3 = 0$ and any two smooth vectors $v, w \in \mathcal{H}$ to say

$$|\langle a \cdot v, w\rangle| \ll e^{-\frac{1}{2}|r_2 - r_1|} S(v) S(w). \tag{9}$$

To obtain this we decompose $v = \sum_{n\in\mathbb{Z}} v_n$ and $w = \sum_{m\in\mathbb{Z}} w_m$ into eigenfunctions for SO(2) – by smoothness these sums converge absolutely. Next notice that

$$a = \begin{pmatrix} e^{r_1 + \frac{1}{2}r_3} & 0 & 0 \\ 0 & e^{r_2 + \frac{1}{2}r_3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{1}{2}r_3} & 0 & 0 \\ 0 & e^{-\frac{1}{2}r_3} & 0 \\ 0 & 0 & e^{r_3} \end{pmatrix} = a_{r_2 + \frac{1}{2}r_3} c,$$

where $a_{r_2+\frac{1}{2}r_3} = a_{\frac{1}{2}(r_2 - r_1)}$ is as in (8) and c commutes with SO(2). The latter implies that v_n is mapped under c again to eigenfunctions of SO(2). Therefore, we may apply (8) to each $c \cdot v_n$ and w_m to get

$$|\langle a \cdot v, w\rangle| \le \sum_{m,n\in\mathbb{Z}} |\langle a_{\frac{1}{2}(r_2 - r_1)} c \cdot v_n, w_m\rangle| \ll e^{-\frac{1}{2}|r_2 - r_1|} \sum_{m,n\in\mathbb{Z}} \|v_n\|\|w_m\|.$$

However, the last sum on the right may be written as the product of $\sum_{n\in\mathbb{Z}} \|v_n\|$ and the corresponding sum for w_m. Notice that the derivative of v_n along some element r of the Lie algebra of SO(2) equals

$$\left(\frac{\partial}{\partial t} \exp(tr) \cdot v_n\right)\Big|_{t=0} = n v_n$$

and so $\left(\frac{\partial}{\partial t}\exp(tr)\cdot v\right)\big|_{t=0} = \sum_{n\in\mathbb{Z}} nv_n$, and that the terms in the last sum are all orthogonal to each other. Hence Cauchy–Schwarz gives

$$\sum_{n\in\mathbb{Z}}\|v_n\| = \|v_0\| + \sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\|nv_n\| \ll \|v_0\| + \left\|\left(\frac{\partial}{\partial t}\exp(tr)\cdot v\right)\bigg|_{t=0}\right\| \ll S(v),$$

where we used that r can be expressed as a linear combination of the basis elements $e_1,\dots,e_8 \in \mathfrak{sl}_3$ that we used to define $S(v)$. This gives (9).

To make (9) closer to (7) we notice that in (9) we could have proven the same statement with either $e^{-\frac{1}{2}|r_3-r_2|}S(v)S(w)$ or with $e^{-\frac{1}{2}|r_3-r_1|}S(v)S(w)$ on the right. We claim that

$$\min(e^{-\frac{1}{2}|r_2-r_1|}, e^{-\frac{1}{2}|r_3-r_2|}, e^{-\frac{1}{2}|r_3-r_1|}) \le \|a\|^{-\frac{3}{4}},$$

which then shows that (7) holds for all diagonal matrices. To prove the above, assume that $r_3 \ge r_2 \ge r_1$. Then $\|a\| = e^{r_3}$ and $e^{r_1} \le e^{\frac{1}{2}(r_1+r_2)}$ which together with $r_1+r_2+r_3 = 0$ gives

$$e^{r_1-r_3} \le e^{\frac{1}{2}(r_1+r_2+r_3)}e^{-\frac{3}{2}r_3} = e^{-\frac{3}{2}r_3} = \|g\|^{-\frac{3}{2}},$$

which proves the claim and so (7) in this case.

To prove (7) for all $g \in \mathrm{SL}(3,\mathbb{R})$ we recall that $g = k_1 a k_2$ for some $k_1, k_2 \in \mathrm{SO}(3)$ and some diagonal matrix a by the Cartan decomposition of g in $\mathrm{SL}(3,\mathbb{R})$. As $\mathrm{SO}(3)$ is compact, $\|g\|$ and $\|a\|$ are bounded by some multiples of each other. Similarly, $S(k_2\cdot v) \ll S(v)$ and $S(k_1^{-1}w) \ll S(w)$. Together this gives using (9) that

$$|\langle g\cdot v, w\rangle| = |\langle a\cdot(k_2\cdot v), k_1^{-1}\cdot w\rangle| \ll \|g\|^{-\frac{3}{4}}S(v)S(w),$$

which proves (7).

4.5. Groups without property (T). As we mentioned before the above argument is the effectivization of the proof that $\mathrm{SL}(3,\mathbb{R})$ has property (T). However, e.g. $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(m,1)(\mathbb{R})$ do not have property (T). For these groups spectral gap (respective effective decay of matrix coefficients) is not an automatic property for any unitary representation. However, Selberg showed that the $\mathrm{SL}(2,\mathbb{R})$-action on congruence quotients $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$ has a spectral gap – in fact there is a uniform spectral gap that is independent of Γ. In this case and in similar cases the spectral gap is a property of the space and not of the group.

5. An effective pointwise ergodic theorem

For the main theorem of [7], which we will discuss in §6, a pointwise ergodic theorem was needed and also proven in [7, Proposition 9.2]. As we outline now this is a consequence of the effective decay of matrix coefficients (6) discussed earlier (but we will refer to [7] for the last step of the argument).

There are two basic types of non-compact one-parameter subgroups of semisimple groups $G \subseteq \mathrm{SL}(n,\mathbb{R})$: Diagonalizable subgroups and unipotent subgroups. The estimate in (6) can be used to establish an effective ergodic theorem for both of them, but as the unipotent case may seem a bit more delicate and at the same time is the case that will be used later, let us focus on that case. Hence suppose $u_t = \exp(tp)$ is a unipotent one-parameter subgroup defined by some nilpotent element p in the Lie algebra of G. Then we notice that $t \ll \|u_t\| \ll t^n$ as the entries of the matrix u_t are polynomials in t and so (6) is the statement that matrix coefficients decay at a polynomial rate with respect to the time parameter of the subgroup. (For diagonalizable subgroups (6) would be exponential decay of matrix coefficients.)

5.1. A single function and a given time first. For $f \in C_c^\infty(X)$ and $T > 0$ we define the *discrepancy* at x by

$$D_T(f)(x) = \frac{1}{T}\int_0^T f(xu_t)\,dt - \int_X f\,dm_X.$$

It measures how far the time average over $[0,T]$ is away from the expected value. Using Fubini's theorem several times as well as that $u_t \in G$ preserves m_X we get

$$\begin{aligned}\int |D_T(f)|^2\,dm_X &= \frac{1}{T^2}\int_0^T\int_0^T\int_X f(xu_{t_1})\bar{f}(xu_{t_2})dm_X\,dt_2dt_1\\ &\quad -2\,\mathrm{Re}\,\frac{1}{T}\int_0^T\int_X f(xu_t)\,dm_X dt\int_X \bar{f}dm_X + \left|\int_X f\,dm_X\right|^2\\ &= \frac{1}{T^2}\int_0^T\int_0^T\left(\langle u_{t_1-t_2}\cdot f, f\rangle - \left|\int_X f\,dm_X\right|^2\right)dt_2dt_1.\end{aligned}$$

Now notice that for most $(t_1,t_2) \in [0,T]^2$ we have that $|t_1-t_2|$ is quite big, and so the expression within the last integral is quite small. More precisely, if $|t_1-t_2| > T^{\frac12}$, then by (6) we have that

$$\left|\langle u_{t_1-t_2}\cdot f, f\rangle - \left|\textstyle\int_X f\,dm_X\right|^2\right| \ll T^{-\frac12\delta}S(f)^2$$

while the integral over the part $|t_1-t_2| \le T^{\frac12}$ is bounded by the area $\le T\,T^{\frac12}$ times the trivial estimate $\ll \|f\|_\infty^2$ of the integrand. Together this gives

$$\int |D_T(f)|^2\,dm_X \ll T^{-\frac12\delta}S(f)^2 + T^{-\frac12}\|f\|_\infty^2.$$

We can simplify this as follows. If we modify our notion of Sobolev norm, we can make sure that

$$\|f\|_\infty \ll S(f),$$

see [7, Lemma 5.1.1]. This is not entirely trivial, because in fact, we claim that one can modify the norm in such a way that $S(f)$ is still the norm of a pre-Hilbert-space structure (i.e., is an Hermitian norm) on $C_c^\infty(X)$ – this is not important right now, but will be for the last step of the argument. If additionally we also assume w.l.o.g. that $\delta \le 1$, then we have

$$\int |D_T(f)|^2 \, dm_X \ll T^{-\frac{1}{2}\delta} S(f)^2.$$

This shows that

$$W^2 m_X\big(\{x \in X : D_T(f)(x) \ge W\}\big) \ll T^{-\frac{1}{2}\delta} S(f)^2 \tag{10}$$

for any $W > 0$. We still have some freedom in W – asking for a better estimate, i.e., a smaller value of W, will make the estimate of the set worse. To achieve a reasonable estimate on the set, we set $W = T^{-\frac{1}{6}\delta} S(f)$ which makes the above into

$$m_X\big(\{x \in X : D_T(f)(x) \ge T^{-\frac{1}{6}\delta} S(f)\}\big) \ll T^{-\frac{1}{6}\delta}. \tag{11}$$

This is already an effective version of the pointwise ergodic theorem: For a given $f \in C_c^\infty(X)$ and $T > 0$ we know that the average of f over the $[0, T]$-orbit of x is $T^{-\frac{1}{6}\delta} S(f)$ close to $\int_X f \, dm_X$ except possibly for a set of points x of measure $\ll T^{-\frac{1}{6}\delta}$. However, this is not yet very satisfactory as the exceptional set is still allowed to depend on f and on T.

5.2. A single function with large enough times. It is relatively easy to obtain the following strengthening of the above: There exists some $\epsilon > 0$ such that for a given $f \in C_c^\infty(X)$ we have that for any T_0 the average of f over the $[0, T]$-orbit of x is $T^{-\epsilon} S(f)$ close to $\int_X f dm_X$ for all $T \ge T_0$ except possible for a set of measure $\ll T_0^{-\epsilon}$. I.e., at some cost in the exponents we can make the set independent of the particular time interval $[0, T]$ that we use to average and still get a very good estimate on the measure of the exceptional points if we only restrict ourself to large enough times $T \ge T_0$.

To prove the above let $M = 12\frac{1}{\delta}$. Then we may apply (11) for $T_n = n^M$ which gives

$$m_X\big(\{x \in X : D_T(f)(x) \ge T_n^{-\frac{1}{6}\delta} S(f)\}\big) \ll n^{-2}.$$

Call this exceptional set E_n, then $m_X\big(\bigcup_{n \ge n_0} E_n\big) \ll n_0^{-1}$ for any real $n_0 > 0$.

Now choose some T_0 and define $n_0 = T_0^{\frac{1}{M}}$. Now let $T \ge T_0$ and let $n = \lceil T^{\frac{1}{M}} \rceil \ge n_0$. Then $|n^M - T| \ll n^{M-1} \ll T^{\frac{M-1}{M}} = T^{1-\frac{1}{M}}$ and from this it is easy to see that

$$\left| \frac{1}{n^M} \int_0^{n^M} f(xu_t)\, dt - \frac{1}{T} \int_0^T f(xu_t)\, dt \right| \ll \|f\|_\infty T^{-\frac{1}{M}}.$$

This gives for $x \notin E_n$ that

$$\left| \frac{1}{T} \int_0^T f(xu_t)\, dt - \int_X f \, dm_X \right| \ll (T^{-\frac{1}{M}} + T^{-\frac{1}{6}\delta}) S(f).$$

Setting $\epsilon = \min\left(\frac{1}{M}, \frac{1}{6}\delta\right) = \frac{1}{12}\delta$ gives the desired estimate.

5.3. Bootstrapping to all functions $f \in C_c^\infty(X)$. We fix some $\epsilon > 0$. We say a point $x \in X$ is (T_0, ϵ)-*generic* if

$$\left| \frac{1}{T} \int_0^T f(xu_t) dt - \int_X f dm_X \right| \ll T^{-\epsilon} S'(f) \tag{12}$$

for all $T \geq T_0$ and all $f \in C_c^\infty(X)$. Then an even stronger effective version of the pointwise ergodic theorem would be that there exists a choice of ϵ for which

$$m_X\left(\left\{x : x \text{ is not } (T_0, \epsilon)\text{-generic}\right\}\right) \ll T_0^{-\epsilon}.$$

This can be obtained by the argument in [7, Section 9]. The hidden cost is that in (12) a different notion of Sobolev norm S' (defined using more derivatives) is used than in (10). Allowing for that, gives us the possibility of making W in (10) also depend on $\frac{S'(f)}{S(f)}$. The argument is in some way then similar to §2.3.2 and §4.4. Using different Sobolev norms one can find an orthonormal basis $f_1, \ldots, f_k, \ldots$ w.r.t. $S'(\cdot)$ such that $\sum_{n=1}^\infty S(f_n)$ is finite. This uses some ideas (relative traces of Hermitian norms) of Bernstein and Reznikov [1]

6. Effective equidistribution for semisimple subgroups

We shall assume the following.

- There is a semisimple $\mathbb{Q}$-group $\boldsymbol{G}$ so that $G = \boldsymbol{G}(\mathbb{R})^\circ$ and Γ is a congruence subgroup of $\boldsymbol{G}(\mathbb{Q})$.
- H is a connected semisimple subgroup without compact factors.

We note that in this context an H-orbit $x_0 H \subset X = \Gamma\backslash G$ is closed *if and only if* it has finite volume.

Two examples of this setup are $H = \mathrm{SO}(2,1)(\mathbb{R})^\circ$ acting on $\mathrm{SL}(3,\mathbb{Z})\backslash \mathrm{SL}(3,\mathbb{R})$, and $H = \mathrm{SL}(k,\mathbb{R})$ embedded diagonally in $\mathrm{SL}(k,\mathbb{R}) \times \mathrm{SL}(k,\mathbb{R})$ acting on $\mathrm{SL}(k,\mathbb{Z}) \times \mathrm{SL}(k,\mathbb{Z})\backslash \mathrm{SL}(k,\mathbb{R}) \times \mathrm{SL}(k,\mathbb{R})$ for $k \geq 2$. In fact in both of these examples H is a maximal subgroup, where a subgroup $H \subset G$ is called *maximal* if there is no subgroup $S \subset G$ containing H with dimension strictly between the dimensions of G and H.

6.1. Maximal subgroup theorem. In joint work with Margulis and Venkatesh we proved [7] the following theorem[8].

Theorem 1 ([7], simpler form). *Let $\Gamma, H \subset G$ be as above. Assume that H is a maximal subgroup of G.*

There exists $\delta > 0$ depending only on G, H so that the Haar measure $m_{x_0 H}$ on a closed orbit $x_0 H$ is $\mathrm{Vol}^{-\delta}$-close to m_X, i.e., for any $f \in C_c^\infty(X)$ we have

$$\left| \int_{x_0 H} f - \int_X f \right| \ll \mathrm{Vol}^{-\delta} S(f),$$

where Vol *denotes the volume*[9] *of the orbit $x_0 H$.*

Crucial input. This theorem has as the major input the *spectral gap* for the *H-action on $L^2(x_0 H)$* in a *uniform* way for all possible closed orbits $x_0 H$, i.e., δ and the implicit constant as in the discussion of effective decay of matrix coefficients (6) are not allowed to depend on x_0.

- If H has property (T) as e.g. for $H = \mathrm{SL}(3, \mathbb{R})$, this holds always.
- If H does not have (T) as e.g. for $H = \mathrm{SO}(2, 1)(\mathbb{R})^\circ$, the required statement is property (τ) as established by Clozel [5] (building on work of Burger and Sarnak [4]). This is where the congruence assumption on Γ is crucial, see [7, Section 6].

6.2. A comment about the proof. Our proof has little to do with the outline for the horocycle flow in §3.2.2, instead may be viewed as an effective version of the measure classification theorem by Ratner and the limiting distribution theorem due to Mozes and Shah (in the semisimple case considered here). It uses a version of the effective ergodic theorem discussed in §5 (where the measure m_X is replaced by $m_{x_0 H}$). The difference of the effective ergodic theorem in [7, Proposition 9.2] and what we discussed in §5 is that in the former the average is not taken over initial intervals $[0, T]$ but rather over long intervals very far away from the origin. More precisely, in [7, Proposition 9.2] an error is obtained for the average of f over the interval $[T^M, (T+1)^M]$ which roughly speaking has length T^{M-1}, and this error holds for all points but those in a set of small measure. This is desirable, as the divergence of two nearby points under a unipotent one-parameter subgroup in H is determined by a polynomial. If this polynomial is uniformly bounded on $[0, (T+1)^M]$ then it is nearly constant on the interval $[T^M, (T+1)^M]$. This allows the effectivization of

[8]The first simpler version of the theorem was presented by Margulis in several talks before our joint work and may also be approachable by other methods. In fact most of the work in [7] goes into the discussion of possible intermediate subgroups where the argument becomes more involved, see Theorem 2.

[9]The volume Vol is calculated in comparison with a fixed Haar measure on H, but the Haar measure $m_{x_0 H}$ is normalized to be a probability measure.

a particular argument that appears in Ratner's work (the combination of the ergodic theorem and polynomial divergence for unipotent orbits, see [14] and [15, pg. 244]), we refer to [6] or [7, Section 2] for the ineffective argument in precisely the context we need here.

6.3. More general version. The more general version of our theorem is as follows.

Theorem 2 ([7], current form). *Let Γ, $H \subset G$ be as above. Assume that H has finite centralizer in G. There exists $\delta > 0$ depending only on G, H and $V_0 > 0$ depending only on Γ, G, H so that, for any $V \geq V_0$ and any closed orbit $x_0 H$ there exists an intermediate subgroup $H \subseteq S \subseteq G$ for which*

- *$x_0 S$ is a closed S-orbit with volume $< V$, and*
- *the Haar measure on $x_0 H$ is $V^{-\delta}$-close to the Haar measure on $x_0 S$, i.e., for any $f \in C_c^\infty(X)$ we have*

$$\left| \int_{x_0 H} f - \int_{x_0 S} f \right| < V^{-\delta} S(f).$$

One may read this statement as follows: If $V = \mathrm{Vol}(x_0 H)$ is very large we may apply the theorem to this parameter and obtain some bigger group $S \supsetneq H$. The orbit $x_0 S$ of the higher dimensional group S has finite volume $V' = \mathrm{Vol}(x_0 S)$ (w.r.t. to a Haar measure on S) and should be thought of as being less complicated since $V' < V$. However, V' may still be large ($x_0 S$ may still be complicated), so that one may want to apply the theorem to the parameter V' to obtain a different group $S' \supsetneq S$ whose orbit $x_0 S'$ has smaller volume (is less complicated) at the cost of obtaining a worse error statement. This may be continued until the volume of the orbit of some group becomes less than V_0 (e.g. if $S'' = G$).

6.3.1. Visualization on $\mathbb{T}^3$. A toy model for this problem of intermediate orbits is the image of long rational line $L \subset \mathbb{R}^3$ in a 3-dimensional torus $L/\mathbb{Z}^3 \subset \mathbb{T}^3$. It is determined $L = \mathbb{R}\boldsymbol{n}$ by a single primitive vector $\boldsymbol{n} \in \mathbb{Z}^3$ and the length of the closed circle $L/\mathbb{Z}^3$ is precisely $\|\boldsymbol{n}\|$. As we mentioned in §3.2 it is quite easy to establish an effective error for the distribution properties of a rational torus in $\mathbb{T}^2$. However, unlike the case of a rational line in $\mathbb{T}^2$ the circle $L/\mathbb{Z}^3$ is contained in rational planes $P \subset \mathbb{R}^3$. A rational plane is determined by a primitive orthogonal vector $\boldsymbol{v} \in \mathbb{Z}^3$, and one may check that $\|\boldsymbol{v}\|$ equals the area of the image torus $P/\mathbb{Z}^3$ – we will also think of $\|\boldsymbol{v}\|$ as a measure of the complexity of $P/\mathbb{Z}^3$ inside $\mathbb{T}^3$. If $\|\boldsymbol{v}\|$ is much smaller than $\|\boldsymbol{n}\|$ for some choice of the plane, then an effective error with an error determined by $\|\boldsymbol{n}\|$ can only be given if we compare the Lebesgue measure on the circle $L/\mathbb{Z}^3$ to the Lebesgue measure on the two-dimensional subtorus $P/\mathbb{Z}^3$. If $\|\boldsymbol{v}\|$ is also big (for all rational planes containing L), then one can also compare the

Lebesgue measure on the circle $L/\mathbb{Z}^3$ to the Lebesgue measure on $\mathbb{T}^3$ but the error would be expressed in terms of the smallest $\|\boldsymbol{v}\|$.

7. Transportation of spectral gap

7.1. Hecke correspondences. The above theorem (in fact the maximal case in Theorem 1) may be used in the context of $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ with Γ equal to the product of $\mathrm{SL}(2,\mathbb{Z})$ with itself. Then the Hecke correspondence T_p^n (roughly speaking) corresponds to big volume orbits inside $X = \Gamma\backslash G$ with respect to the diagonal subgroup isomorphic to $\mathrm{SL}(2,\mathbb{R})$, i.e., $H_\Delta = \{(g,g) : g \in \mathrm{SL}_2(\mathbb{R})\}$. These orbits are isomorphic to congruence quotients of $\mathrm{SL}(2,\mathbb{R})$. The uniform effective decay of matrix coefficients (which comes from Selberg's theorem) for the action of H_Δ then implies a bound for the eigenvalue of the Hecke operator T_p. In that sense, the theorem allows us to *transport* the spectral gap from one place to another (in this case from ∞ to p).

7.2. General setup. Another instance of this *transportation of spectral gap* can be set up as follows.

Let G_1, G_2 be simple groups, and suppose G_2 has (T) but G_1 has not. Let Γ be an irreducible lattice in $G = G_1 \times G_2$, e.g. this is possible for $G_1 = \mathrm{SU}(2,1)(\mathbb{R})$ and $G_2 = \mathrm{SL}(3,\mathbb{R})$. As we discussed in §4 (resp. §4.1 in the case of $\mathrm{SL}(3,\mathbb{R})$) we then have effective decay of matrix coefficients for the action of G_2.

We wish to bound the matrix coefficients of G_1 acting on $X = \Gamma\backslash G$. Let $H_\Delta = \{(g,g) : g \in G\}$. Notice that the diagonal orbit $\Gamma \times \Gamma H_\Delta \subset X \times X$ is 'responsible' for the inner product in the sense that the integral of $f_1 \otimes \bar{f}_2$ over this orbit equals the inner product $\langle f_1, f_2\rangle$. In the same sense is the deformed orbit $\Gamma \times \Gamma H_\Delta(g,e)$ responsible for the matrix coefficients of g, i.e.,

$$\int f_1 \otimes \bar{f}_2 dm_{\Gamma\times\Gamma H_\Delta(g,e)} = \int_X f_1(xg)\bar{f}_2(x) d_{m_X}(x) = \langle g \cdot f_1, f_2\rangle.$$

The volume of the deformed orbit $\Gamma \times \Gamma H_\Delta(g,e)$ is roughly speaking a power of $\|g\|$, more precisely bounded from above and below by multiples of powers of $\|g\|$. Hence effective equidistribution of the Haar measures on these orbits to the Haar measure on $X \times X$ gives effective decay of matrix coefficients.

However, notice that the theorem does not apply as the group giving the closed orbit has been conjugated and does not remain fixed. (In the theorem the rate of equidistribution is allowed to depend on the group H, which is changing in this case.)

On a positive side, if $g = (g_1, e)$ then the simple factor of H_Δ corresponding to G_2 remains (as a subgroup of $G \times G$) fixed and this is the part with known effective decay. In this case the method behind the proof of the theorem can be used to show effective equidistribution and so decay of matrix coefficients for the G_1-action. In all of this, the rate (i.e., the δ appearing in the discussion) of decay of matrix coefficients for G_1 only depends on the spectral gap for G_2 (but not on Γ).

7.3. Effective equidistribution implies a weak form of (τ). Using the above construction for a p-adic group G_2, one can prove a weaker version of property (τ) for all simple algebraic groups of absolute higher rank (i.e., all groups except forms of SL_2 for which property (τ) has been known much longer).

So let $\boldsymbol{G}$ be a simple, simply connected algebraic $\mathbb{Q}$-group of absolute rank ≥ 2. Let $G_1 = \boldsymbol{G}(\mathbb{R})$, $G_2 = \boldsymbol{G}(\mathbb{Q}_p)$, and let Γ be commensurable with $\boldsymbol{G}(\mathbb{Z}[\frac{1}{p}])$, then $L^2(\Gamma_1\backslash G)$ (with $\Gamma_1 = \Gamma \cap \boldsymbol{G}(\mathbb{Z}_p)$) is contained in $L^2(\Gamma\backslash G_1 \times G_2)$. We choose p such that G_2 has $\mathbb{Q}_p$-rank ≥ 2. This gives that G_2 has property (T), and so also effective decay of matrix coefficients. The latter is the only input to the method which establishes the result.

Hence the spectral gap of the G_2-action and its independence from Γ gives also some spectral gap of the G_1-action on $\Gamma_1\backslash G_1$ and in a uniform way (as long as the lattice in G_1 can be obtained from a lattice in $G_1 \times G_2$ by intersection which is always possible for congruence subgroups). We then obtain a proof of uniform spectral gap of the action of G_1, a version of property (τ). We note that the gap hereby obtained is probably quite bad in comparison to what Clozel obtained in [5]. This is part of an ongoing joint work with Margulis and Venkatesh.

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5ECM Amsterdam 2008

Survey on aspherical manifolds

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Abstract. This is a survey on known results and open problems about closed aspherical manifolds, i.e., connected closed manifolds whose universal coverings are contractible. Many examples come from certain kinds of non-positive curvature conditions. The property aspherical, which is a purely homotopy theoretical condition, implies many striking results about the geometry and analysis of the manifold or its universal covering, and the ring theoretic properties and the K- and L-theory of the group ring associated to its fundamental group. The Borel Conjecture predicts that closed aspherical manifolds are topologically rigid. The article contains new results about product decompositions of closed aspherical manifolds and an announcement of a result joint with Arthur Bartels and Shmuel Weinberger about hyperbolic groups with spheres of dimension ≥ 6 as boundary. At the end we describe (winking) our universe of closed manifolds.

Mathematics Subject Classification (2000). 57N99, 19A99, 19B99, 19D99, 19G24, 20C07, 20F25, 57P10.

Keywords. Aspherical closed manifolds, topological rigidity, conjectures due to Borel, Novikov, Hopf, Singer, non-positively curved spaces.

Contents

Introduction

A space X is called *aspherical* if it is path connected and all its higher homotopy groups vanish, i.e., $\pi_n(X)$ is trivial for $n \geq 2$. This survey article is devoted to aspherical closed manifolds. These are very interesting objects for many reasons. Often interesting geometric constructions or examples lead to aspherical closed manifolds. The study of the question which groups occur as fundamental groups of closed aspherical manifolds is intriguing. The condition aspherical is of purely homotopy theoretical nature. Nevertheless there are some interesting questions and conjectures about curvature properties of a closed aspherical Riemann manifold and about the spectrum of the Laplace operator on its universal covering. The Borel Conjecture predicts that aspherical closed topological manifolds are topologically rigid and that aspherical compact Poincaré complexes are homotopy equivalent to closed manifolds. We discuss the status of some of these questions and conjectures. Examples of exotic aspherical closed manifolds come from hyperbolization techniques and we list certain examples. At the end we describe (winking) our universe of closed manifolds.

The results about product decompositions of closed aspherical manifolds in Section 6 are new and Section 8 contains an announcement of a result joint with Arthur Bartels and Shmuel Weinberger about hyperbolic groups with spheres of dimension ≥ 6 as boundary.

The author wants to the thank the Max-Planck-Institute for Mathematics in Bonn for its hospitality during his stay from October 2007 until December 2007 when parts of this paper were written. The work was financially supported by the Sonderforschungsbereich 478 – Geometrische Strukturen in der Mathematik – the Max-Planck-Forschungspreis and the Leibniz-Preis of the author. The author wants to thank the referee for his valuable suggestions.

1. Homotopy theory of aspherical manifolds

From the homotopy theory point of view an aspherical CW-complex is completely determined by its fundamental group. Namely

Theorem 1.1 (Homotopy classification of aspherical spaces).

(i) *Two aspherical CW-complexes are homotopy equivalent if and only if their fundamental groups are isomorphic.*

(ii) *Let X and Y be connected CW-complexes. Suppose that Y is aspherical. Then we obtain a bijection*

$$[X, Y] \xrightarrow{\cong} [\Pi(X), \Pi(Y)], \quad [f] \mapsto [\Pi(f))],$$

where $[X, Y]$ is the set of homotopy classes of maps from X to Y, $\Pi(X)$, $\Pi(Y)$ are the fundamental groupoids, $[\Pi(X), \Pi(Y)]$ is the set of natural equivalence classes of functors from $\Pi(X)$ to $\Pi(Y)$ and $\Pi(f)\colon \Pi(X) \to \Pi(Y)$ is the functor induced by $f : X \to Y$.

Proof. (ii) One easily checks that the map is well-defined. For the proof of surjectivity and injectivity one constructs the desired preimage or the desired homotopy inductively over the skeletons of the source.

(i) This follows directly from assertion (ii). □

The description using fundamental groupoids is elegant and base point free, but a reader may prefer its more concrete interpretation in terms of fundamental groups, which we will give next: Choose base points $x \in X$ and $y \in Y$. Let $\hom(\pi_1(X, x), \pi_1(Y, y))$ be the set of group homomorphisms from $\pi_1(X, x)$ to $\pi_1(Y, y)$. The group $\mathrm{Inn}\big(\pi_1(Y, y)\big)$ of inner automorphisms of $\pi_1(Y, y)$ acts on $\hom\big(\pi_1(X, x), \pi_1(Y, y)\big)$ from the left by composition. We leave it to the reader to check that we obtain a bijection

$$\mathrm{Inn}\big(\pi_1(Y, y)\big)\backslash \hom\big(\pi_1(X, x), \pi_1(Y, y)\big) \xrightarrow{\cong} [\Pi(X), \Pi(Y)],$$

under which the bijection appearing in Lemma 1.1 (ii) sends $[f]$ to the class of $\pi_1(f, x)$ for any choice of representative of f with $f(x) = y$. In the sequel we will often ignore base points especially when dealing with the fundamental group.

Lemma 1.2. *A CW-complex X is aspherical if and only if it is connected and its universal covering $\widetilde{X}$ is contractible.*

Proof. The projection $p\colon \widetilde{X} \to X$ induces isomorphisms on the homotopy groups π_n for $n \geq 2$ and a connected CW-complex is contractible if and only if all its homotopy groups are trivial (see [99, Theorem IV.7.17 on page 182]. □

An aspherical CW-complex X with fundamental group π is the same as an *Eilenberg–Mac Lane space* $K(\pi, 1)$ *of type* $(\pi, 1)$ and the same as the *classifying space* $B\pi$ *for the group* π.

2. Examples of aspherical manifolds

In this section we give examples and constructions of aspherical closed manifolds.

2.1. Non-positive curvature. Let M be a closed smooth manifold. Suppose that it possesses a Riemannian metric whose sectional curvature is non-positive, i.e., is ≤ 0 everywhere. Then the universal covering $\widetilde{M}$ inherits a complete Riemannian metric whose sectional curvature is non-positive. Since $\widetilde{M}$ is simply-connected and has non-positive sectional curvature, the Hadamard–Cartan Theorem (see [45, 3.87 on page 134]) implies that $\widetilde{M}$ is diffeomorphic to $\mathbb{R}^n$ and hence contractible. We conclude that $\widetilde{M}$ and hence M is aspherical.

2.2. Low-dimensions. A connected closed 1-dimensional manifold is homeomorphic to S^1 and hence aspherical.

Let M be a connected closed 2-dimensional manifold. Then M is either aspherical or homeomorphic to S^2 or $\mathbb{RP}^2$. The following statements are equivalent: i.) M is aspherical. ii.) M admits a Riemannian metric which is *flat*, i.e., with sectional curvature constant 0, or which is *hyperbolic*, i.e., with sectional curvature constant -1. iii) The universal covering of M is homeomorphic to $\mathbb{R}^2$.

A connected closed 3-manifold M is called *prime* if for any decomposition as a connected sum $M \cong M_0 \sharp M_1$ one of the summands M_0 or M_1 is homeomorphic to S^3. It is called *irreducible* if any embedded sphere S^2 bounds a disk D^3. Every irreducible closed 3-manifold is prime. A prime closed 3-manifold is either irreducible or an S^2-bundle over S^1 (see [53, Lemma 3.13 on page 28]). A closed orientable 3-manifold is aspherical if and only if it is irreducible and has infinite fundamental group. A closed 3-manifold is aspherical if and only if it is irreducible and its fundamental group is infinite and contains no element of order 2. This follows from the Sphere Theorem [53, Theorem 4.3 on page 40].

Thurston's Geometrization Conjecture implies that a closed 3-manifold is aspherical if and only if its universal covering is homeomorphic to $\mathbb{R}^3$. This follows from [53, Theorem 13.4 on page 142] and the fact that the 3-dimensional geometries which have compact quotients and whose underlying topological spaces are contractible have as underlying smooth manifold $\mathbb{R}^3$ (see [89]).

A proof of Thurston's Geometrization Conjecture is given in [74] following ideas of Perelman.

There are examples of closed orientable 3-manifolds that are aspherical but do not support a Riemannian metric with non-positive sectional curvature (see [66]).

For more information about 3-manifolds we refer for instance to [53], [89].

2.3. Torsionfree discrete subgroups of almost connected Lie groups. Let L be a Lie group with finitely many path components. Let $K \subseteq L$ be a maximal compact subgroup. Let $G \subseteq L$ be a discrete torsionfree subgroup. Then $M = G\backslash L/K$ is a closed aspherical manifold with fundamental group G since its universal covering L/K is diffeomorphic to $\mathbb{R}^n$ for appropriate n (see [52, Theorem 1. in Chapter VI]).

2.4. Hyperbolization. A very important construction of aspherical manifolds comes from the *hyperbolization technique* due to Gromov [49]. It turns a cell complex into a non-positively curved (and hence aspherical) polyhedron. The rough idea is to define this procedure for simplices such that it is natural under inclusions of simplices and then define the hyperbolization of a simplicial complex by gluing the results for the simplices together as described by the combinatorics of the simplicial complex. The goal is to achieve that the result shares some of the properties of the simplicial complexes one has started with, but additionally to produce a non-positively curved and hence aspherical polyhedron. Since this construction preserves local structures, it turns manifolds into manifolds.

We briefly explain what the *orientable hyperbolization procedure* gives. Further expositions of this construction can be found in [19], [22], [24], [25]. We start with a finite-dimensional simplicial complex Σ and a assign to it a cubical cell complex $h(\Sigma)$ and a natural map $c\colon h(\Sigma) \to \Sigma$ with the following properties:

(i) $h(\Sigma)$ is non-positively curved and in particular aspherical.

(ii) The natural map $c\colon h(\Sigma) \to \Sigma$ induces a surjection on the integral homology.

(iii) $\pi_1(f)\colon \pi_1(h(\Sigma)) \to \pi_1(\Sigma)$ is surjective.

(iv) If Σ is an orientable manifold, then

   (a) $h(\Sigma)$ is a manifold;

   (b) the natural map $c\colon h(\Sigma) \to \Sigma$ has degree one;

   (c) there is a stable isomorphism between the tangent bundle $Th(\Sigma)$ and the pullback $c^*T\Sigma$;

Remark 2.1 (Characteristic numbers and aspherical manifolds). Suppose that M is a closed manifold. Then the pullback of the characteristic classes of M under the natural map $c\colon h(M) \to M$ yield the characteristic classes of $h(M)$, and M and $h(M)$ have the same characteristic numbers. This shows that the condition aspherical does not impose any restrictions on the characteristic numbers of a manifold.

Remark 2.2 (Bordism and aspherical manifolds). The conditions above say that c is a normal map in the sense of surgery. One can show that c is normally bordant to the identity map on M. In particular M and $h(M)$ are oriented bordant.

Consider a bordism theory Ω_* for PL-manifolds or smooth manifolds which is given by imposing conditions on the stable tangent bundle. Examples are unoriented bordism, oriented bordism, framed bordism. Then any bordism class can be represented by an aspherical closed manifold. If two closed aspherical manifolds represent the same bordism class, then one can find an aspherical bordism between them. See [22, Remarks 15.1] and [25, Theorem B].

2.5. Exotic aspherical manifolds. The following result is taken from Davis–Januszkiewicz [25, Theorem 5a.1].

Theorem 2.3. *There is a closed aspherical* 4*-manifold* N *with the following properties:*

(i) N *is not homotopy equivalent to a PL-manifold.*

(ii) N *is not triangulable, i.e., not homeomorphic to a simplicial complex.*

(iii) *The universal covering* $\widetilde{N}$ *is not homeomorphic to* $\mathbb{R}^4$.

(iv) N *is homotopy equivalent to a piecewise flat, non-positively curved polyhedron.*

The next result is due to Davis–Januszkiewicz [25, Theorem 5a.4].

Theorem 2.4 (Non-PL-example). *For every* $n \geq 4$ *there exists a closed aspherical* n*-manifold which is not homotopy equivalent to a PL-manifold*

The proof of the following theorem can be found in [23], [25, Theorem 5b.1].

Theorem 2.5 (Exotic universal covering). *For each* $n \geq 4$ *there exists a closed aspherical* n*-dimensional manifold such that its universal covering is not homeomorphic to* $\mathbb{R}^n$.

By the Hadamard–Cartan Theorem (see [45, 3.87 on page 134]) the manifold appearing in Theorem 2.5 above cannot be homeomorphic to a smooth manifold with Riemannian metric with non-positive sectional curvature.

The following theorem is proved in [25, Theorem 5c.1 and Remark on page 386] by considering the ideal boundary, which is a quasiisometry invariant in the negatively curved case.

Theorem 2.6 (Exotic example with hyperbolic fundamental group). *For every* $n \geq 5$ *there exists an aspherical closed smooth* n*-dimensional manifold* N *which is homeomorphic to a strictly negatively curved polyhedron and has in particular a hyperbolic fundamental group such that the universal covering is homeomorphic to* $\mathbb{R}^n$ *but* N *is not homeomorphic to a smooth manifold with Riemannian metric with negative sectional curvature.*

The next results are due to Belegradek [8, Corollary 5.1], Mess [71] and Weinberger (see [22, Section 13]).

Theorem 2.7 (Exotic fundamental groups).

(i) *For every* $n \geq 4$ *there is a closed aspherical manifold of dimension* n *whose fundamental group contains an infinite divisible abelian group.*

(ii) *For every* $n \geq 4$ *there is a closed aspherical manifold of dimension* n *whose fundamental group has an unsolvable word problem and whose simplicial volume is non-zero.*

Notice that a finitely presented group with unsolvable word problem is not a CAT(0)-group, not hyperbolic, not automatic, not asynchronously automatic, not residually finite and not linear over any commutative ring (see [8, Remark 5.2]).

The proof of Theorem 2.7 is based on the *reflection group trick* as it appears for instance in [22, Sections 8,10 and 13]. It can be summarized as follows.

Theorem 2.8 (Reflection group trick). *Let G be a group which possesses a finite model for BG. Then there is a closed aspherical manifold M and a map $i\colon BG \to M$ and $r\colon M \to BG$ such that $r \circ i = \mathrm{id}_{BG}$.*

Remark 2.9 (Reflection group trick and various conjectures). Another interesting immediate consequence of the reflection group trick is (see also [22, Sections 11]) that many well-known conjectures about groups hold for every group which possesses a finite model for BG if and only if it holds for the fundamental group of every closed aspherical manifold. This applies for instance to the Kaplansky Conjecture, Unit Conjecture, Zero-divisor Conjecture, Baum–Connes Conjecture, Farrell–Jones Conjecture for algebraic K-theory for regular R, Farrell–Jones Conjecture for algebraic L-theory, the vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ and of $\mathrm{Wh}(G) = 0$. For information about these conjectures and their links we refer for instance to [6], [68] and [70]. Further similar consequences of the reflection group trick can be found in Belegradek [8].

3. Non-aspherical closed manifolds

A closed manifold of dimension ≥ 1 with finite fundamental group is never aspherical. So prominent non-aspherical manifolds are spheres, lens spaces, real projective spaces and complex projective spaces.

Lemma 3.1. *The fundamental group of an aspherical finite-dimensional CW-complex X is torsionfree.*

Proof. Let $C \subseteq \pi_1(X)$ be a finite cyclic subgroup of $\pi_1(X)$. We have to show that C is trivial. Since X is aspherical, $C\backslash\widetilde{X}$ is a finite-dimensional model for BC. Hence $H_k(BC) = 0$ for large k. This implies that C is trivial. □

Lemma 3.2. *If M is a connected sum $M_1 \sharp M_2$ of two closed manifolds M_1 and M_2 of dimension $n \geq 3$ which are not homotopy equivalent to a sphere, then M is not aspherical.*

Proof. We proceed by contradiction. Suppose that M is aspherical. The obvious map $f\colon M_1 \sharp M_2 \to M_1 \vee M_2$ given by collapsing S^{n-1} to a point is $(n-1)$-connected, where n is the dimension of M_1 and M_2. Let $p\colon \widetilde{M_1 \vee M_2} \to M_1 \vee M_2$

be the universal covering. By the Seifert–van Kampen Theorem the fundamental group of $\pi_1(M_1 \vee M_2)$ is $\pi_1(M_1) * \pi_1(M_2)$ and the inclusion of $M_k \to M_1 \vee M_2$ induces injections on the fundamental groups for $k = 1, 2$. We conclude that $p^{-1}(M_k) = \pi_1(M_1 \vee M_2) \times_{\pi_1(M_k)} \widetilde{M_k}$ for $k = 1, 2$. Since $n \geq 3$, the map f induces an isomorphism on the fundamental groups and an $(n-1)$-connected map $\tilde{f} \colon \widetilde{M_1 \sharp M_2} \to \widetilde{M_1 \vee M_2}$. Since $\widetilde{M_1 \sharp M_2}$ is contractible, $H_m(\widetilde{M_1 \vee M_2}) = 0$ for $1 \leq m \leq n-1$. Since $p^{-1}(M_1) \cup p^{-1}(M_2) = \widetilde{M_1 \vee M_2}$ and $p^{-1}(M_1) \cap p^{-1}(M_2) = p^{-1}(\{\bullet\}) = \pi_1(M_1 \vee M_2)$, we conclude $H_m(p^{-1}(M_k)) = 0$ for $1 \leq m \leq n-1$ from the Mayer–Vietoris sequence. This implies $H_m(\widetilde{M_k}) = 0$ for $1 \leq m \leq n-1$ since $p^{-1}(M_k)$ is a disjoint union of copies of $\widetilde{M_k}$.

Suppose that $\pi_1(M_k)$ is finite. Since $\pi_1(M_1 \sharp M_2)$ is torsionfree by Lemma 3.1, $\pi_1(M_k)$ must be trivial and $M_k = \widetilde{M_k}$. Since M_k is simply connected and $H_m(M_k) = 0$ for $1 \leq m \leq n-1$, M_k is homotopy equivalent to S^n. Since we assume that M_k is not homotopy equivalent to a sphere, $\pi_1(M_k)$ is infinite. This implies that the manifold $\widetilde{M_k}$ is non-compact and hence $H_n(\widetilde{M_k}) = 0$. Since $\widetilde{M_k}$ is n-dimensional, we conclude $H_m(\widetilde{M_k}) = 0$ for $m \geq 1$. Since $\widetilde{M_k}$ is simply connected, all homotopy groups of $\widetilde{M_k}$ vanish by the Hurewicz Theorem [99, Corollary IV.7.8 on page 180]. We conclude from Lemma 1.2 that M_1 and M_2 are aspherical. Using the Mayer–Vietoris argument above one shows analogously that $M_1 \vee M_2$ is aspherical. Since M is by assumption aspherical, $M_1 \sharp M_2$ and $M_1 \vee M_2$ are homotopy equivalent by Lemma 1.1 (i). Since they have different Euler characteristics, namely $\chi(M_1 \sharp M_2) = \chi(M_1) + \chi(M_2) - (1 + (-1)^n)$ and $\chi(M_1 \vee M_2) = \chi(M_1) + \chi(M_2) - 1$, we get a contradiction. □

4. The Borel Conjecture

In this section we deal with

Conjecture 4.1 (Borel Conjecture for a group G). *If M and N are closed aspherical manifolds of dimensions ≥ 5 with $\pi_1(M) \cong \pi_1(N) \cong G$, then M and N are homeomorphic and any homotopy equivalence $M \to N$ is homotopic to a homeomorphism.*

Definition 4.2 (Topologically rigid). We call a closed manifold N *topologically rigid* if any homotopy equivalence $M \to N$ with a closed manifold M as source is homotopic to a homeomorphism.

If the Borel Conjecture holds for all finitely presented groups, then every closed aspherical manifold is topologically rigid.

The main tool to attack the Borel Conjecture is surgery theory and the Farrell–Jones Conjecture. We consider the following special version of the Farrell–Jones Conjecture.

Conjecture 4.3 (Farrell–Jones Conjecture for torsionfree groups and regular rings). *Let G be a torsionfree group and let R be a regular ring, e.g., a principal ideal domain, a field, or $\mathbb{Z}$. Then*

(i) $K_n(RG) = 0$ *for* $n \le -1$.

(ii) *The change of rings homomorphism $K_0(R) \to K_0(RG)$ is bijective.* (*This implies in the case $R = \mathbb{Z}$ that the reduced projective class group $\widetilde{K}_0(\mathbb{Z}G)$ vanishes*).

(iii) *The obvious map $K_1(R) \times G/[G,G] \to K_1(RG)$ is surjective.* (*This implies in the case $R = \mathbb{Z}$ that the Whitehead group* $\mathrm{Wh}(G)$ *vanishes*).

(iv) *For any orientation homomorphism $w\colon G \to \{\pm 1\}$ the w-twisted L-theoretic assembly map*
$$H_n(BG; {}^w\boldsymbol{L}^{\langle -\infty\rangle}) \xrightarrow{\cong} L_n^{\langle -\infty\rangle}(RG, w)$$
is bijective.

Lemma 4.4. *Suppose that the torsionfree group G satisfies the version of the Farrell–Jones Conjecture stated in Conjecture 4.3 for $R = \mathbb{Z}$.*

Then the Borel Conjecture is true for closed aspherical manifolds of dimension ≥ 5 with G as fundamental group. Its is true for closed aspherical manifolds of dimension 4 with G as fundamental group if G is good in the sense of Freedman (*see* [42], [43]).

Sketch of the proof. We treat the orientable case only. The *topological structure set* $\mathcal{S}^{\mathrm{top}}(M)$ of a closed topological manifold M is the set of equivalence classes of homotopy equivalences $M' \to M$ with a topological closed manifold as source and M as target under the equivalence relation, for which $f_0\colon M_0 \to M$ and $f_1\colon M_1 \to M$ are equivalent if there is a homeomorphism $g\colon M_0 \to M_1$ such that $f_1 \circ g$ and f_0 are homotopic. The Borel Conjecture 4.1 for a group G is equivalent to the statement that for every closed aspherical manifold M with $G \cong \pi_1(M)$ its topological structure set $\mathcal{S}^{\mathrm{top}}(M)$ consists of a single element, namely, the class of $\mathrm{id}\colon M \to M$.

The *surgery sequence* of a closed orientable topological manifold M of dimension $n \ge 5$ is the exact sequence
$$\cdots \longrightarrow \mathcal{N}_{n+1}\big(M \times [0,1], M \times \{0,1\}\big) \xrightarrow{\sigma} L_{n+1}^s\big(\mathbb{Z}\pi_1(M)\big)$$
$$\xrightarrow{\partial} \mathcal{S}^{\mathrm{top}}(M) \xrightarrow{\eta} \mathcal{N}_n(M) \xrightarrow{\sigma} L_n^s\big(\mathbb{Z}\pi_1(M)\big),$$
which extends infinitely to the left. It is the basic tool for the classification of topological manifolds. (There is also a smooth version of it.) The map σ appearing in the sequence sends a normal map of degree one to its surgery obstruction. This map can be identified with the version of the L-theory assembly map where one works with the 1-connected cover $\boldsymbol{L}^s(\mathbb{Z})\langle 1\rangle$ of $\boldsymbol{L}^s(\mathbb{Z})$. The map $H_k\big(M; \boldsymbol{L}^s(\mathbb{Z})\langle 1\rangle\big) \to H_k\big(M; \boldsymbol{L}^s(\mathbb{Z})\big)$

is injective for $k = n$ and an isomorphism for $k > n$. Because of the K-theoretic assumptions we can replace the s-decoration with the $\langle -\infty \rangle$-decoration. Therefore the Farrell–Jones Conjecture implies that the maps $\sigma \colon \mathcal{N}_n(M) \to L_n^s\big(\mathbb{Z}\pi_1(M)\big)$ and $\mathcal{N}_{n+1}\big(M \times [0,1], M \times \{0,1\}\big) \xrightarrow{\sigma} L_{n+1}^s\big(\mathbb{Z}\pi_1(M)\big)$ are injective respectively bijective and thus by the surgery sequence that $\mathcal{S}^{\mathrm{top}}(M)$ is a point and hence the Borel Conjecture 4.1 holds for M. More details can be found, e.g., in [39, pages 17, 18, 28], [87, Chapter 18]. □

Remark 4.5 (The Borel Conjecture in low dimensions). The Borel Conjecture is true in dimension ≤ 2 by the classification of closed manifolds of dimension 2. It is true in dimension 3 if Thurston's Geometrization Conjecture is true. This follows from results of Waldhausen (see Hempel [53, Lemma 10.1 and Corollary 13.7]) and Turaev (see [93]) as explained for instance in [65, Section 5]. A proof of Thurston's Geometrization Conjecture is given in [74] following ideas of Perelman.

Remark 4.6 (Topological rigidity for non-aspherical manifolds). Topological rigidity phenomenons do hold also for some non-aspherical closed manifolds. For instance the sphere S^n is topologically rigid by the Poincaré Conjecture. The Poincaré Conjecture is known to be true in all dimensions. This follows in high dimensions from the h-cobordism theorem, in dimension four from the work of Freedman [42], in dimension three from the work of Perelman as explained in [62], [73], and in dimension two from the classification of surfaces.

Many more examples of classes of manifolds which are topologically rigid are given and analyzed in Kreck–Lück [65]. For instance the connected sum of closed manifolds of dimension ≥ 5 which are topologically rigid and whose fundamental groups do not contain elements of order two, is again topologically rigid and the connected sum of two manifolds is in general not aspherical (see Lemma 3.2). The product $S^k \times S^n$ is topologically rigid if and only if k and n are odd. An integral homology sphere of dimension $n \geq 5$ is topologically rigid if and only if the inclusion $\mathbb{Z} \to \mathbb{Z}[\pi_1(M)]$ induces an isomorphism of simple L-groups $L_{n+1}^s(\mathbb{Z}) \to L_{n+1}^s\big(\mathbb{Z}[\pi_1(M)]\big)$.

Remark 4.7 (The Borel Conjecture does not hold in the smooth category). The Borel Conjecture 4.1 is false in the smooth category, i.e., if one replaces topological manifold by smooth manifold and homeomorphism by diffeomorphism. The torus T^n for $n \geq 5$ is an example (see [97, 15A]). Other counterexample involving negatively curved manifolds are constructed by Farrell–Jones [31, Theorem 0.1].

Remark 4.8 (The Borel Conjecture versus Mostow rigidity). The examples of Farrell–Jones [31, Theorem 0.1] give actually more. Namely, it yields for given $\epsilon > 0$ a closed Riemannian manifold M_0 whose sectional curvature lies in the interval $[1-\epsilon, -1+\epsilon]$

and a closed hyperbolic manifold M_1 such that M_0 and M_1 are homeomorphic but no diffeomorphic. The idea of the construction is essentially to take the connected sum of M_1 with exotic spheres. Notice that by definition M_0 were hyperbolic if we would take $\epsilon = 0$. Hence this example is remarkable in view of *Mostow rigidity*, which predicts for two closed hyperbolic manifolds N_0 and N_1 that they are isometrically diffeomorphic if and only if $\pi_1(N_0) \cong \pi_1(N_1)$ and any homotopy equivalence $N_0 \to N_1$ is homotopic to an isometric diffeomorphism.

One may view the Borel Conjecture as the topological version of Mostow rigidity. The conclusion in the Borel Conjecture is weaker, one gets only homeomorphisms and not isometric diffeomorphisms, but the assumption is also weaker, since there are many more aspherical closed topological manifolds than hyperbolic closed manifolds.

Remark 4.9 (The work of Farrell–Jones). Farrell–Jones have made deep contributions to the Borel Conjecture. They have proved it in dimension ≥ 5 for non-positively curved closed Riemannian manifolds, for compact complete affine flat manifolds and for closed aspherical manifolds whose fundamental group is isomorphic to the fundamental group of a complete non-positively curved Riemannian manifold which is A-regular (see [32], [33], [35], [36]).

The following result is due to Bartels and Lück [4].

Theorem 4.10. *Let $\mathcal{C}$ be the smallest class of groups satisfying the following conditions:*

- *Every hyperbolic group belongs to $\mathcal{C}$.*
- *Every group that acts properly, isometrically and cocompactly on a complete proper* $\mathrm{CAT}(0)$*-space belongs to $\mathcal{C}$.*
- *If G_1 and G_2 belong to $\mathcal{C}$, then both $G_1 * G_2$ and $G_1 \times G_2$ belong to $\mathcal{C}$.*
- *If H is a subgroup of G and $G \in \mathcal{C}$, then $H \in \mathcal{C}$.*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{C}$ for every $i \in I$. Then the directed colimit $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{C}$.*

Then every group G in $\mathcal{C}$ satisfies the version of the Farrell–Jones Conjecture stated in Conjecture 4.3.

Remark 4.11 (Exotic closed aspherical manifolds). Theorem 4.10 implies that the exotic aspherical manifolds mentioned in Section 2.5 satisfy the Borel Conjecture in dimension ≥ 5 since their universal coverings are $\mathrm{CAT}(0)$-spaces.

Remark 4.12 (Directed colimits of hyperbolic groups). There are also a variety of interesting groups such as *lacunary groups* in the sense of Olshanskii–Osin–Sapir [80] or *groups with expanders* as they appear in the counterexample to the

Baum–Connes Conjecture with coefficients due to Higson–Lafforgue–Skandalis [54] and which have been constructed by Arzhantseva–Delzant [2, Theorem 7.11 and Theorem 7.12]. Since these arise as colimits of directed systems of hyperbolic groups, they do satisfy the Farrell–Jones Conjecture and the Borel Conjecture in dimension ≥ 5 by Theorem 4.10.

The *Bost Conjecture* has also been proved for colimits of hyperbolic groups by Bartels–Echterhoff–Lück [3].

The original source for the (Fibered) Farrell–Jones Conjecture is the paper by Farrell–Jones [34, 1.6 on page 257 and 1.7 on page 262]. The C^*-analogue of the Farrell–Jones Conjecture is the Baum–Connes Conjecture whose formulation can be found in [7, Conjecture 3.15 on page 254]. For more information about the Baum–Connes Conjecture and the Farrell–Jones Conjecture and literature about them we refer for instance to the survey article [70].

5. Poincaré duality groups

The following definition is due to Johnson–Wall [59].

Definition 5.1 (Poincaré duality group). A group G is called a *Poincaré duality group of dimension n* if the following conditions holds:

(i) The group G is of type FP, i.e., the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ possesses a finite-dimensional projective $\mathbb{Z}G$-resolution by finitely generated projective $\mathbb{Z}G$-modules.

(ii) We get an isomorphism of abelian groups

$$H^i(G;\mathbb{Z}G) \cong \begin{cases} \{0\} & \text{for } i \neq n; \\ \mathbb{Z} & \text{for } i = n. \end{cases}$$

The next definition is due to Wall [96]. Recall that a CW-complex X is called *finitely dominated* if there exists a finite CW-complex Y and maps $i\colon X \to Y$ and $r\colon Y \to X$ with $r \circ i \simeq \mathrm{id}_X$.

Definition 5.2 (Poincaré complex). Let X be a finitely dominated connected CW-complex with fundamental group π.

It is called a *Poincaré complex of dimension n* if there exists an orientation homomorphism $w\colon \pi \to \{\pm 1\}$ and an element

$$[X] \in H_n^{\pi}(\widetilde{X};{}^w\mathbb{Z}) = H_n\big(C_*(\widetilde{X}) \otimes_{\mathbb{Z}\pi} {}^w\mathbb{Z}\big)$$

in the n-th π-equivariant homology of its universal covering $\widetilde{X}$ with coefficients in the $\mathbb{Z}G$-module ${}^w\mathbb{Z}$, such that the up to $\mathbb{Z}\pi$-chain homotopy equivalence unique

$\mathbb{Z}\pi$-chain map

$$- \cap [X]\colon C^{n-*}(\widetilde{X}) = \hom_{\mathbb{Z}\pi}\bigl(C_{n-*}(\widetilde{X}), \mathbb{Z}\pi\bigr) \to C_*(\widetilde{X})$$

is a $\mathbb{Z}\pi$-chain homotopy equivalence. Here ${}^w\mathbb{Z}$ is the $\mathbb{Z}G$-module, whose underlying abelian group is $\mathbb{Z}$ and on which $g \in \pi$ acts by multiplication with $w(g)$.

If in addition X is a finite CW-complex, we call X a *finite Poincaré duality complex of dimension n*.

A topological space X is called an *absolute neighborhood retract* or briefly ANR if for every normal space Z, every closed subset $Y \subseteq Z$ and every (continuous) map $f : Y \to X$ there exists an open neighborhood U of Y in Z together with an extension $F : U \to Z$ of f to U. A *compact n-dimensional homology* ANR-*manifold* X is a compact absolute neighborhood retract such that it has a countable basis for its topology, has finite topological dimension and for every $x \in X$ the abelian group $H_i(X, X - \{x\})$ is trivial for $i \neq n$ and infinite cyclic for $i = n$. A closed n-dimensional topological manifold is an example of a compact n-dimensional homology ANR-manifold (see [21, Corollary 1A in V.26 page 191]).

Theorem 5.3 (Homology ANR-manifolds and finite Poincaré complexes). *Let M be a closed topological manifold, or more generally, a compact homology* ANR-*manifold of dimension n. Then M is homotopy equivalent to a finite n-dimensional Poincaré complex.*

Proof. A closed topological manifold, and more generally a compact ANR, has the homotopy type of a finite CW-complex (see [61, Theorem 2.2]. [98]). The usual proof of Poincaré duality for closed manifolds carries over to homology manifolds. □

Theorem 5.4 (Poincaré duality groups). *Let G be a group and $n \geq 1$ be an integer.*

(i) *The following assertions are equivalent:*

  (a) *G is finitely presented and a Poincaré duality group of dimension n.*

  (b) *There exists an n-dimensional aspherical Poincaré complex with G as fundamental group.*

(ii) *Suppose that $\widetilde{K}_0(\mathbb{Z}G) = 0$. Then the following assertions are equivalent:*

  (a) *G is finitely presented and a Poincaré duality group of dimension n.*

  (b) *There exists a finite n-dimensional aspherical Poincaré complex with G as fundamental group.*

(iii) *A group G is a Poincaré duality group of dimension 1 if and only if $G \cong \mathbb{Z}$.*

(iv) *A group G is a Poincaré duality group of dimension 2 if and only if G is isomorphic to the fundamental group of a closed aspherical surface.*

Proof. (i) Every finitely dominated CW-complex has a finitely presented fundamental group since every finite CW-complex has a finitely presented group and a group which is a retract of a finitely presented group is again finitely presented [94, Lemma 1.3]. If there exists a CW-model for BG of dimension n, then the cohomological dimension of G satisfies $\mathrm{cd}(G) \leq n$ and the converse is true provided that $n \geq 3$ (see [14, Theorem 7.1 in Chapter VIII.7 on page 205], [29], [94], [95]). This implies that the implication (i)(b) $\Longrightarrow$ (i)(a) holds for all $n \geq 1$ and that the implication (i)(a) $\Longrightarrow$ (i)(b) holds for $n \geq 3$. For more details we refer to [59, Theorem 1]. The remaining part to show the implication (i)(a) $\Longrightarrow$ (i)(b) for $n = 1, 2$ follows from assertions (iii) and (iv).

(ii) This follows in dimension $n \geq 3$ from assertion (i) and Wall's results about the finiteness obstruction which decides whether a finitely dominated CW-complex is homotopy equivalent to a finite CW-complex and takes values in $\widetilde{K}_0(\mathbb{Z}\pi)$ (see [37], [72], [94], [95]). The implication (ii)(b) $\Longrightarrow$ (ii)(a) holds for all $n \geq 1$. The remaining part to show the implication (ii)(a) $\Longrightarrow$ (ii)(b) follows from assertions (iii) and (iv).

(iii) Since $S^1 = B\mathbb{Z}$ is a 1-dimensional closed manifold, $\mathbb{Z}$ is a finite Poincaré duality group of dimension 1 by Theorem 5.3. We conclude from the (easy) implication (i)(b) $\Longrightarrow$ (i)(a) appearing in assertion (i) that $\mathbb{Z}$ is a Poincaré duality group of dimension 1. Suppose that G is a Poincaré duality group of dimension 1. Since the cohomological dimension of G is 1, it has to be a free group (see [91], [92]). Since the homology group of a group of type FP is finitely generated, G is isomorphic to a finitely generated free group F_r of rank r. Since $H^1(BF_r) \cong \mathbb{Z}^r$ and $H_0(BF_r) \cong \mathbb{Z}$, Poincaré duality can only hold for $r = 1$, i.e., G is $\mathbb{Z}$.

(iv) This is proved in [27, Theorem 2]. See also [10], [11], [26], [28]. □

Conjecture 5.5 (Aspherical Poincaré complexes). *Every finite aspherical Poincaré complex is homotopy equivalent to a closed manifold.*

Conjecture 5.6 (Poincaré duality groups). *A finitely presented group is an n-dimensional Poincaré duality group if and only if it is the fundamental group of a closed aspherical n-dimensional topological manifold.*

Because of Theorem 5.3 and Theorem 5.4 (i), Conjecture 5.5 and Conjecture 5.6 are equivalent.

The *disjoint disk property* says that for any $\epsilon > 0$ and maps $f, g \colon D^2 \to M$ there are maps $f', g' \colon D^2 \to M$ so that the distance between f and f' and the distance between g and g' are bounded by ϵ and $f'(D^2) \cap g'(D^2) = \emptyset$.

Lemma 5.7. *Suppose that the torsionfree group G and the ring $R = \mathbb{Z}$ satisfy the version of the Farrell–Jones Conjecture stated in Theorem 4.3. Let X be a Poincaré complex of dimension ≥ 6 with $\pi_1(X) \cong G$. Then X is homotopy equivalent to a compact homology* ANR*-manifold satisfying the disjoint disk property.*

Proof. See [87, Remark 25.13 on page 297], [15, Main Theorem on page 439 and Section 8] and [16, Theorem A and Theorem B]. □

Remark 5.8 (Compact homology ANR-manifolds versus closed topological manifolds). In the following all manifolds have dimension ≥ 6. One would prefer if in the conclusion of Lemma 5.7 one could replace "compact homology ANR-manifold" by "closed topological manifold". The problem is that in the geometric exact surgery sequence one has to work with the 1-connective cover $\boldsymbol{L}\langle 1\rangle$ of the L-theory spectrum $\boldsymbol{L}$, whereas in the assembly map appearing in the Farrell–Jones setting one uses the L-theory spectrum $\boldsymbol{L}$. The L-theory spectrum $\boldsymbol{L}$ is 4-periodic, i.e., $\pi_n(\boldsymbol{L}) \cong \pi_{n+4}(\boldsymbol{L})$ for $n \in \mathbb{Z}$. The 1-connective cover $\boldsymbol{L}\langle 1\rangle$ comes with a map of spectra $\boldsymbol{f} \colon \boldsymbol{L}\langle 1\rangle \to \boldsymbol{L}$ such that $\pi_n(\boldsymbol{f})$ is an isomorphism for $n \geq 1$ and $\pi_n(\boldsymbol{L}\langle 1\rangle) = 0$ for $n \leq 0$. Since $\pi_0(\boldsymbol{L}) \cong \mathbb{Z}$, one misses a part involving $L_0(\mathbb{Z})$ of the so called *total surgery obstruction* due to Ranicki, i.e., the obstruction for a finite Poincaré complex to be homotopy equivalent to a closed topological manifold, if one deals with the periodic L-theory spectrum $\boldsymbol{L}$ and picks up only the obstruction for a finite Poincaré complex to be homotopy equivalent to a compact homology ANR-manifold, the so called *four-periodic total surgery obstruction*. The difference of these two obstructions is related to the *resolution obstruction* of Quinn which takes values in $L_0(\mathbb{Z})$. Any element of $L_0(\mathbb{Z})$ can be realized by an appropriate compact homology ANR-manifold as its *resolution obstruction*. There are compact homology ANR-manifolds that are not homotopy equivalent to closed manifolds. But no example of an aspherical compact homology ANR-manifold that is not homotopy equivalent to a closed topological manifold is known. For an aspherical compact homology ANR-manifold M, the total surgery obstruction and the resolution obstruction carry the same information. So we could replace in the conclusion of Lemma 5.7 "compact homology ANR-manifold" by "closed topological manifold" if and only if every aspherical compact homology ANR-manifold with the disjoint disk property admits a resolution.

We refer for instance to [15], [38], [85], [86], [87] for more information about this topic.

Question 5.9 (Vanishing of the resolution obstruction in the aspherical case). Is every aspherical compact homology ANR-manifold homotopy equivalent to a closed manifold?

6. Product decompositions

In this section we show that, roughly speaking, a closed aspherical manifold M is a product $M_1 \times M_2$ if and only if its fundamental group is a product $\pi_1(M) = G_1 \times G_2$ and that such a decomposition is unique up to homeomorphism.

Theorem 6.1 (Product decomposition). *Let M be a closed aspherical manifold of dimension n with fundamental group $G = \pi_1(M)$. Suppose we have a product decomposition*

$$p_1 \times p_2 \colon G \xrightarrow{\cong} G_1 \times G_2.$$

Suppose that G, G_1 and G_2 satisfy the version of the Farrell–Jones Conjecture stated in Theorem 4.3 in the case $R = \mathbb{Z}$.

Then G, G_1 and G_2 are Poincaré duality groups whose cohomological dimensions satisfy

$$n = \operatorname{cd}(G) = \operatorname{cd}(G_1) + \operatorname{cd}(G_2).$$

Suppose in the sequel that

- *the cohomological dimension $\operatorname{cd}(G_i)$ is different from 3, 4 and 5 for $i = 1, 2$,*
- *$n \geq 5$ or $n \leq 2$ or ($n = 4$ and G is good in the sense of Freedman).*

Then:

(i) *There are topological closed aspherical manifolds M_1 and M_2 together with isomorphisms*

$$v_i : \pi_1(M_i) \xrightarrow{\cong} G_i$$

and maps

$$f_i \colon M \to M_i$$

for $i = 1, 2$ such that

$$f = f_1 \times f_2 \colon M \to M_1 \times M_2$$

is a homeomorphism and $v_i \circ \pi_1(f_i) = p_i$ (up to inner automorphisms) for $i = 1, 2$.

(ii) *Suppose we have another such choice of topological closed aspherical manifolds M_1' and M_2' together with isomorphisms*

$$v_i' \colon \pi_1(M_i') \xrightarrow{\cong} G_i$$

and maps

$$f_i' \colon M \to M_i'$$

for $i = 1, 2$ such that the map $f' = f_1' \times f_2'$ is a homotopy equivalence and $v_i' \circ \pi_1(f_i') = p_i$ (up to inner automorphisms) for $i = 1, 2$. Then there are for $i = 1, 2$ homeomorphisms $h_i \colon M_i \to M_i'$ such that $h_i \circ f_i \simeq f_i'$ and $v_i \circ \pi_1(h_i) = v_i'$ holds for $i = 1, 2$.

Proof. In the sequel we identify $G = G_1 \times G_2$ by $p_1 \times p_2$. Since the closed manifold M is a model for BG and $\mathrm{cd}(G) = n$, we can choose BG to be an n-dimensional finite Poincaré complex in the sense of Definition 5.2 by Theorem 5.3.

From $BG = B(G_1 \times G_2) \simeq BG_1 \times BG_2$ we conclude that there are finitely dominated CW-models for BG_i for $i = 1, 2$. Since $\widetilde{K}_0(\mathbb{Z}G_i)$ vanishes for $i = 0, 1$ by assumption, we conclude from the theory of the finiteness obstruction due to Wall [94], [95] that there are finite models for BG_i of dimension $\max\{\mathrm{cd}(G_i), 3\}$. We conclude from [47], [84] that BG_1 and BG_2 are Poincaré complexes. One easily checks using the Künneth formula that

$$n = \mathrm{cd}(G) = \mathrm{cd}(G_1) + \mathrm{cd}(G_2).$$

If $\mathrm{cd}(G_i) = 1$, then BG_i is homotopy equivalent to a manifold, namely S^1, by Theorem 5.4 (iii). If $\mathrm{cd}(G_i) = 2$, then BG_i is homotopy equivalent to a manifold by Theorem 5.4 (iv). Hence it suffices to show for $i = 1, 2$ that BG_i is homotopy equivalent to a closed aspherical manifold, provided that $\mathrm{cd}(G_i) \geq 6$.

Since by assumption G_i satisfies the version of the Farrell–Jones Conjecture stated in Theorem 4.3 in the case $R = \mathbb{Z}$, there exists a compact homology ANR-manifold M_i that satisfies the disjoint disk property and is homotopy equivalent to BG_i (see Lemma 5.7). Hence it remains to show that Quinn's resolution obstruction $I(M_i) \in (1 + 8 \cdot \mathbb{Z})$ is 1 (see [86, Theorem 1.1]). Since this obstruction is multiplicative (see [86, Theorem 1.1]), we get $I(M_1 \times M_2) = I(M_1) \cdot I(M_2)$. In general the resolution obstruction is not a homotopy invariant, but it is known to be a homotopy invariant for aspherical compact ANR-manifolds if the fundamental group satisfies the Novikov Conjecture 7.2 (see [15, Proposition on page 437]). Since G_i satisfies the version of the Farrell–Jones Conjecture stated in Theorem 4.3 in the case $R = \mathbb{Z}$, it satisfies the Novikov Conjecture by Lemma 4.4 and Remark 7.4. Hence $I(M_1 \times M_2) = I(M)$. Since $I(M)$ is a closed manifold, we have $I(M) = 1$. Hence $I(M_i) = 1$ and M_i is homotopy equivalent to a closed manifold. This finishes the proof of assertion (i).

Assertion (ii) follows from Lemma 4.4. □

Remark 6.2 (Product decompositions and non-positive sectional curvature). The following result has been proved by Gromoll–Wolf [48, Theorem 2]. Let M be a closed Riemannian manifold with non-positive sectional curvature. Suppose that we are given a splitting of its fundamental group $\pi_1(M) = G_1 \times G_2$ and that the center of $\pi_1(M)$ is trivial. Then this splitting comes from an isometric product decomposition of closed Riemannian manifolds of non-positive sectional curvature $M = M_1 \times M_2$.

7. Novikov Conjecture

Let G be a group and let $u\colon M \to BG$ be a map from a closed oriented smooth manifold M to BG. Let

$$\mathcal{L}(M) \in \bigoplus_{k\in\mathbb{Z},k\geq 0} H^{4k}(M;\mathbb{Q})$$

be the *L-class of M*. Its k-th entry $\mathcal{L}(M)_k \in H^{4k}(M;\mathbb{Q})$ is a certain homogeneous polynomial of degree k in the rational Pontrjagin classes $p_i(M;\mathbb{Q}) \in H^{4i}(M;\mathbb{Q})$ for $i = 1, 2, \ldots, k$ such that the coefficient s_k of the monomial $p_k(M;\mathbb{Q})$ is different from zero. The L-class $\mathcal{L}(M)$ is determined by all the rational Pontrjagin classes and vice versa. The L-class depends on the tangent bundle and thus on the differentiable structure of M. For $x \in \prod_{k\geq 0} H^k(BG;\mathbb{Q})$ define the *higher signature of M associated to x and u* to be the integer

$$\operatorname{sign}_x(M,u) := \langle \mathcal{L}(M) \cup f^*x, [M]\rangle. \tag{7.1}$$

We say that sign_x for $x \in H^*(BG;\mathbb{Q})$ is *homotopy invariant* if for two closed oriented smooth manifolds M and N with reference maps $u\colon M \to BG$ and $v\colon N \to BG$ we have

$$\operatorname{sign}_x(M,u) = \operatorname{sign}_x(N,v),$$

whenever there is an orientation preserving homotopy equivalence $f : M \to N$ such that $v \circ f$ and u are homotopic. If $x = 1 \in H^0(BG)$, then the higher signature $\operatorname{sign}_x(M,u)$ is by the Hirzebruch signature formula (see [56], [57]) the signature of M itself and hence an invariant of the oriented homotopy type. This is one motivation for the following conjecture.

Conjecture 7.2 (Novikov Conjecture). *Let G be a group. Then sign_x is homotopy invariant for all $x \in \prod_{k\in\mathbb{Z},k\geq 0} H^k(BG;\mathbb{Q})$.*

This conjecture appears for the first time in the paper by Novikov [78, §11]. A survey about its history can be found in [39]. More information can be found for instance in [39], [40], [64].

We mention the following deep result due to Novikov [75], [76], [77].

Theorem 7.3 (Topological invariance of rational Pontrjagin classes). *The rational Pontrjagin classes $p_k(M,\mathbb{Q}) \in H^{4k}(M;\mathbb{Q})$ are topological invariants, i.e., for a homeomorphism $f : M \to N$ of closed smooth manifolds we have*

$$H_{4k}(f;\mathbb{Q})\big(p_k(M;\mathbb{Q})\big) = p_k(N;\mathbb{Q})$$

for all $k \geq 0$ and in particular $H_*(f;\mathbb{Q})(\mathcal{L}(M)) = \mathcal{L}(N)$.

The rational Pontrjagin classes are not homotopy invariants and the integral Pontrjagin classes $p_k(M)$ are not homeomorphism invariants (see for instance [64, Example 1.6 and Theorem 4.8]).

Remark 7.4 (The Novikov Conjecture and aspherical manifolds). Let $f: M \to N$ be a homotopy equivalence of closed aspherical manifolds. Suppose that the Borel Conjecture 4.1 is true for $G = \pi_1(N)$. This implies that f is homotopic to a homeomorphism and hence by Theorem 7.3

$$f_*(\mathcal{L}(M)) = \mathcal{L}(N).$$

But this is equivalent to the conclusion of the Novikov Conjecture in the case $N = BG$.

Conjecture 7.5. *A closed aspherical smooth manifold does not admit a Riemannian metric of positive scalar curvature.*

Proposition 7.6. *Suppose that the strong Novikov Conjecture is true for the group G, i.e., the assembly map*

$$K_n(BG) \to K_n(C_r^*(G))$$

is rationally injective for all $n \in \mathbb{Z}$. Let M be a closed aspherical smooth manifold whose fundamental group is isomorphic to G.

Then M carries no Riemannian metric of positive scalar curvature.

Proof. See [88, Theorem 3.5]. □

Proposition 7.7. *Let G be a group. Suppose that the assembly map*

$$K_n(BG) \to K_n(C_r^*(G))$$

is rationally injective for all $n \in \mathbb{Z}$. Let M be a closed aspherical smooth manifold whose fundamental group is isomorphic to G.

Then M satisfies the Zero-in-the-spectrum Conjecture 9.5.

Proof. See [67, Corollary 4]. □

We refer to [70, Section 5.1.3] for a discussion about the large class of groups for which the assembly map $K_n(BG) \to K_n(C_r^*(G))$ is known to be injective or rationally injective.

8. Boundaries of hyperbolic groups

We announce the following two theorems joint with Arthur Bartels and Shmuel Weinberger. For the notion of the boundary of a hyperbolic group and its main properties we refer for instance to [60].

Theorem 8.1. *Let G be a torsion-free hyperbolic group and let n be an integer ≥ 6.*

(i) *The following statements are equivalent:*

  (a) *The boundary ∂G is homeomorphic to S^{n-1}.*

  (b) *There is a closed aspherical topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n.*

(ii) *The aspherical manifold M appearing in the assertion above is unique up to homeomorphism.*

The proof depends strongly on the surgery theory for compact homology ANR-manifolds due to Bryant–Ferry–Mio–Weinberger [15] and the validity of the K- and L-theoretic Farrell–Jones Conjecture for hyperbolic groups due to Bartels–Reich–Lück [5] and Bartels–Lück [4]. It seems likely that this result holds also if $n = 5$. Our methods can be extended to this case if the surgery theory from [15] can be extended to the case of 5-dimensional compact homology ANR-manifolds.

We do not get information in dimensions $n \leq 4$ for the usual problems about surgery. For instance, our methods give no information in the case, where the boundary is homeomorphic to S^3, since virtually cyclic groups are the only hyperbolic groups which are known to be good in the sense of Friedman [43]. In the case $n = 3$ there is the conjecture of Cannon [17] that a group G acts properly, isometrically and cocompactly on the 3-dimensional hyperbolic plane $\mathbb{H}^3$ if and only if it is a hyperbolic group whose boundary is homeomorphic to S^2. Provided that the infinite hyperbolic group G occurs as the fundamental group of a closed irreducible 3-manifold, Bestvina–Mess [9, Theorem 4.1] have shown that its universal covering is homeomorphic to $\mathbb{R}^3$ and its compactification by ∂G is homeomorphic to D^3, and the Geometrization Conjecture of Thurston implies that M is hyperbolic and G satisfies Cannon's conjecture. The problem is solved in the case $n = 2$, namely, for a hyperbolic group G its boundary ∂G is homeomorphic to S^1 if and only if G is a Fuchsian group (see [18], [41], [44]).

For every $n \geq 5$ there exists a strictly negatively curved polyhedron of dimension n whose fundamental group G is hyperbolic, which is homeomorphic to a closed aspherical smooth manifold and whose universal covering is homeomorphic to $\mathbb{R}^n$, but the boundary ∂G is not homeomorphic to S^{n-1}, see [25, Theorem 5c.1 on page 384 and Remark on page 386]. Thus the condition that ∂G is a sphere for a torsion-free hyperbolic group is (in high dimensions) not equivalent to the existence of an aspherical manifold whose fundamental group is G.

Theorem 8.2. *Let G be a torsion-free hyperbolic group and let n be an integer ≥ 6.*

(i) *The following statements are equivalent:*

   (a) *The boundary ∂G has the integral Čech cohomology of S^{n-1}.*

   (b) *G is a Poincaré duality group of dimension n;*

   (c) *There exists a compact homology* ANR*-manifold M homotopy equivalent to BG. In particular, M is aspherical and $\pi_1(M) \cong G$.*

(ii) *If the statements in assertion (i) hold, then the compact homology* ANR*-manifold M appearing there is unique up to s-cobordism of compact* ANR*-homology manifolds.*

The discussion of compact homology ANR-manifolds versus closed topological manifolds of Remark 5.8 and Question 5.9 are relevant for Theorem 8.2 as well.

In general the boundary of a hyperbolic group is not locally a Euclidean space but has a fractal behavior. If the boundary ∂G of an infinite hyperbolic group G contains an open subset homeomorphic to Euclidean n-space, then it is homeomorphic to S^n. This is proved in [60, Theorem 4.4], where more information about the boundaries of hyperbolic groups can be found.

9. L^2-invariants

Next we mention some prominent conjectures about aspherical manifolds and L^2-invariants. For more information about these conjectures and their status we refer to [68] and [69].

9.1. The Hopf and the Singer Conjecture

Conjecture 9.1 (Hopf Conjecture). *If M is an aspherical closed manifold of even dimension, then*

$$(-1)^{\dim(M)/2} \cdot \chi(M) \geq 0.$$

If M is a closed Riemannian manifold of even dimension with sectional curvature $\sec(M)$*, then*

$$\begin{aligned}
(-1)^{\dim(M)/2} \cdot \chi(M) > 0 &\quad \textit{if } \sec(M) < 0;\\
(-1)^{\dim(M)/2} \cdot \chi(M) \geq 0 &\quad \textit{if } \sec(M) \leq 0;\\
\chi(M) = 0 &\quad \textit{if } \sec(M) = 0;
\end{aligned}$$

$$\chi(M) \geq 0 \quad \textit{if } \sec(M) \geq 0;$$
$$\chi(M) > 0 \quad \textit{if } \sec(M) > 0.$$

Conjecture 9.2 (Singer Conjecture). *If M is an aspherical closed manifold, then*

$$b_p^{(2)}(\widetilde{M}) = 0 \quad \textit{if } 2p \neq \dim(M).$$

If M is a closed connected Riemannian manifold with negative sectional curvature, then

$$b_p^{(2)}(\widetilde{M}) \begin{cases} = 0 & \textit{if } 2p \neq \dim(M); \\ > 0 & \textit{if } 2p = \dim(M). \end{cases}$$

9.2. L^2-torsion and aspherical manifolds

Conjecture 9.3 (L^2-torsion for aspherical manifolds). *If M is an aspherical closed manifold of odd dimension, then $\widetilde{M}$ is* det-*L^2-acyclic and*

$$(-1)^{\frac{\dim(M)-1}{2}} \cdot \rho^{(2)}(\widetilde{M}) \geq 0.$$

If M is a closed connected Riemannian manifold of odd dimension with negative sectional curvature, then $\widetilde{M}$ is det-*L^2-acyclic and*

$$(-1)^{\frac{\dim(M)-1}{2}} \cdot \rho^{(2)}(\widetilde{M}) > 0.$$

If M is an aspherical closed manifold whose fundamental group contains an amenable infinite normal subgroup, then $\widetilde{M}$ is det-*L^2-acyclic and*

$$\rho^{(2)}(\widetilde{M}) = 0.$$

9.3. Simplicial volume and L^2-invariants

Conjecture 9.4 (Simplicial volume and L^2-invariants). *Let M be an aspherical closed orientable manifold. Suppose that its simplicial volume $\|M\|$ vanishes. Then $\widetilde{M}$ is of determinant class and*

$$b_p^{(2)}(\widetilde{M}) = 0 \quad \textit{for } p \geq 0;$$
$$\rho^{(2)}(\widetilde{M}) = 0.$$

9.4. Zero-in-the-spectrum Conjecture

Conjecture 9.5 (Zero-in-the-spectrum Conjecture). *Let $\widetilde{M}$ be a complete Riemannian manifold. Suppose that $\widetilde{M}$ is the universal covering of an aspherical closed*

Riemannian manifold M (with the Riemannian metric coming from M). Then for some $p \geq 0$ zero is in the spectrum of the minimal closure

$$(\Delta_p)_{\min}\colon \operatorname{dom}\big((\Delta_p)_{\min}\big) \subset L^2\Omega^p(\widetilde{M}) \to L^2\Omega^p(\widetilde{M})$$

of the Laplacian acting on smooth p-forms on $\widetilde{M}$.

Remark 9.6 (Non-aspherical counterexamples to the Zero-in-the-spectrum Conjecture). For all of the conjectures about aspherical spaces stated in this article it is obvious that they cannot be true if one drops the condition aspherical except for the Zero-in-the-spectrum Conjecture 9.5. Farber and Weinberger [30] gave the first example of a closed Riemannian manifold for which zero is not in the spectrum of the minimal closure $(\Delta_p)_{\min}\colon \operatorname{dom}\big((\Delta_p)_{\min}\big) \subset L^2\Omega^p(\widetilde{M}) \to L^2\Omega^p(\widetilde{M})$ of the Laplacian acting on smooth p-forms on $\widetilde{M}$ for each $p \geq 0$. The construction by Higson, Roe and Schick [55] yields a plenty of such counterexamples. But there are no aspherical counterexamples known.

10. The universe of closed manifolds

At the end we describe (winking) our universe of closed manifolds.

The idea of a random group has successfully been used to construct groups with certain properties, see for instance [2], [46], [50, 9.B on pages 273ff], [51], [79], [82], [90] and [100]. In a precise statistical sense almost all finitely presented groups are hyperbolic see [81]. One can actually show that in a precise statistical sense almost all finitely presented groups are torsionfree hyperbolic and in particular have a finite model for their classifying space. In most cases it is given by the limit for $n \to \infty$ of the quotient of the number of finitely presented groups with a certain property (P) which are given by a presentation satisfying a certain condition C_n by the number of all finitely presented groups which are given by a presentation satisfying condition C_n.

It is not clear what it means in a precise sense to talk about a random closed manifold. Nevertheless, the author's intuition is that almost all closed manifolds are aspherical. (A related question would be whether a random closed smooth manifold admits a Riemannian metric with non-positive sectional curvature.) This intuition is supported by Remark 2.1. It is certainly true in dimension 2 since only finitely many closed surfaces are not aspherical. The characterization of closed 3-dimensional manifolds in Section 2.2 seems to fit as well. In the sequel we assume that this (vague) intuition is correct.

If we combine these considerations, we get that almost all closed manifolds are aspherical and have a hyperbolic fundamental group. Since except in dimension 4 the Borel Conjecture is known in this case by Lemma 4.4, Remark 4.5 and Theorem 4.10, we get as a consequence that almost all closed manifolds are aspherical and topologically rigid.

A closed manifold M is called *asymmetric* if every finite group which acts effectively on M is trivial. This is equivalent to the statement that for any choice of Riemannian metric on M the group of isometries is trivial (see [63, Introduction]). A survey on asymmetric closed manifolds can be found in [83]. The first constructions of asymmetric closed aspherical manifolds are due to Connor–Raymond–Weinberger [20]. The first simply-connected asymmetric manifold has been constructed by Kreck [63] answering a question of Raymond and Schultz [13, page 260] which was repeated by Adem and Davis [1] in their problem list. Raymond and Schultz expressed also their feeling that a random manifold should be asymmetric. Borel has shown that an aspherical closed manifold is asymmetric if its fundamental group is centerless and its outer automorphism group is torsionfree (see the manuscript "On periodic maps of certain $K(\pi, 1)$" in [12, pages 57–60]).

This leads to the intuitive statement:

Almost all closed manifolds are aspherical, topologically rigid and asymmetric.

In particular almost every closed manifold is determined up to homeomorphism by its fundamental group.

This is — at least on the first glance — surprising since often our favorite manifolds are not asymmetric and not determined by their fundamental group. There are prominent manifolds such as lens spaces which are homotopy equivalent but not homeomorphic. There seem to be plenty of simply connected manifolds. So why do human beings may have the feeling that the universe of closed manifolds described above is different from their expectation?

If one asks people for the most prominent closed manifold, most people name the standard sphere. It is interesting that the n-dimensional standard sphere S^n can be characterized among (simply connected) closed Riemannian manifolds of dimension n by the property that its isometry group has maximal dimension. More precisely, if M is a closed n-dimensional smooth manifold, then the dimension of its isometry group for any Riemannian metric is bounded by $n(n + 1)/2$ and the maximum $n(n + 1)/2$ is attained if and only if M is diffeomorphic to S^n or $\mathbb{RP}^n$; see Hsiang [58], where the Ph.D Thesis of Eisenhart is cited and the dimension of the isometry group of exotic spheres is investigated. It is likely that the human taste whether a geometric object is beautiful is closely related to the question how many symmetries it admits. In general it seems to be the case that a human being is attracted by unusual representatives among mathematical objects such as groups or closed manifolds and not by the generic ones. In group theory it is clear that random groups can have very strange properties and that these groups are to some extend scary. The analogous statement seems to hold for closed topological manifolds.

At the time of writing the author cannot really name a group which could be a potential counterexample to the Farrell–Jones Conjecture or other conjectures discussed in this article. But the author has the feeling that nevertheless the class of groups, for which we can prove the conjecture and which is for "human standards" quite large, is only a very tiny portion of the whole universe of groups and the question whether these conjectures are true for all groups is completely open.

Here is an interesting parallel to our actual universe. If you materialize at a random point in the universe it will be very cold and nothing will be there. There is no interaction between different random points, i.e., it is rigid. A human being will not like this place, actually even worse, it cannot exist at such a random place. But there are unusual rare non-generic points in the universe, where human beings can exist such as the surface of our planet and there a lot of things and interactions are happening. And human beings tend to think that the rest of the universe looks like the place they are living in and cannot really comprehend the rest of the universe.

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5ECM Amsterdam 2008

Wheeled props in algebra, geometry and quantization

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Abstract. Wheeled props is one the latest species found in the world of operads and props. We attempt to give an elementary introduction to the main ideas of the theory of wheeled props for beginners, and also a survey of its most recent major applications (ranging from algebra and geometry to deformation theory and Batalin–Vilkovisky quantization) which might be of interest to experts.

Mathematics Subject Classification (2000). 18D50, 46L65.

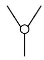
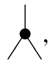
Keywords. Wheeled props, Poisson manifolds, Feynman diagrams.

1. Introduction

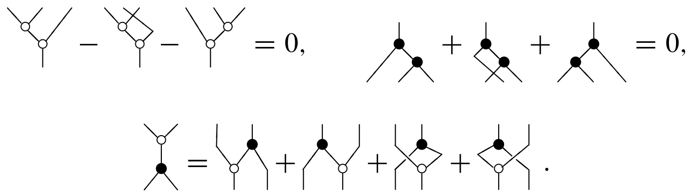
The theory of operads and props undergoes a rapid development in recent years; its applications can be seen nowadays almost everywhere – in algebraic topology, in homological algebra, in differential geometry, in non-commutative geometry, in string topology, in deformation theory, in quantization theory, etc. The theory demonstrates a remarkable unity of mathematics; for example, one and the same *operad of little* 2*-disks* solves the recognition problem for based 2-loop spaces in algebraic topology, describes homotopy Gerstenhaber structure on the Hochschild deformation complex in homological algebra, and also controls diffeomorphism invariant Hertling–Manin's integrability equations in differential geometry!

First examples of operads and props were constructed in the 1960s in the classical papers by Gerstenhaber on deformation theory of algebras and by Stasheff on homotopy theory of loop spaces. The notion of prop was introduced by MacLane already in 1963 as a useful way to code axioms for operations with many inputs and outputs. The notion of operad was ultimately coined 10 years later by P. May through axiomatization of properties of earlier discovered *associahedra* polytopes and the associated A_∞*-spaces* by Stasheff and of the *little cubes operad* by Boardman and Vogt.

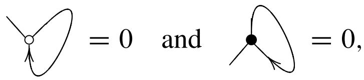
In this paper we attempt to explain the main ideas and constructions of the theory of wheeled operads and props and illustrate them with some of the most recent applications [Gr1], [Gr2], [Me1]–[Me7], [MMS], [MeVa], [Mn], [Str1], [Str2] to geometry, deformation theory and Batalin–Vilkovisky quantization formalism of theoretical physics. In the heart of these applications lies the fact that some categories of local geometric and theoretical physics structures can be identified with the derived categories of surprisingly simple algebraic structures. The language of graphs is essential for the proof of this fact and permits us to reformulate it as follows. Solution spaces of several important highly non-linear differential equations in geometry and physics are controlled by (wheeled) props which are resolutions of very compact graphical data, a kind of "genome". For example, the "genome" of the species *local Poisson structures* is the prop of Lie 1-bialgebras built from two "genes"

 and ,

subject to the following engineering rules (see §2 for precise details):



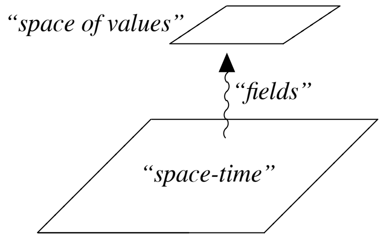
We shall explain how a slight modification of the above rules by addition of two extra conditions,



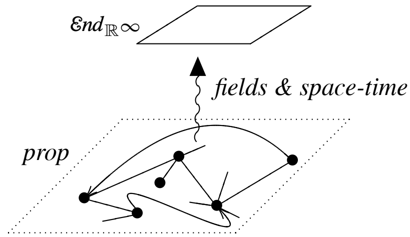
changes the resulting "species" dramatically: instead of the category of local Poisson structures one gets the category of *quantum BV manifolds with split quasi-classical limit* which, for example, naturally emerges in the study of quantum master equations [BaVi], [Sc] for BF-type quantum field theories (see §5 for precise details). Moreover, in the homotopy theory sense, this category is as perfect as, for example, the nowadays famous category of $\mathcal{L}ie_\infty$-algebras: quasi-isomorphisms of quantum BV manifolds turn out to be equivalence relations.

It is yet to see how non-trivial topology can be incorporated into the current pro(p)file of *local* differential geometry, but it is worth stressing already now that this approach to geometry and physics turns space-time – "the background of everything" – into an ordinary observable, a certain function (representation) on a prop, and hence unveils a possibility for a new architecture.

In fact, some elements of this architecture have been envisaged long ago by Roger Penrose [Pe] in his "abstract index calculus".



Classical architecture:
A space-time is the fundamental
background for geometric structures.



A new architecture of geometry and physics:
A prop is the fundamental background
for both a space-time and structures.

The paper is organized as follows. In Section 2 we give a short but self-contained introduction into the theory of (wheeled) operads and props. Sections 3 and 4 aim to give an account of most recent applications of that theory to geometry and, respectively, deformation theory. In Section 5 we explain some ideas of Koszul duality theory and its relation to the homotopy transfer formulae and Batalin–Vilkovisky formalism.

A few words about notation. The symbol $\mathbb{S}_n$ stands for the permutation group, i.e., for the group of all bijections, $[n] \to [n]$, where $[n]$ denotes (here and everywhere) the set $\{1, 2, \ldots, n\}$. Given a partition, $[n] = I_1 \sqcup \cdots \sqcup I_k$, the symbol $\sigma(I_1, \ldots, I_k)$ denotes the sign of the permutation $[n] \to \{I_1, \ldots, I_k\}$. If $V = \bigoplus_{i \in \mathbb{Z}} V^i$ is a graded vector space, then $V[k]$ is a graded vector space with $V[k]^i := V^{i+k}$. We work throughout over a field $\mathbb{K}$ of characteristic 0.

2. An introduction to operads, dioperads, properads and props

2.1. Directed graphs. Let m and n be arbitrary non-negative integers. A *directed (m, n)-graph* is a triple $(G, f_{\text{in}}, f_{\text{out}})$, where G is a finite 1-dimensional CW-complex whose 1-dimensional cells (“edges”) are oriented (“directed”), and

$$f_{\text{in}} \colon [m] \to \left\{ \begin{array}{c} \text{the set of all 0-cells, } v \text{, of } G \\ \text{which have precisely one} \\ \text{adjacent edge directed from } v \end{array} \right\},$$

$$f_{\text{out}} \colon [n] \to \left\{ \begin{array}{c} \text{the set of all 0-cells, } v \text{, of } G \\ \text{which have precisely one} \\ \text{adjacent edge directed towards } v \end{array} \right\}$$

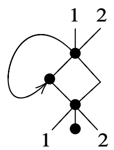
are injective maps of finite sets (called *labelling maps* or simply *labellings*) such that $\mathrm{Im}\, f_{\mathrm{in}} \cap \mathrm{Im}\, f_{\mathrm{out}} = \emptyset$. The set $\mathfrak{G}^{\circlearrowright}(m,n)$ of all possible directed (m,n)-graphs carries an action, $(G, f_{\mathrm{in}}, f_{\mathrm{out}}) \to (G, f_{\mathrm{in}} \circ \sigma^{-1}, f_{\mathrm{out}} \circ \tau)$, of the group $\mathbb{S}_m \times \mathbb{S}_n$ (more precisely, the *right* action of $\mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n$ we omit this detail from now). We often abbreviate a triple $(G, f_{\mathrm{in}}, f_{\mathrm{out}})$ to G. For any $G \in \mathfrak{G}^{\circlearrowright}(m,n)$ the set

$$V(G) := \{\text{all 0-cells of } G\} \setminus \{\mathrm{Im}\, f_{\mathrm{in}} \cup \mathrm{Im}\, f_{\mathrm{out}}\}$$

of all unlabelled 0-cells is called the set of *vertices* of G. The edges attached to labelled 0-cells, i.e., the ones lying in $\mathrm{Im}\, f_{\mathrm{in}}$ or in $\mathrm{Im}\, f_{\mathrm{out}}$ are called *incoming* or, respectively, *outgoing legs* of the graph G. The set

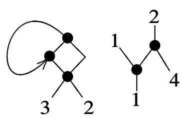
$$E(G) := \{\text{all 1-cells of } G\} \setminus \{\text{legs}\}$$

is called the set of (*internal*) *edges* of G. Legs and edges of G incident to a vertex $v \in V(G)$ are often called *half-edges* of v; the set of half-edges of v splits naturally into two disjoint sets, In_v and Out_v, consisting of incoming and, respectively, outgoing half-edges. In all our pictures the vertices of a graph will be denoted by bullets, the edges by intervals (or sometimes curves) connecting the vertices, and legs by intervals attached from one side to vertices. A choice of orientation on an edge or a leg will be visualized by the choice of a particular direction (arrow) on the associated interval/curve; unless otherwise explicitly shown the direction of each edge in all our pictures is assumed to go *from bottom to the top*. For example, the graph



$\in \mathfrak{G}^{\circlearrowright}(2,2)$

has four vertices, four legs and five edges; the orientation of all legs and of four internal edges is *not* shown explicitly and hence, by default, flows *upwards*. Sometimes we skip showing explicitly labellings of legs (as in Table 1, for example). We set $\mathfrak{G}^{\circlearrowright} := \sqcup_{m,n \geq 0} \mathfrak{G}^{\circlearrowright}(m,n)$. Note that elements of $\mathfrak{G}^{\circlearrowright}$ are not necessarily connected, e.g.,



$\in \mathfrak{G}^{\circlearrowright}(2,4).$

2.2. Decorated directed graphs. Let E be an $\mathbb{S}$-*bimodule*, that is, a family $\{E(p,q)\}_{p,q \geq 0}$ of vector spaces on which the group $\mathbb{S}_p$ acts on the left and the group $\mathbb{S}_q$ acts on the right, and both actions commute with each other. We shall use elements of E to decorate vertices of an arbitrary graph $G \in \mathfrak{G}^{\circlearrowright}$ as follows. First, for each vertex $v \in V(G)$ we construct a vector space

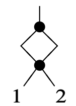
$$E(\mathrm{Out}_v, \mathrm{In}_v) := \langle \mathrm{Out}_v \rangle \otimes_{\mathbb{S}_p} E(p,q) \otimes_{\mathbb{S}_q} \langle \mathrm{In}_v \rangle,$$

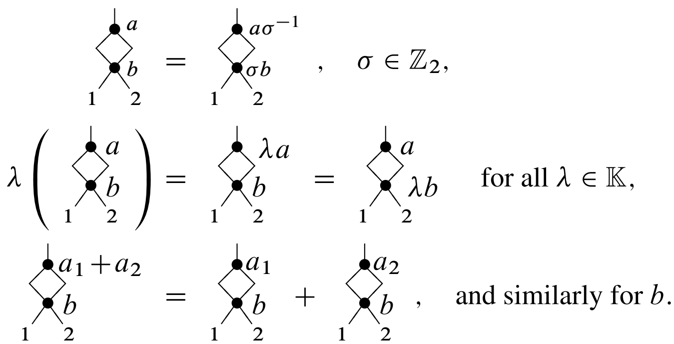
where $\langle \mathrm{Out}_v \rangle$ (resp., $\langle \mathrm{In}_v \rangle$) is the vector space spanned by all bijections $[\#\mathrm{Out}_v] \to \mathrm{Out}_v$ (resp., $\mathrm{In}_v \to [\#\mathrm{In}_v]$). It is (non-canonically) isomorphic to $E(p,q)$ as a vector space and carries natural actions of the automorphism groups of the sets Out_v and In_v. These actions make the following *unordered tensor product* over the set $V(G)$ (of cardinality, say, k),

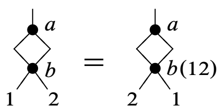
$$\bigotimes_{v \in V(G)} E(\mathrm{Out}_v, \mathrm{In}_v) := \Big(\bigoplus_{i:[k] \to V(G)} E(\mathrm{Out}_{i(1)}, \mathrm{In}_{i(1)}) \otimes \ldots \otimes E(\mathrm{Out}_{i(k)}, \mathrm{In}_{i(k)}) \Big)_{\mathbb{S}_k},$$

into a representation space of the automorphism group $\mathrm{Aut}(G)$ of the graph G which, by definition, is the subgroup of the symmetry group of the 1-dimensional CW-complex underlying the graph G which fixes its legs. Hence with an arbitrary graph $G \in \mathfrak{G}^{\circlearrowright}$ and an arbitrary $\mathbb{S}$-bimodule E one can associate a vector space

$$G\langle E \rangle := \big(\otimes_{v \in V(G)} E(\mathrm{Out}_v, \mathrm{In}_v) \big)_{\mathrm{Aut} G}$$

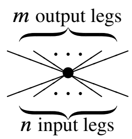
whose elements are called *decorated* (*by* E) *graphs*. For example, the automorphism group of the graph $G =$  is $\mathbb{Z}_2$ so that $G\langle E \rangle = E(1,2) \otimes_{\mathbb{Z}_2} E(2,2)$. It is useful to think of an element in $G\langle E \rangle$ as of the graph G whose vertices are literally decorated by some elements $a \in E(1,2)$ and $b \in E(2,1)$ and subject to the following relations:



It also follows from the definition that  , $(12) \in \mathbb{Z}_2$.

2.2.1. Remark. If $E = \{E(p,q)\}$ is a *differential graded* (dg, for short) $\mathbb{S}$-bimodule, i.e., if each vector space $E(p,q)$ is a complex equipped with an $\mathbb{S}_p \times \mathbb{S}_q$-equivariant differential δ, then, for any graph $G \in \mathfrak{G}^{\circlearrowright}(m,n)$, the associated graded vector space $G\langle E\rangle$ comes equipped with an induced $\mathbb{S}_m \times \mathbb{S}_n$-equivariant differential δ_G so that the collection, $\{\bigoplus_{G\in\mathfrak{G}^{\circlearrowright}(m,n)} G\langle E\rangle\}_{m,n\geq 0}$, is again a *dg* $\mathbb{S}$-bimodule. We sometimes abbreviate δ_G with δ.

2.2.2. Remark. The one vertex graph

$$\mathfrak{C}_{m,n} :=$$


$$\in \mathfrak{G}^{\circlearrowright}(m,n)$$

is often called *the* (m,n)*-corolla*. It is clear that for any $\mathbb{S}$-bimodule E one has $G\langle E\rangle = E(m,n)$.

2.3. Wheeled props. A *wheeled prop* is an $\mathbb{S}$-bimodule $\mathcal{P} = \{\mathcal{P}(m,n)\}$ together with a family of linear $\mathbb{S}_m \times \mathbb{S}_n$-equivariant maps,

$$\{\mu_G : G\langle\mathcal{P}\rangle \to \mathcal{P}(m,n)\}_{G\in\mathfrak{G}^{\circlearrowright}(m,n)}, \quad m,n \geq 0,$$

parameterized by elements $G \in \mathfrak{G}^{\circlearrowright}$, which satisfy a "three-dimensional" associativity condition,

$$\mu_G = \mu_{G/H} \circ \mu'_H, \tag{1}$$

for any subgraph $H \subset G$. Here G/H is the graph obtained from G by shrinking the whole subgraph H into a single internal vertex, and $\mu'_H : G\langle E\rangle \to (G/H)\langle E\rangle$ stands for the map which equals μ_H on the decorated vertices lying in H and which is identity on all other vertices of G.

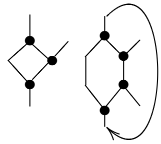
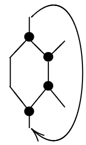
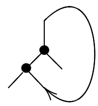
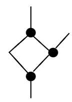
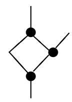
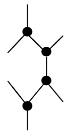
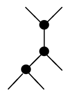
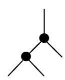
If the $\mathbb{S}$-bimodule $\mathcal{P}$ underlying a wheeled prop has a differential δ satisfying, for any $G \in \mathfrak{G}^{\circlearrowright}$, the condition $\delta \circ \mu_G = \mu_G \circ \delta_G$, then the wheeled prop $\mathcal{P}$ is called *differential*.

By Remark 2.2.2, the values of the maps μ_G can be identified with decorated corollas, and hence the maps themselves can be visually understood as *contraction* maps, $\mu_{G\in\mathfrak{G}^{\circlearrowright}(m,n)} : G\langle\mathcal{P}\rangle \to \mathfrak{C}_{m,n}\langle\mathcal{P}\rangle$, contracting all the edges and vertices of G into a single vertex.

2.3.1. Remark. Strictly speaking, the notion introduced in § 2.3 should be called a wheeled prop *without unit*. A wheeled prop *with unit* can be defined as in §2.1.1 provided one enlarges $\mathfrak{G}^{\circlearrowright}$ by adding a family of graphs, $\{\uparrow\uparrow\cdots\uparrow\circlearrowright\circlearrowright\cdots\circlearrowright\}$, *without vertices* [MMS].

2.4. Props, properads, operads, etc. as $\mathfrak{G}$-algebras. Let $\mathfrak{G} = \sqcup_{m,n}\mathfrak{G}(m,n)$ be a subset of the set $\mathfrak{G}^{\circlearrowright}$, say, one of the subsets defined in Table 1 below. A subgraph H of a graph $G \in \mathfrak{G}$ is called *admissible* if $H \in \mathfrak{G}$ and $G/H \in \mathfrak{G}$, i.e., a contraction of a graph from $\mathfrak{G}$ by a subgraph belonging to $\mathfrak{G}$ gives a graph which again belongs to $\mathfrak{G}$.

Table 1. A list of $\mathfrak{G}$-algebras.

$\mathfrak{G}$	**Definition**	**$\mathfrak{G}$-algebra**	**Typical examples**
$\mathfrak{G}^{\circlearrowright}$	All possible directed graphs	Wheeled prop	
$\mathfrak{G}_c^{\circlearrowright}$	A subset $\mathfrak{G}_c^{\circlearrowright} \subset \mathfrak{G}^{\circlearrowright}$ consisting of all *connected* graphs	Wheeled properad	
$\mathfrak{G}_{\mathrm{oper}}^{\circlearrowright}$	A subset $\mathfrak{G}_{\mathrm{oper}}^{\circlearrowright} \subset \mathfrak{G}_c^{\circlearrowright}$ consisting of graphs whose vertices have at most one output leg	Wheeled operad	
$\mathfrak{G}^{\uparrow}$	A subset $\mathfrak{G}^{\uparrow} \subset \mathfrak{G}^{\circlearrowright}$ consisting of graphs with no *wheels*, i.e., with no directed closed paths of edges	Prop	
$\mathfrak{G}_c^{\uparrow}$	A subset $\mathfrak{G}_c^{\uparrow} \subset \mathfrak{G}^{\uparrow}$ consisting of all *connected* graphs	Properad	
$\mathfrak{G}_{c,0}^{\uparrow}$	A subset $\mathfrak{G}_{c,0}^{\uparrow} \subset \mathfrak{G}_c^{\uparrow}$ consisting of graphs of genus zero	Dioperad	
$\mathfrak{G}^{\frac{1}{2}}$	A subset $\mathfrak{G}^{\frac{1}{2}} \subset \mathfrak{G}_{c,0}^{\uparrow}$ consisting of all (m,n)-graphs with the number of directed paths from input legs to the output legs equal to mn	$\frac{1}{2}$-Prop	
$\mathfrak{G}^{\curlywedge}$	A subset $\mathfrak{G}^{\curlywedge} \subset \mathfrak{G}_{c,0}^{\uparrow}$ consisting of graphs whose vertices have precisely one output leg	Operad	
$\mathfrak{G}^{\mid}$	A subset $\mathfrak{G}^{\mid} \subset \mathfrak{G}^{\curlywedge}$ consisting of graphs whose vertices have precisely one input leg	Associative algebra	

A $\mathfrak{G}$-*algebra* is, by definition, an $\mathbb{S}$-bimodule $\mathcal{P} = \{\mathcal{P}(m,n)\}$ together with a family of linear $\mathbb{S}_m \times \mathbb{S}_n$-equivariant maps, $\{\mu_G : G\langle\mathcal{P}\rangle \to \mathcal{P}(m,n)\}_{G\in\mathfrak{G}^{\circlearrowright}(m,n)}$, parameterized by elements $G \in \mathfrak{G}$, which satisfy condition (1) for any admissible subgraph $H \subset G$ (cf. § 2.3). Applying this idea to the subfamilies $\mathfrak{G} \subset \mathfrak{G}^{\circlearrowright}$ from Table 1 gives us, in the chronological order, the notions of *prop, operad, dioperad, properad,* $\frac{1}{2}$*-prop* and their *wheeled* versions which have been introduced, respectively, in the papers [Mc], [May], [Ga], [Va1], [Ko1], [Me5], [MMS].

We leave it as an exercise to the reader to check that $\mathfrak{G}^{|}$-algebra structures on an $\mathbb{S}$-bimodule E with only $E(1,1)$ non-zero are precisely associative algebra structures on $E(1,1)$. This fact implies that, for any $\mathfrak{G}$-algebra $E = \{E(m,n)\}_{m,n\geq 0}$, the space $E(1,1)$ is an associative algebra.

2.5. Basic examples of $\mathfrak{G}$-algebras. (i) For any $\mathfrak{G}$ and any finite-dimensional vector space V the $\mathbb{S}$-bimodule $\mathcal{E}nd_V = \{\mathrm{Hom}(V^{\otimes n}, V^{\otimes m})\}$ is naturally a $\mathfrak{G}$-algebra with contraction maps $\mu_{G\in\mathfrak{G}}$ being ordinary compositions and, possibly, traces of linear maps; it is called the *endomorphism* $\mathfrak{G}$*-algebra of* V. In the cases $\mathfrak{G} \neq \mathfrak{G}^{\circlearrowright}, \mathfrak{G}_c^{\circlearrowright}$ the assumption of finite-dimensionality of V can be dropped (as the defining operations μ_G do not employ traces).

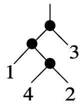
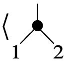
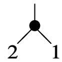
(ii) With any $\mathbb{S}$-bimodule, $E = \{E(m,n)\}$, there is associated another $\mathbb{S}$-bimodule, $\mathcal{F}^{\mathfrak{G}}\langle E\rangle = \{\mathcal{F}^{\mathfrak{G}}\langle E\rangle(m,n)\}$ with $\mathcal{F}^{\mathfrak{G}}\langle E\rangle(m,n) := \bigoplus_{G\in\mathfrak{G}(m,n)} G\langle E\rangle$, which has a natural $\mathfrak{G}$-algebra structure with the contraction maps μ_G being tautological. The $\mathfrak{G}$-algebra $\mathcal{F}^{\mathfrak{G}}\langle E\rangle$ is called *the free* $\mathfrak{G}$*-algebra generated by the* $\mathbb{S}$*-bimodule* E. We often abbreviate notations by replacing $\mathcal{F}^{\mathfrak{G}^{\circlearrowright}}$ by $\mathcal{F}^{\circlearrowright}$, $\mathcal{F}^{\mathfrak{G}^{\curlywedge}}$ by $\mathcal{F}^{\curlywedge}$, etc.

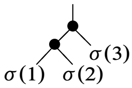
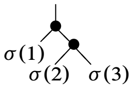
(iii) Definitions of $\mathfrak{G}$-*sub*algebras, $\mathcal{Q} \subset \mathcal{P}$, of $\mathfrak{G}$-algebras, of their ideals, $\mathcal{I} \subset \mathcal{P}$, and the associated quotient $\mathfrak{G}$-algebras, $\mathcal{P}/\mathcal{I}$, are straightforward. We omit the details.

2.6. Morphisms and resolutions of $\mathfrak{G}$-algebras. A morphisms of $\mathfrak{G}$-algebras, $\rho\colon \mathcal{P}_1 \to \mathcal{P}_2$, is a morphism of the underlying $\mathbb{S}$-bimodules such that, for any graph G, one has $\rho \circ \mu_G = \mu_G \circ (\rho^{\otimes G})$, where $\rho^{\otimes G}$ is a map, $G\langle\mathcal{P}_1\rangle \to G\langle\mathcal{P}_2\rangle$, which changes decorations of each vertex in G in accordance with ρ. A morphism $\mathcal{P} \to \mathcal{E}nd\langle V\rangle$ of $\mathfrak{G}$-algebras is called a *representation* of the $\mathfrak{G}$-algebra $\mathcal{P}$ in a graded vector space V.

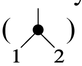
A *free resolution* of a dg $\mathfrak{G}$-algebra $\mathcal{P}$ is, by definition, a dg free $\mathfrak{G}$-algebra, $(\mathcal{F}^{\mathfrak{G}}\langle E\rangle, \delta)$, together with a morphism, $\pi : (\mathcal{F}\langle E\rangle, \delta) \to \mathcal{P}$, which induces a cohomology isomorphism. If the differential δ in $\mathcal{F}\langle\mathcal{E}\rangle$ is decomposable with respect to compositions μ_G, then it is called a *minimal model* of $\mathcal{P}$ and is often denoted by $\mathcal{P}_\infty$.

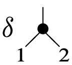
3. Applications to algebra and geometry

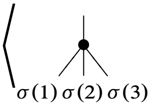
3.1. The operad of associative algebras. Let $A_0 = \{A_0(m,n)\}$ be an $\mathbb{S}$-bimodule with all $A_0(m,n) = 0$ except $A_0(1,2) := \mathbb{K}[\mathbb{S}_2]$. The associated free operad $\mathcal{F}^{\curlywedge}\langle A_0\rangle$ can be identified with the vector space spanned by all connected planar graphs of the form  . In particular, $\mathcal{F}^{\curlywedge}\langle A_0\rangle(1,2) \cong A_0(1,2) \cong \mathrm{span}\langle$  ,  $\rangle$. Let I_0 be an ideal of $\mathcal{F}^{\curlywedge}\langle A_0\rangle$ generated by the following 6 planar graphs:

 $-$  $\in \mathcal{F}^{\curlywedge}\langle A_0\rangle(1,3)$ for all $\sigma \in \mathbb{S}_3$. (2)

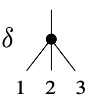
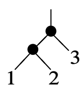
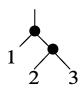
3.1.1. Claim. *There is a* 1-1 *correspondence between representations* $\rho\colon \mathcal{A}ss \to \mathcal{E}nd_V$ *of the quotient operad* $\mathcal{A}ss := \mathcal{F}^{\curlywedge}\langle A_0\rangle/\langle I_0\rangle$ *in a space* V *and associative algebra structures on* V.

Proof. The values of ρ on arbitrary (equivalence classes of) planar graphs is uniquely determined by its value, $\rho($  $) \in \mathrm{Hom}(V^{\otimes 2}, V)$, on one of the two generators. Denote this value by μ. As ρ sends any of the graphs (2) to zero, the multiplication in V given by μ must be associative. $\square$

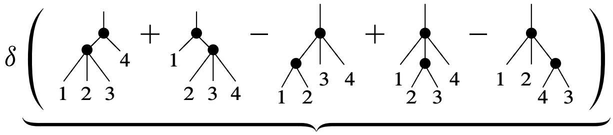
Thus the operad $\mathcal{A}ss$ can be called the *operad of associative algebras*. What could be a (minimal) free resolution of $\mathcal{A}ss$? By the definition in § 2.6, this must be a *free* operad, $\mathcal{F}^{\curlywedge}\langle A\rangle$, generated by some $\mathbb{S}$-bimodule $A = \{A(1,n)\}_{n\geq 2}$ equipped with a differential δ and a projection $\pi\colon \mathcal{F}^{\curlywedge}\langle A\rangle \to \mathcal{A}ss$ inducing an isomorphism, $H(\mathcal{F}^{\curlywedge}\langle A\rangle, \delta) = \mathcal{A}ss$, at the cohomology level. The latter condition suggests that we can choose $A(1,2)$ to be identical to $A_0(1,2)$ and set a differential δ to satisfy δ  $= 0$. Then the graphs (2) are cocycles in $\mathcal{F}^{\curlywedge}\langle A\rangle(1,3)$. In view of the cohomology isomorphism $\mathcal{F}^{\curlywedge}\langle A\rangle \to \mathcal{A}ss$, we have to make them coboundaries, and hence are forced to introduce an $\mathbb{S}_3$-module,

$A(1,3) := \mathbb{K}[\mathbb{S}_3][1] = \mathrm{span}\Big\langle$  $\Big\rangle_{\sigma\in\mathbb{S}_3}$,

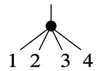
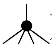
and set

δ  $=$  $-$  . (3)

We get in this way a well-defined dg *free* operad together with a well-defined epimorphism, $\left(\mathcal{F}^{\curlywedge}(\right.$  ,  $\left.) \,,\ \delta\right) \to (\mathcal{A}ss, 0)$, sending $(1,3)$-corollas to zero. However, this epimorphisms fails to be a quasi-isomorphism as

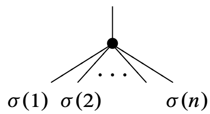
 $= 0.$

a nontrivial cohomology class in $H^{-1}(\mathcal{F}^{\curlywedge}($  ,  $)\ ,\delta)$

To kill this cohomology class we have to introduce a new generating $(1,4)$-corolla,  , of degree -2 and set the value of the differential on it to be equal to the underbraced expression above. Again we get a well-defined dg free operad together with a natural homomorphism, $\left(\mathcal{F}\right.($  ,  ,  $\left.), \delta\right) \to \mathcal{A}ss$, which, again, fails to be a quasi-isomorphism. To treat the new problem one has to introduce a new generating corolla of degree -3 with 5 input legs and so on.

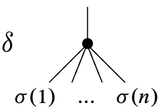
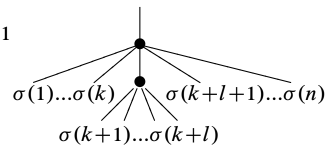
3.1.2. Theorem ([Sta]). *The minimal resolution of* $\mathcal{A}ss$ *is a dg free operad,* $\mathcal{A}ss_\infty := (\mathcal{F}^{\curlywedge}\langle A\rangle, \delta)$*, generated by the* $\mathbb{S}$*-bimodule* $A = \{A(1,n)\}$*,*

$$A(1,n) := \mathbb{K}[\mathbb{S}_n][n-2] = \mathrm{span}\left\langle \quad \right\rangle_{\sigma\in\mathbb{S}_n}, \qquad (4)$$



and with the differential given on the generators by

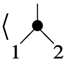
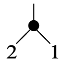
$$\delta \quad = \sum_{k=0}^{n-2}\sum_{l=2}^{n-k}(-1)^{k+l(n-k-l)+1} \quad . \qquad (5)$$

3.1.3. Definition. Representations, $\mathcal{A}ss_\infty \to \mathcal{E}nd_V$, of the dg operad $(\mathcal{A}ss_\infty, \delta)$ in a dg vector space V are called A_∞*-structures* in V.

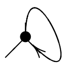
3.1.4. Remark. We now suggest the reader to re-read Stasheff's Theorem 3.1.2 from the end to the beginning: *given an infinite dimensional graph complex,* $(\mathcal{A}ss_\infty, \delta)$*, spanned by all possible planar graphs (without wheels) built from* $(1,n)$*-corollas with* $n \geq 2$ *and equipped with differential* (5)*, then its cohomology,* $H(\mathcal{A}ss_\infty, \delta)$*, is generated by only* $(1,2)$*-corollas*, i.e., it is surprisingly small. It is often impossible to obtain such a result by a direct computation. One of the main theorem-proving technique in the theory of operads and props is called the *Koszul duality theory*, and a result of type 3.1.2 often requires a combination of ideas from homological algebra, algebraic topology, the theory of Cohen–Macaulay posets [Va2] and so on. Stasheff [Sta] proved Theorem 3.1.2 by constructing a remarkable family of polytopes called nowadays *associahedra*; in his approach the surprising smallness of $H(\mathcal{A}ss_\infty, \delta)$ gets nicely explained by the obvious contractibility of Stasheff's polytopes as topological spaces. We shall review some theorem-proving techniques in Section 5 and continue this section with a list of examples which are most relevant to differential geometry.

3.2. The wheeled operad of finite-dimensional associative algebras. Theorem 3.1.2 has been obtained in the category of algebras over the family of graphs, $\mathfrak{G}^{\curlywedge}$, which contain *no* closed directed paths of internal edges. What happens if we keep the *same* family of generators as in the case of $\mathcal{A}ss$,

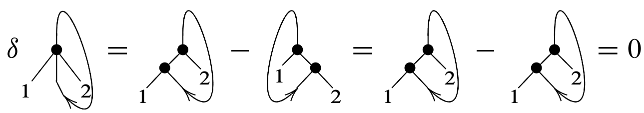
$A_0(m,n) = \begin{cases} \mathbb{K}[\mathbb{S}_2] = \operatorname{span}\langle$  ,  $\rangle$ for $m=1,\ n=2$,
0 otherwise,

the *same* family of relations (1), but enlarge the family of graphs we work over from $\mathfrak{G}^{\curlywedge}$ to $\mathfrak{G}^{\circlearrowright}$? The associated quotient wheeled operad, $\mathcal{A}ss^{\circlearrowright} := \mathcal{F}^{\circlearrowright}\langle A_0\rangle/\langle I_0\rangle$, can be called the *operad of finite-dimensional associative algebras*. Indeed, one has the following

3.2.1. Claim. *There is a one-to-one correspondence between representations* $\rho\colon \mathcal{A}ss^{\circlearrowright} \to \mathcal{E}nd_V$ *of* $\mathcal{A}ss^{\circlearrowright}$ *in a finite-dimensional vector space* V *and associative algebra structures on* V.

Proof. We need to explain only the subjective *finite-dimensional*, and that follows from the fact that representations of graphs $\in \mathcal{A}ss^{\circlearrowright}$ which have wheels involve traces. For example, the element  $\in \mathcal{A}ss^{\circlearrowright}(0,1)$ gets represented in V as the image of the multiplication map $\rho($$) \in \operatorname{Hom}(V\otimes V, V)$ under a natural trace map $\operatorname{Hom}(V\otimes V, V) \to \operatorname{Hom}(V,\mathbb{K})$. $\square$

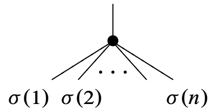
It is easy to see that the straightforward analogue of Theorem 3.1.2 can *not* hold true for the operad of *finite-dimensional* associative algebras as, for example, formula (3) implies

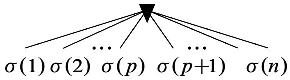


and hence provides us with a non-trivial cohomology class in $H^{-1}(\mathcal{F}^{\circlearrowright}\langle A\rangle,\delta)$ which maps under the natural projection $\mathcal{F}^{\circlearrowright}\langle A\rangle \to \mathcal{A}ss^{\circlearrowright}$ to zero. The correct analogue of Stasheff's result for finite-dimensional associative algebras was found in [MMS].

3.2.2. Theorem. *The minimal resolution of $\mathcal{A}ss^{\circlearrowright}$ is a dg free wheeled operad, $(\mathcal{A}ss^{\circlearrowright})_\infty := (\mathcal{F}^{\circlearrowright}\langle \hat{A}\rangle,\delta)$ generated by an $\mathbb{S}$-bimodule $\hat{A} = \{\hat{A}(m,n)\}$,*

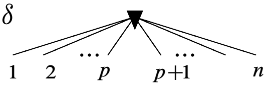
$$\hat{A}(m,n) := \begin{cases} \mathbb{K}[\mathbb{S}_n][n-2] = \mathrm{span}\left\langle \text{ [tree 1] } \right\rangle_{\sigma\in\mathbb{S}_n} \\ \qquad \textit{for } m=1, n\geq 2, \\ \bigoplus_{p=1}^{n-1}\mathbb{K}[\mathbb{S}_n]_{C_p\times C_{n-p}}[n] = \mathrm{span}\left\langle \text{ [tree 2] } \right\rangle_{\sigma\in\mathbb{S}_n} \\ \qquad \textit{for } m=0, n\geq 2, \\ 0 \qquad \textit{otherwise,} \end{cases}$$

[tree 1]: 

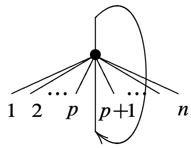
[tree 2]: 

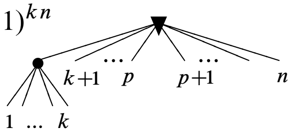
where $C_p\times C_{n-p}$ is the subgroup of $\mathbb{S}_n$ generated by two commuting cyclic permutations $\zeta := (12\dots p)$ and $\xi := (p+1\dots n)$, and $k[\mathbb{S}_n]_{C_p\times C_{n-p}}$ stands for coinvariants.

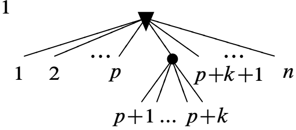
The differential is given on the generators of $\hat{A}(1,n)$ by (5) and on the generators of $\hat{A}(0,n)$ by

δ 

$$= \oint_{(1\dots p)}\oint_{(p+1\dots n)}\left(\text{[tree 4]} + \sum_{k=2}^{p}(-1)^{kn}\ \text{[tree 5]} + \sum_{k=2}^{n-2}(-1)^{p+k(1+n-p)+1}\ \text{[tree 6]} \right)$$

[tree 4]: 

[tree 5]: 

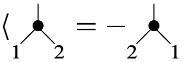
[tree 6]: 

where the symbol $\oint_{(i_1\dots i_k)}$ stands for the cyclic skewsymmetrization of the indices $(i_1\dots i_k)$.

Thus the minimal resolution, $(\mathcal{A}ss^{\circlearrowright})_\infty$, of the operad of finite-dimensional associative algebras is different from the naive "wheelification", $(\mathcal{A}ss_\infty)^{\circlearrowright}$, of the Stasheff minimal resolution of the operad, $\mathcal{A}ss$, of arbitrary associative algebras. A similar phenomenon occurs for the operad of commutative algebras [MMS]. In contrast, the operad, $\mathcal{L}ie$, of Lie algebras is rigid with respect to the wheelification:

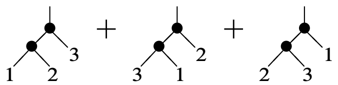
3.2.3. Fact ([Me5]). $(\mathcal{L}ie^{\circlearrowright})_\infty = (\mathcal{L}ie_\infty)^{\circlearrowright}$, i.e., wheeled L_∞-algebras are exactly the same as ordinary finite-dimensional L_∞-algebras.

3.2.4. Reminder on L_∞-algebras and their homotopy classification. For future reference we recall here a few useful facts about Lie and L_∞-algebras [Ko2]. The operad, $\mathcal{L}ie$, of Lie algebras is the quotient operad, $\mathcal{L}ie := \mathcal{F}^{\curlywedge}\langle L_0\rangle/I$, of the free operad generated by an $\mathbb{S}$-bimodule $L_0 = \{L_0(m,n)\}$,



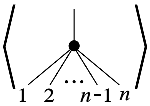
$$L_0(m,n) = \begin{cases} \mathrm{sgn}_2 = \mathrm{span}\langle \quad = - \quad \rangle & \text{for } m=1,\ n=2,\\ 0 & \text{otherwise,}\end{cases} \tag{6}$$

modulo the ideal I generated by the following relations:



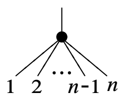
$$= 0. \tag{7}$$

Its minimal resolution $\mathcal{L}ie_\infty$ is a dg free operad $\mathcal{F}^{\curlywedge}\langle L\rangle$ generated by an $\mathbb{S}_n$-bimodule

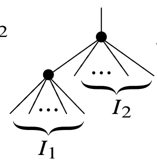


$$L(m,n) := \begin{cases} \mathrm{sgn}_n[n-2] = \mathrm{span}\left\langle \quad \right\rangle & \text{for } m=1,\ n\geq 2,\\ 0 & \text{otherwise,}\end{cases}$$

with the differential given by



$$\delta \quad = \sum_{\substack{[n]=I_1\sqcup I_2\\ \#I_1\geq 2,\#I_2\geq 1}} (-1)^{\sigma(I_1,I_2)+(\#I_1+1)\#I_2} \quad . \tag{8}$$



Here (and elsewhere) sgn_n stands for the 1-dimensional sign representation of $\mathbb{S}_n$.

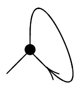
With an arbitrary graded vector space V one can associate a formal graded manifold $\mathcal{M}_V$, whose structure sheaf $\mathcal{O}_{\mathcal{M}_V}$ is, by definition, the completed graded cocommutative coalgebra $\widehat{\odot}(V[1])$; if V is finite dimensional, then one can equivalently view $\mathcal{M}_V$ as a small neighbourhood of zero in the space $V[1]$ equipped with the algebra (rather than coalgebra), $\widehat{\odot}(V^*[-1])$, of ordinary smooth formal functions. It is well known (see, e.g., [Ko2]) that L_∞-structures in a dg space V, that is, representations $\mathcal{L}_\infty \to \mathcal{E}nd_V$, are in one-to-one correspondence with degree 1 vector fields, $\eth$, on $\mathcal{M}_V$ which vanish at the distinguished point, $\eth\,|_{0\in\mathcal{M}_V} = 0$, and satisfy the condition $[\eth,\eth] = 0$ (such vector fields are called *cohomological*). The pairs $(\mathcal{M}_V, \eth)$ are often called *dg manifolds*. This interpretation of L_∞-structures permits us to use simple and concise geometric instruments to describe notions which, in the pure algebraic translation, look awkwardly large. For example, a *morphism* of L_∞-algebras $V \to W$ is nothing but a smooth map, $f : \mathcal{M}_V \to \mathcal{M}_W$, of the associated formal manifolds such that $f_*(\eth_V) = \eth_W$.

An L_∞-algebra $(\mathcal{M}_V, \eth)$ is called *minimal* if the first Taylor coefficient, $\eth_{(1)}$, of the homological vector field $\eth$ at the distinguished point $0 \in \mathcal{M}_V$ vanishes. It is called *linear contractible* if the higher Taylor coefficients $\eth_{(\geq 2)}$ vanish and the first one $\eth_{(1)}$ has trivial cohomology when viewed as a differential in V. According to Kontsevich [Ko2], *any L_∞-algebra (or, better, the associated dg manifold) is isomorphic to the direct product of a minimal and of a linear contractible one*. This fact implies that quasi-isomorphisms in the category of L_∞-algebras are equivalence relations. A dg manifold is called *contractible* if it is isomorphic to a linear contractible one.

3.3. Unimodular Lie algebras. Many important Lie algebras $\mathfrak{g}$ (e.g., all semisimple Lie algebras) have the additional property that, for any $g \in \mathfrak{g}$, the trace of the associated adjoint action

$$\begin{aligned} \mathrm{Ad}_g : \mathfrak{g} &\longrightarrow \mathfrak{g}, \\ e &\longmapsto [g,e] \end{aligned}$$

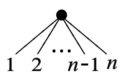
vanishes. Lie algebras with this property are called *unimodular*. The wheeled operad, $\mathcal{ULie}$, controlling unimodular Lie algebras is the quotient of the free wheeled operad, $\mathcal{F}^{\circlearrowright}\langle L_0\rangle$, generated by the $\mathbb{S}$-bimodule (6) modulo the ideal generated by the Jacobi relations (7) and the unimodularity relation



$$= 0.$$

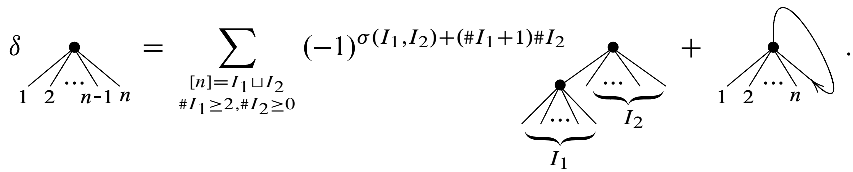
Its minimal resolution has been found in [Gr1]:

3.3.1. Theorem. *The operad $\mathcal{ULie}_\infty$ is a dg free operad, $\mathcal{F}^{\circlearrowright}\langle \hat{L}\rangle$ generated by the $\mathbb{S}$-bimodule*



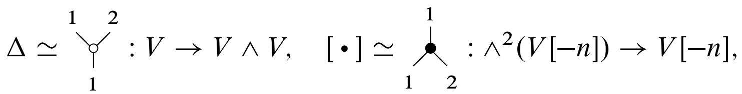
$$\hat{L}(m,n) := \begin{cases} \mathrm{sgn}_n[n-2] & \text{for } m = 1,\ n \geq 2, \\ \mathrm{sgn}_n[n] = \mathrm{span}\left\langle \qquad \right\rangle & \text{for } m = 0,\ n \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

with the differential on the generators of $\hat{L}(1,n)$ *given by* (8) *and on the generators of* $\hat{L}(1,n)$ *by*



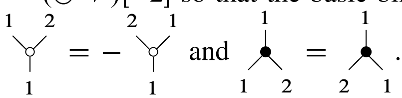
Geometrically, unimodular L_∞-structures in V can be interpreted as pairs $(\eth, \omega)$, where $\eth$ is a cohomological vector field and ω a Q-invariant section of the Berezinian bundle on $V^*[1]$ (see [Gr1]).

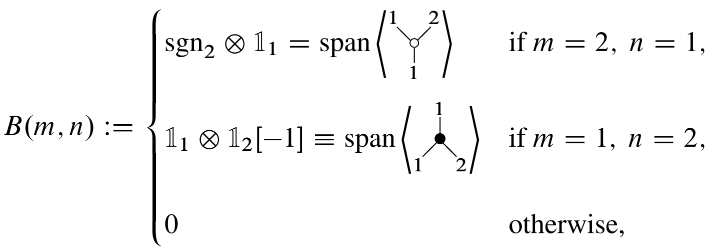
3.4. Lie 1-bialgebras and Poisson geometry. A *Lie n-bialgebra* on a graded vector space V is a pair of linear maps,



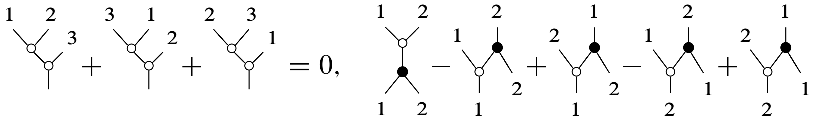
making the space V into a Lie coalgebra and the space $V[-n]$ into a Lie algebra and satisfying, for any $a, b \in V$, the compatibility condition

$$\Delta[a \bullet b] = \sum a_1 \otimes [a_2 \bullet b] + [a \bullet b_1] \otimes b_2 \\ + (-1)^{|a||b|+n|a|+n|b|}([b \bullet a_1] \otimes a_2 + b_1 \otimes [b_2 \bullet a]).$$

Here $\Delta a =: \sum a_1 \otimes a_2$ and $\Delta b =: \sum b_1 \otimes b_2$. The case $n = 0$ gives the notion of a Lie bialgebra which was introduced by Drinfeld [Dr] in the context of quantum groups. The case $n = 1$, as we shall see below, is relevant to Poisson geometry. In this case one has $\wedge^2(V[-1]) = (\odot^2 V)[-2]$ so that the basic binary operations have the following symmetries:  Thus the prop of Lie 1-bialgebras $\mathcal{L}ie^1\mathcal{B}$ is the quotient of the free prop $\mathcal{F}^\uparrow\langle B\rangle$ generated by an $\mathbb{S}$-bimodule

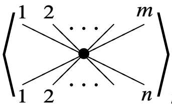

(9)

modulo the ideal generated by Jacobi relations (7) and the following ones:

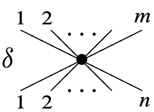
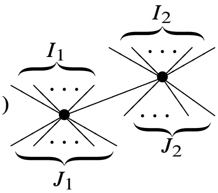

(10)

Its minimal resolution, $\mathcal{L}ie^1\mathcal{B}_\infty$, has been computed in [Me3].

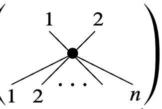
3.4.1. Theorem. (i) *$\mathcal{L}ie^1\mathcal{B}_\infty$ is a dg free prop, $\mathcal{F}^\uparrow\langle X\rangle$, generated by an $\mathbb{S}$-bimodule*

$$X(m,n)[-1] = \mathrm{sgn}_m \otimes \mathbb{1}_n[m-2] = \mathrm{span}\left\langle \right.$$

$$\left.\right\rangle_{m\geq 1, n\geq 1, m+n\geq 3} \qquad (11)$$

and with the differential given on the generators as follows:

δ 
$$= \sum_{\substack{[m]=I_1\sqcup I_2, [n]=J_1\sqcup J_2\\ |I_1|\geq 0, |I_2|\geq 1, |J_1|\geq 1, |J_2|\geq 0}} (-1)^{\sigma(I_1\sqcup I_2)+|I_1|(|I_2|+1)}$$


(ii) *For any $d \in \mathbb{N}$, there is a one-to-one correspondence between representations of the dg prop $\mathcal{L}ie^1\mathcal{B}_\infty$ in $\mathbb{R}^p$ and formal Poisson structures, π, on $\mathbb{R}^d$ vanishing at the origin.*

Proof. The proof of (i) is straightforward (see, e.g., [MeVa], [Me5], [Va1]) once one uses rather non-straightforward Koszul duality theory for dioperads, [GiKa], [Ga], and Kontsevich's ideas of $\frac{1}{2}$-props and path filtrations [Ko1], [MaVo]. We shall discuss some of these ideas in §5 and show here now only the proof of (ii). Since $\mathbb{R}^p$ is concentrated in degree zero, an arbitrary representation $\rho\colon \mathcal{L}ie^1\mathcal{B}_\infty \to \mathcal{E}nd_{\mathbb{R}^p}$ can have non-zero values only on (m,n)-corollas with $m=2$. Denote these values, $\rho\Bigg($  $\Bigg) \in \mathrm{Hom}(\odot^n\mathbb{R}^p, \wedge^2\mathbb{R}^p)$, by π_n. As the tangent space, $\mathcal{T}_0$, to $\mathbb{R}^p$ at zero can be identified with $\mathbb{R}^p$ itself, we can identify the total sum $\pi := \sum_{n\geq 1}\pi_n \in \mathrm{Hom}(\odot^{\geq 1}\mathbb{R}^p, \wedge^2\mathcal{T}_0)$ with a formal bi-vector field on $\mathbb{R}^p$. Then the equation $\rho\circ\delta = \delta\circ\rho$ becomes precisely the Poisson equation $[\pi,\pi]_S = 0$, where $[\ ,\ ]_S$ denotes the Schouten bracket. $\square$

It is worth pointing out that the vanishing condition $\pi|_{0\in\mathbb{R}^p} = 0$ in Theorem 3.4.1(ii) is no serious restriction: given an arbitrary formal or analytic Poisson structure π on $\mathbb{R}^p$ (not necessary vanishing at $0\in\mathbb{R}^p$), then, for any parameter λ viewed as a coordinate on $\mathbb{R}$, the product $\lambda\pi$ is a Poisson structure on $\mathbb{R}^{p+1} = \mathbb{R}^p\times\mathbb{R}$ vanishing at zero $0\in\mathbb{R}^{n+1}$ and hence is a representation of the prop $\mathcal{L}ie^1\mathcal{B}_\infty$.

3.4.2. Bi-Hamiltonian geometry. The prop profile of a *pair of compatible Poisson structures* (which is an important concept in the theory of integrable systems) has been computed by Strohmayer in [Str2] with the help of an earlier result of Dotsenko and Khoroshkin [DoKh].

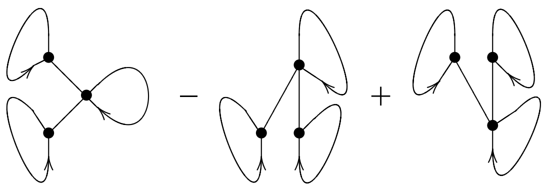
3.4.3. Wheeled Poisson structures? Theorem 3.4.1 says that the minimal resolution

$$\mathcal{L}ie^1\mathcal{B}_\infty = (\mathcal{F}^{\uparrow}\langle X\rangle, \delta)$$

of the prop $\mathcal{L}ie^1\mathcal{B}$ of arbitrary Lie 1-bialgebras controls the category of local (formal) smooth Poisson structures. What can be said about a minimal resolution, $(\mathcal{L}ie^1\mathcal{B}^{\circlearrowright})_\infty$, of the wheeled prop, $\mathcal{L}ie^1\mathcal{B}^{\circlearrowright}$, of *finite dimensional* Lie 1-bialgebras whose representations can, in view of Theorem 3.4.1(ii), be called *wheeled Poisson structures*? Note that $\mathcal{L}ie^1\mathcal{B}^{\circlearrowright}$ has the same generators and relations as $\mathcal{L}ie^1\mathcal{B}$, the only difference being that graphs now might have wheels. As in the case of associative algebra, the naive wheelification,

$$(\mathcal{L}ie^1\mathcal{B}_\infty)^{\circlearrowright} := (\mathcal{F}^{\circlearrowright}\langle X\rangle, \delta),$$

creates new *non-trivial* cohomology classes, as e.g. this one [Me5],



$$\in (\mathcal{L}ie^1\mathcal{B}_\infty)^{\circlearrowright}, \qquad (12)$$

which map under the natural projection $(\mathcal{L}ie^1\mathcal{B}_\infty)^{\circlearrowright} \to \mathcal{L}ie^1\mathcal{B}^{\circlearrowright}$ to zero. Thus the set of generators of a minimal resolution, $(\mathcal{L}ie^1\mathcal{B}^{\circlearrowright})_\infty$, of $\mathcal{L}ie^1\mathcal{B}^{\circlearrowright}$ must be larger than the set (11), and at present its computation is beyond reach. All we can say now about mysterious *wheeled* Poisson structures on a graded formal manifold M is that (i) they are Maurer–Cartan elements of a certain L_∞-algebra extension of the ordinary Schouten bracket on M which involves divergence operators (in fact, graph (12) gives us a glimpse of the μ_3 composition in that L_∞-algebra), and (ii) they can be deformation quantized in exactly the same sense as ordinary Poisson structures; moreover, it is proven in [Me6] with the help of the theory of wheeled props that there exist universal formulae for deformation quantization of wheeled Poisson structures which involve only rational numbers $\mathbb{Q}$.

3.5. Pre-Lie algebras, Nijenhuis geometry and contractible dg manifolds. A *pre-Lie* algebra is a vector space together with a binary operation, $\circ\colon V^{\otimes 2} \to V$, satisfying the condition

$$(a \circ b) \circ c - a \circ (b \circ c) - (-1)^{|b||c|}(a \circ c) \circ b + (-1)^{|b||c|} a \circ (c \circ b) = 0$$

for any $a, b, c \in V$. Any pre-Lie algebra is naturally a Lie algebra with the bracket, $[a, b] := a \circ b - (-1)^{|a||b|} b \circ a$. Let us consider the following extension of this notion: a *pre-Lie*^{2} algebra is a pre-Lie algebra $(V, \circ)$ equipped with a compatible Lie bracket in degree 1, i.e., with a linear map[1] $[\bullet]\colon \wedge^2 (V[-1]) \to V[-1]$ satisfying the Jacobi identities and the following compatibility condition for all $a, b, c \in V$:

$$\begin{aligned}
&[a \bullet b] \circ c + (-1)^{|b|} a \circ [b \bullet c] + (-1)^{|b||a|+|b|} b \circ [a \bullet c] \\
&\quad = (-1)^{|b||c|+|c|} [(a \circ c) \bullet b] + (-1)^{(|a|+1)(|b|+|c|)+|a|} [(b \circ c) \bullet a].
\end{aligned}$$

This compatibility condition can be understood as follows. The vector space $V \oplus V[-1]$ is naturally a complex with trivial cohomology. If we write elements of $V \oplus V[-1]$ as $a + \Pi b$, where $a, b \in V$ and Π is a formal symbol of degree 1, then the natural differential in $V \oplus V[-1]$ is given by $d(a + \Pi b) = 0 + \Pi a$. Given two arbitrary binary operations,

$$\circ\colon V \otimes V \to V, \quad [\bullet]\colon \odot^2 V \to V[1],$$

define a degree zero map, $[\ ,\ ]\colon \wedge^2 (V \oplus V[-1]) \to V \oplus V[-1]$, by setting

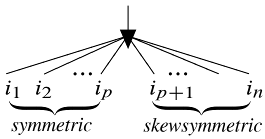
$$[a, b] := a \circ b - (-1)^{|a||b|} b \circ a, \quad [\Pi a, b] := -(-1)^{|a|} [a \bullet b] + \Pi a \circ b,$$
$$[\Pi a, \Pi b] := \Pi\, [a \bullet b]\,.$$

3.5.1. Proposition ([Me4]). *The data $(V \oplus V[-1], d, [\ ,\ ])$ is a (contractible) dg Lie algebra if and only if $(V, \circ, [\bullet])$ is a pre-Lie^2 algebra.*

Rather surprisingly, the minimal resolution, *pre-*$\mathcal{L}ie^2_\infty$, of the operad of pre-Lie^2-algebras has much to do with the famous Nijenhuis integrability condition in differential geometry. The following result is based on the works [ChLi], [Me4], [Str2].

3.5.2. Theorem. (i) *The operad pre-*$\mathcal{L}ie^2_\infty$ *is a free operad, $\mathcal{F}^\curlywedge\langle N\rangle$, generated by an $\mathbb{S}$-bimodule N with all $N(m, n) = 0$ except the following ones,*

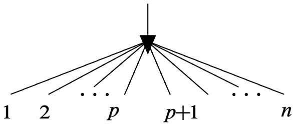
$$N(1, n) := \bigoplus_{p=1}^{n} \mathrm{Ind}^{\mathbb{S}_n}_{\mathbb{S}_p \times \mathbb{S}_{n-p}} \mathbb{1}_p \otimes \mathrm{sgn}_{n-p}[n-p-1]$$

$= \mathrm{span}\Big\langle$ 

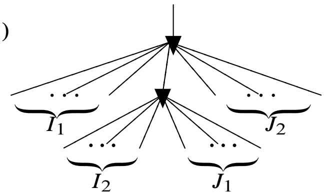
$\Big\rangle, \quad n \geq 2,$

[1] Equivalently, a linear map $[\bullet]\colon \odot^2 V \to V[1]$.

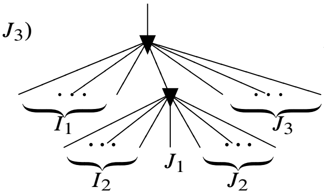
and equipped with a differential given on the generators by

d 

$$= \sum_{\substack{I_1 \sqcup I_2 = (1,\dots,p)\\ J_1 \sqcup J_2 = (p+1,\dots,n)\\ \#I_2 \geq 1, \#I_1 + \#J_2 \geq 1\\ \#I_2 + \#J_1 \geq 2}} (-1)^{\#J_2 + \sigma(J_1,J_2)}$$



$$- \sum_{\substack{I_1 \sqcup I_2 = (1,\dots,n-p)\\ J_1 \sqcup J_2 \sqcup J_3 = (n-p+1,\dots,n)\\ \#I_1 \geq 1, \#I_2 \geq 1\\ \#I_1 + \#J_3 \geq 1, \#I_2 + \#J_2 \geq 1}} (-1)^{\#J_2 + \#J_3 + \sigma(J_1,J_2,J_3)}$$

 .

(ii) *For any* $d \in \mathbb{N}$, *there is a one-to-one correspondence between representations of pre-*$\mathcal{L}ie^2_\infty$ *in* $\mathbb{R}^d$ *and endomorphisms,* $J : \mathcal{T}_{\mathbb{R}^d} \to \mathcal{T}_{\mathbb{R}^d}$, *of the tangent bundle on the affine space* $\mathbb{R}^d$ *satisfying the Nijenhuis integrability condition,* $N_J = 0$, *and the vanishing condition* $J|_{0 \in \mathbb{R}^d} = 0$.

We recall that the Nijenhuis tensor of an endomorphism $J : \mathcal{T}_M \to \mathcal{T}_M$ of the tangent bundle of an arbitrary smooth manifold M (in particular, of $\mathbb{R}^m$) can be defined as a map

$$\begin{aligned} N_J :\ \wedge^2 \mathcal{T}_{\mathbb{R}^m} &\longrightarrow \mathcal{T}_{\mathbb{R}^m}, \\ X \otimes Y &\longmapsto N_J(X,Y) := [JX, JY] + J^2[X,Y] - J[X,JY] - J[JX,Y], \end{aligned}$$

and that its beauty is hidden in the far from being obvious fact that it is linear not only over $\mathbb{R}$ but also over arbitrary smooth functions, $f \in \mathcal{O}_M$, on M, that is, $N_J(fX,Y) = N_J(X,fY) = fN_J(X,Y)$.

A representation of this dg operad in an arbitrary *graded* vector space V might be called a *graded* or *extended* Nijenhuis structure on V (viewed as a formal manifold). Interestingly, the category of these *extended Nijenhuis manifolds* is almost identical (see [Me4]) to the category of *contractible dg manifolds* which we first met in §3.2.4 when discussing Kontsevich's homotopy classification of dg manifolds. Proposition 3.5.1 above is in fact one of the simplest manifestations of this more general phenomenon.

3.6. Gerstenhaber algebras and Hertling–Manin geometry. We conclude this section with an example which was actually the first one to reveal strong interconnections between derived (via minimal resolutions) categories of rather simple algebraic structures and solution sets of highly non-linear *diffeomorphism covariant* differential equations on ordinary smooth manifolds.

A *Gerstenhaber* algebra is, by definition, a graded vector space V together with two linear maps $\circ\colon \odot^2 V \to V$ and $[\bullet]\colon \odot^2 V \to V[1]$ such that $(V, \circ)$ is a graded commutative algebra, $(V[-1], [\ \bullet\ ])$ is a graded Lie algebra, and the compatibility equation

$$[(a \circ b) \bullet c] = a \circ [b \bullet c] + (-1)^{|b|(|c|+1)}[a \bullet c] \circ b \quad \text{for all } a, b, c \in V$$

holds. The operad of Gerstenhaber algebras is often denoted by $\mathcal{G}$. Its minimal resolution, $\mathcal{G}_\infty$, has been computed in [GeJo]; it is one of the most important operads in mathematics which found many applications in homological algebra, algebraic topology and deformation quantization. It was shown in [Me2] that $\mathcal{G}_\infty$ has also a differential geometric dimension:

3.6.1. Theorem. *For any $d \in \mathbb{N}$, there is a one-to-one correspondence between representations of the dg operad $\mathcal{G}_\infty$ in $\mathbb{R}^d$ (concentrated in degree 0) and linear maps $\mu\colon \odot^2 \mathcal{T}_{\mathbb{R}^d} \to \mathcal{T}_{\mathbb{R}^d}$ making the tangent sheaf $\mathcal{T}_{\mathbb{R}^d}$ into a commutative and associative algebra, and satisfying the Hertling–Manin integrability condition, $R_\mu = 0$, and the vanishing condition $\mu\mid_{0\in\mathbb{R}^d} = 0$.*

We recall that the Hertling–Manin tensor R_μ of an arbitrary commutative and associative product, $\mu\colon \mathcal{T}_M \odot \mathcal{T}_M \to \mathcal{T}_M$, on the tangent sheaf of an arbitrary smooth manifold M is a map [HeMa]

$$\begin{aligned} R_\mu\colon\ \otimes^4 \mathcal{T}_M &\longrightarrow \mathcal{T}_M, \\ X \otimes Y \otimes Z \otimes W &\longmapsto R_\mu(X, Y, Z, W), \end{aligned}$$

where

$$\begin{aligned} R_\mu(X, Y, Z, W) = [\mu(X, Y), \mu(Z, W)] &- \mu([\mu(X, Y), Z], W) \\ &- \mu(Z, [\mu(X, Y), W]) - \mu(X, [Y, \mu(Z, W)]) \\ &- \mu[X, \mu(Z, W)], Y) + \mu(X, \mu(Z, [Y, W])) \\ &+ \mu(X, \mu([Y, Z], W)) + \mu([X, Z], \mu(Y, W)) \\ &+ \mu([X, W], \mu(Y, Z)). \end{aligned}$$

A remarkable fact is that this map is linear not only over $\mathbb{R}$ but also over arbitrary smooth functions $f \in \mathcal{O}_M$ on M, that is, $R_\mu(fX, Y, Z, W) = fR_\mu(X, Y, Z, W)$, $R_\mu(X, fY, Z, W) = fR_\mu(X, Y, Z, W)$, etc. One can view the Hertling–Manin integrability equation as a diffeomorphism covariant version of the WDVV equation [HeMa], [HMT].

4. Applications to deformation theory

4.1. From minimal resolutions to L_∞-algebras. One of the advantages of knowing a dg free resolution, $\mathcal{P}_\infty$, of a $\mathfrak{G}$-algebra controlling a mathematical structure $\mathcal{P}$ is that $\mathcal{P}_\infty$ paves a direct way to the deformation theory of $\mathcal{P}$-structures. In the heart of this approach to the deformation theory of many algebraic and geometric structures is the observation 4.1.2 (see below) which was proven in [MeVa] in several ways. For its precise formulation we need the following notion.

4.1.1. Definitions. An L_∞-algebra $(\mathfrak{g}, \{\mu_n : \wedge^n \mathfrak{g} \to \mathfrak{g}[2-n]\}_{n\geq 1})$ is called *filtered* if $\mathfrak{g}$ admits a non-negative decreasing Hausdorff filtration,

$$\mathfrak{g}_0 = \mathfrak{g} \supseteq \mathfrak{g}_1 \supseteq \cdots \supseteq \mathfrak{g}_i \supseteq \cdots,$$

such that $\operatorname{Im} \mu_n \subset \mathfrak{g}_n$ for all $n \geq n_0$ beginning with some $n_0 \in \mathbb{N}$. In this case it makes sense to define the associated set $\mathcal{MC}(\mathfrak{g})$ of *Maurer–Cartan elements* as a subset of $\mathfrak{g}$ consisting of degree 1 elements Γ satisfying the equation $\sum_{n\geq 1} \frac{1}{n!}\mu_n(\Gamma, \dots, \Gamma) = 0$.

A very useful fact is that to every Maurer–Cartan element $\Gamma \in \mathcal{MC}(\mathfrak{g})$ of a filtered L_∞-algebra $(\mathfrak{g}, \{\mu_n : \wedge^n \mathfrak{g} \to \mathfrak{g}\}_{n\geq 1})$ there corresponds a Γ-*twisted* L_∞-algebra structure, $\{\mu_n^\Gamma : \wedge^n \mathfrak{g} \to \mathfrak{g}\}_{n\geq 1}$, on $\mathfrak{g}$. If one thinks of the original L_∞-algebra as of a dg manifold $(\mathcal{M}_\mathfrak{g}, \eth)$ (see §4.1.2), then the set $\mathcal{MC}(\mathfrak{g})$ can be identified with the zero set of the homological vector field $\eth$, and the Γ-twisted $\mathcal{L}_\infty$-algebra structure on $\mathfrak{g}$ corresponds to that homological vector field $\eth^\Gamma$ on $\mathcal{M}_\mathfrak{g}$ which is obtained from $\eth$ by the translation diffeomorphism $x \to x + \Gamma$ for all $x \in \mathcal{M}_\mathfrak{g}$.

4.1.2. Theorem ([MeVa]). *Let $(\mathcal{F}^{\mathfrak{G}}\langle E\rangle, \delta)$ be a dg free $\mathfrak{G}$-algebra (see Table 1) generated by an $\mathbb{S}$-bimodule E, and let $(\mathcal{Q}, \delta_\mathcal{Q})$ be an arbitrary dg $\mathfrak{G}$-algebra. Then*

(i) *the graded vector space $\mathfrak{g} := \operatorname{Hom}_\mathbb{S}(E, \mathcal{Q})[-1]$ is canonically a filtered L_∞-algebra;*

(ii) *the set of all morphisms $\{\mathcal{F}^{\mathfrak{G}}\langle E\rangle \to \mathcal{Q}\}$ of dg $\mathfrak{G}$-algebras is canonically isomorphic to the Maurer–Cartan set $\mathcal{MC}(\mathfrak{g})$ of the L_∞-algebra in* (i).

Proof. As an illustration we show an elementary proof of the theorem in the simplest case $\mathfrak{G} = \mathfrak{G}^{|}$ (see Table 1), i.e., for the case when $\mathcal{F}^{\mathfrak{G}}\langle E\rangle$ is the free associative algebra, $\otimes^\bullet E$, generated by a graded vector space E and $(\mathcal{Q}, \delta_\mathcal{Q})$ is an arbitrary dg associative algebra (we refer the reader to [MeVa] for all other cases from Table 1 except $\mathfrak{G}^{\circlearrowright}$, and to [Gr2] for the case $\mathfrak{G}^{\circlearrowright}$). With these data we shall associate a cohomological vector field, $\eth$, on the space $\mathfrak{g}[1] = \operatorname{Hom}(E, \mathcal{Q}) = \mathcal{Q} \otimes E^*$, and we shall do it in local coordinates by assuming further (only for simplicity of sign factors in formulae) that the graded vector spaces E and $\mathcal{Q}$ are free modules over some graded commutative ring, $R = \bigoplus_{i\in\mathbb{Z}} R^i$, with *degree* 0 generators $\{e_a\}_{a\in I}$ and, respectively, $\{e_\alpha\}_{\alpha\in J}$. Then the differentials in $\otimes^\bullet E$ and $\mathcal{Q}$, as well as the multiplication $\circ$ in $\mathcal{Q}$, have, respectively, the following coordinate representations:

$$\delta e_a = \sum_{\substack{k\geq 1\\ a_1\ldots a_k\in I}} \delta_a^{a_1\ldots a_k} e_{a_1}\otimes\ldots\otimes e_{a_k}, \quad \delta_{\mathcal{Q}} e_\alpha = \sum_{\beta\in J} Q_\alpha^\beta e_\beta,$$

$$e_\alpha \circ a_\beta = \sum_{\gamma\in J} \mu_{\alpha\beta}^\gamma e_\gamma$$

for some coefficients $\delta_a^{a_1\ldots a_k}\in R^1$, $Q_\alpha^\beta\in R^1$ and $\mu_{\alpha\beta}^\gamma\in R^0$. The vector space of all $\mathcal{R}$-linear maps, $\mathrm{Hom}(E,\mathcal{Q})$, is naturally graded, $\mathrm{Hom}(E,\mathcal{Q}) = \bigoplus_{i\in\mathbb{Z}}\mathrm{Hom}^i(E,\mathcal{Q})$, with $\mathrm{Hom}^i(E,\mathcal{Q})$ denoting the space of all homogeneous linear maps of degree i. In the chosen bases a generic element $\gamma\in\mathrm{Hom}^i(E,\mathcal{Q})$ gets a coordinate representation, $\gamma(e_a) = \sum_{\alpha\in J}\gamma_{a(i)}^\alpha e_\alpha$, for some coefficients $\gamma_{a(i)}^\alpha\in R^i$. The family of parameters $\{\gamma_{a(i)}^\alpha\}_{a\in I,\alpha\in J,i\in\mathbb{Z}}$ provides us with a coordinate system on the formal manifold $\mathcal{M}_\mathfrak{g}\simeq\mathrm{Hom}(E,\mathcal{Q})$. In these coordinates the required homological vector field on $\mathcal{M}_\mathfrak{g}$, that is, a $\mathcal{L}_\infty$-structure on $\mathrm{Hom}(E,\mathcal{Q})[-1]$, is given explicitly by

$$\eth = \Big(\sum_{\alpha,\beta,a,i} Q_\beta^\alpha\gamma_{a(i)}^\beta - \sum_{a,a_\bullet,\alpha,i}(-1)^i\delta_a^{a_1\ldots a_k}\gamma_{a_1\ldots a_k(i)}^\alpha\Big)\frac{\partial}{\partial\gamma_{a(i)}^\alpha},$$

where for $k\geq 2$,

$$\gamma_{a_1a_2\ldots a_k(i)}^\alpha = \sum_{\substack{\beta_\bullet,\gamma_\bullet\in J\\ i_1+\cdots+i_k=i}} \mu_{\beta_1\gamma_1}^\alpha\mu_{\beta_2\gamma_2}^{\gamma_1}\cdots\mu_{\beta_{k-1}\beta_k}^{\gamma_{k-2}}\gamma_{a_1(i_1)}^{\beta_1}\gamma_{a_2(i_2)}^{\beta_2}\cdots\gamma_{a_k(i_k)}^{\beta_k}.$$

The equation $[\eth,\eth]=0$ follows straightforwardly from the assumptions that $\delta^2=0$, $\delta_{\mathcal{Q}}^2=0$, as well as from the associativity of the product $\circ$ and its compatibility with $\delta_{\mathcal{Q}}$. This proves (i).

The Maurer–Cartan set $\mathcal{MC}(\mathfrak{g})$ is precisely the set $\{\gamma\in\mathrm{Hom}^0(E,\mathcal{Q}) : \eth|_\gamma = 0\}$ and, therefore, consists of all points in $\mathrm{Hom}(E,\mathcal{Q})$ which have all the coordinates $\{\gamma_{a(i)}^\alpha\}_{i\neq 0}$ vanishing, and the coordinate $\gamma_{a(0)}^\alpha$ satisfying the equations

$$\sum_{\beta\in J} Q_\beta^\alpha\gamma_{a(0)}^\beta - \sum_{a_1,\ldots,a_k\in I}\delta_a^{a_1\ldots a_k}\gamma_{a_1\ldots a_k(0)}^\alpha = 0.$$

This just says that the map of associative algebras $\odot^\bullet E\to\mathcal{Q}$ associated to $\gamma_{a(0)}^\alpha$ commutes with the differentials δ and $\delta_{\mathcal{Q}}$, defining thereby a morphism of *dg* algebras. This proves claim (ii). □

4.2. Deformation theory. The theory of operads and props gives a universal approach to the deformation theory of many algebraic and geometric structures and provides us with a conceptual explanation of the well-known "experimental" observation that a deformation theory is controlled by a differential graded Lie or, more generally, an L_∞-algebra. What happens is the following [KoSo], [MeVa], [vdL]:

(I) An algebraic or a (germ of) geometric structure, $\mathfrak{s}$, in a vector space V (which is an *object* in the corresponding category, $\mathfrak{S}$, of algebraic or geometric structures) can often be interpreted as a *representation*, $\alpha_{\mathfrak{s}}\colon \mathcal{S} \to \mathcal{E}nd_V$, of a $\mathfrak{G}$-algebra $\mathcal{S}$ uniquely associated to the category of $\mathfrak{s}$-structures.

(II) A dg resolution, $\pi\colon \mathcal{S}_\infty = (\mathcal{F}^{\mathfrak{G}}\langle E\rangle, \delta) \to \mathcal{S}$, of the $\mathfrak{G}$-algebra $\mathcal{S}$ gives rise, by Theorem 4.1.2, to a filtered L_∞-algebra on the vector space $\mathfrak{g} = \mathrm{Hom}_{\mathbb{S}}(E, \mathcal{E}nd_V)[-1])$ whose Maurer–Cartan elements correspond to all possible representations $\mathcal{S}_\infty \to \mathcal{E}nd_V$; in particular, our original algebraic or geometric structure $\mathfrak{s}$ defines a Maurer–Cartan element $\Gamma_{\mathfrak{s}} := \alpha_{\mathfrak{s}} \circ \pi$ in $\mathcal{MC}(\mathfrak{g})$.

(III) The $\Gamma_{\mathfrak{s}}$-twisted L_∞-algebra structure on $\mathfrak{g}$ is precisely the one which controls, in Deligne's sense, the deformation theory of $\mathfrak{s}$.

For example, if $\mathfrak{s}$ is the structure of associative algebra on a vector space V, then

(i) there is an operad, $\mathcal{A}ss$, uniquely associated to the category of associative algebras such that $\mathfrak{s}$ corresponds to a morphism, $\alpha_{\mathfrak{s}}\colon \mathcal{A}ss \to \mathcal{E}nd_V$, of operads (see §2.1);

(ii) there is a unique minimal resolution (see Theorem 3.1.2), $\mathcal{A}ss_\infty$, of $\mathcal{A}ss$ which is generated by the $\mathbb{S}$-module $E = \{\mathbb{K}[\mathbb{S}_n][n-2]\}$ and whose representations, $\pi\colon \mathcal{A}ss_\infty \to \mathcal{E}nd_V$, in a dg space V are in one-to-one correspondence with Maurer–Cartan elements in the Lie algebra

$$\Big(\mathcal{G} := \mathrm{Hom}_{\mathbb{S}}(E, \mathcal{E}nd_V)[-1] = \bigoplus_{n\geq 1} \mathrm{Hom}_{\mathbb{K}}(V^{\otimes n}, V)[1-n], [\ ,\ ]_G\Big),$$

where $[\ ,\ ]_G$ is the Gerstenhaber bracket;

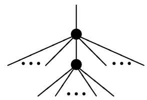
(iii) the particular associative algebra structure $\mathfrak{s}$ on V gives, therefore, rise to the associated Maurer–Cartan element $\gamma_{\mathfrak{s}} := \alpha_{\mathfrak{s}} \circ \pi$ in $\mathcal{G}$; twisting $\mathcal{G}$ by $\gamma_{\mathfrak{s}}$ gives the Hochschild dg Lie algebra, $\mathcal{G}_{\mathfrak{s}} = (\bigoplus_n \mathrm{Hom}_{\mathbb{K}}(V^{\otimes n}, V)[1-n], [\ ,\ ]_G, d_H := [\gamma_{\mathfrak{s}}, \ ]_G)$ which indeed controls the deformation theory of $\mathfrak{s}$.

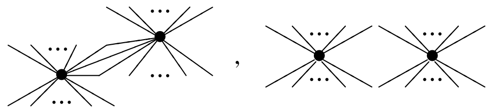
This is a classical example illustrating how the machine works. For some new applications of this approach to deformation theory (e.g. to the proof of Deligne's conjecture or to the deformation theory of associative bialgebras) we refer to [KoSo], [MeVa] and to many references cited there.

5. Koszul duality theory, quantum BV manifolds and effective *BF*-actions

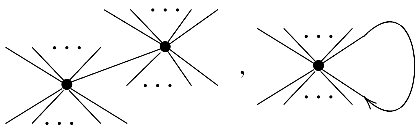
5.1. Quadratic $\mathfrak{G}$-algebras and their Koszul duals. Koszul duality theory of quadratic $\mathfrak{G}$-algebras is one of the most powerful theorem-proving techniques in the theory of (wheeled) operads and properads and their applications.

What is a *quadratic* $\mathfrak{G}$-algebra? Every family of graphs $\mathfrak{G}$ from Table 1 has a uniquely defined subfamily $\mathfrak{G}_{\mathrm{gen}}$ of *generating* graphs, which, by definition, is the smallest subset of $\mathfrak{G}$ with the defining property that for every $G \in \mathfrak{G}$ and any $\mathfrak{G}$-algebra $\mathcal{P}$, the associated "contraction" composition $\mu_{\mathfrak{G}}\colon G\langle\mathcal{P}\rangle \to \mathcal{P}$ can be represented as an iteration (in the sense of (1)) of compositions μ_{G_i} for some $G_i \in \mathfrak{G}_{\mathrm{gen}}$, $i \in I$. For example,

$\mathfrak{G}^{\curlywedge}_{\mathrm{gen}} = \Big\{$  $\Big\}$,

$\mathfrak{G}^{\uparrow}_{\mathrm{gen}} = \Big\{$  $\Big\}$,

and

$\mathfrak{G}^{\circlearrowright}_{c,\mathrm{gen}} = \Big\{$  $\Big\}$.

5.1.1. Weight gradation. Let $\mathfrak{G}$ be a family of graphs from Table 1. For any genus q graph $G \in \mathfrak{G}$ with p vertices we set $\|G\| := p+q$ if the family $\mathfrak{G}$ contains wheels and set $\|G\| := p$ otherwise. This number is called the *weight* of G. Thus $\mathfrak{G}_{\mathrm{gen}} \subset \mathfrak{G}$ consists precisely of graphs of weight 2.

For an $\mathbb{S}$-bimodule E let $\mathcal{F}^{\mathfrak{G}}_{(\lambda)}\langle E\rangle$ stand for a subspace of the free $\mathfrak{G}$-algebra $\mathcal{F}^{\mathfrak{G}}\langle E\rangle$ spanned by decorated graphs of weight λ. Operadic compositions in $\mathcal{F}^{\mathfrak{G}}\langle E\rangle$ are homogeneous with respect to the weight gradation.

5.1.2. Definition. A $\mathfrak{G}$-algebra $\mathcal{P}$ is defined to be *quadratic* if it is the quotient $\mathcal{F}^{\mathfrak{G}}\langle E\rangle/\langle\mathcal{R}\rangle$ of a free $\mathfrak{G}$-algebra (generated by an $\mathbb{S}$-bimodule E) modulo the ideal generated by a subspace $\mathcal{R} \subset \mathcal{F}^{\mathfrak{G}}_{(2)}\langle E\rangle = \bigoplus_{G\in\mathfrak{G}_{\mathrm{gen}}} G\langle E\rangle$. It comes equipped with an induced weight gradation, $\mathcal{P} = \bigoplus_{\lambda\geq 1}\mathcal{P}_{(\lambda)}$, where $\mathcal{P}_{(\lambda)} = \mathcal{F}^{\mathfrak{G}}_{(\lambda)}\langle E\rangle/\langle\mathcal{R}\rangle$. In particular, $\mathcal{P}_{(1)} = E$ and $\mathcal{P}_{(2)} = \mathcal{F}^{\mathfrak{G}}_{(2)}\langle E\rangle/\mathcal{R}$.

5.2. Koszul duality. Let $\mathfrak{G}_c$ be any family of *connected* graphs from Table 1. In this case one can associate to any quadratic $\mathfrak{G}_c$-algebra $\mathcal{P}$ its Koszul dual $\mathfrak{G}_c$-coalgebra

$\mathcal{P}^{¡}$. We omit technical details (referring to [GiKa], [GeJo], [Ga], [Va1], [MMS], [Me7]) and explain just the working scheme:

(i) The notion of $\mathfrak{G}_c$*-coproperad* is obtained by an obvious dualization of the notion of $\mathfrak{G}_c$-algebra (see §2.3): this is an $\mathbb{S}$-bimodule $\mathcal{P} = \{\mathcal{P}(m,n)\}$ together with a family of linear $\mathbb{S}_m \times \mathbb{S}_n$-equivariant maps,

$$\{\Delta_G : \mathcal{P}(m,n) \to G\langle \mathcal{P}\rangle\}_{G\in\mathfrak{G}_c(m,n), m,n\geq 0},$$

which satisfy the coassociativity condition, $\Delta_G = \Delta'_H \circ \Delta_{G/H}$, for any subgraph $H \subset G$ which belongs to the family $\mathfrak{G}$. Here $\Delta'_H : (G/H)\langle E\rangle \to G\langle E\rangle$ is the map which equals Δ_H on the distinguished vertex of G/H and which is the identity on all other vertices of G.

(ii) There exists a pair of adjoint *exact* functors

$$B\colon \text{the category of dg } \mathfrak{G}_c\text{-algebras} \rightleftarrows \text{the category of dg } \mathfrak{G}_c\text{-coalgebras: } B^c,$$

$$\mathcal{P} \longmapsto (B(\mathcal{P}), \partial_{\mathcal{P}}),$$

$$(B^c(\mathcal{Q}), \partial_{\mathcal{Q}}) \longleftarrow \mathcal{Q},$$

such that for any dg $\mathfrak{G}_c$-algebra $\mathcal{P}$ the composition $B^c(B(\mathcal{P}))$ is a dg free resolution of $\mathcal{P}$. The differential $\partial_{\mathcal{P}}$ in $B(\mathcal{P})$ encodes both the differential and all the generating contraction compositions, $\{\mu_{\mathfrak{G}} : G\langle\mathcal{P}\rangle \to \mathcal{P}\}_{G\in\mathfrak{G}_{\rm gen}}$, in the $\mathfrak{G}_c$-algebra $\mathcal{P}$ (and similarly for $\partial_{\mathcal{Q}}$).

(iii) As a vector space $B(\mathcal{P})$ is isomorphic to the free $\mathfrak{G}_c$-algebra, $\mathcal{F}^{\mathfrak{G}_c}\langle\widehat{\mathcal{P}}\rangle$, generated by an $\mathbb{S}$-bimodule $\widehat{\mathcal{P}}$ which is linearly isomorphic to $\mathcal{P}$ and hence comes equipped with an induced weight gradation. The subspace $\mathcal{B}(\mathcal{P}_{(1)}) := \mathcal{F}^{\mathfrak{G}_c}\langle\widehat{\mathcal{P}}_{(1)}\rangle$ of $\mathcal{B}(\mathcal{P})$ is obviously a sub-coproperad. On the other hand, $B(\mathcal{P})$ has its own "outer" weight gradation, $B(\mathcal{P}) = \bigoplus_{\mu\geq 1} B_{(\mu)}(\mathcal{P})$, induced from the weight gradation of the free algebra $B_{(\mu)}(\mathcal{P}) := \mathcal{F}^{\mathfrak{G}_c}_{(\mu)}\langle\widehat{\mathcal{P}}\rangle$; the cobar differential $\partial_{\mathcal{P}}$ has weight -1 with respect to this outer weight gradation.

5.2.1. Definition. Given a quadratic $\mathfrak{G}_c$-algebra $\mathcal{P}$, the $\mathfrak{G}_c$-coalgebra

$$\mathcal{P}^{¡} = \bigoplus_{\mu\geq 1} \mathcal{P}^{¡}_{(\mu)}$$

with $\mathcal{P}^{¡}_{(\mu)} := B_{(\mu)}(\mathcal{P}_{(1)}) \cap \mathrm{Ker}\,\partial_{\mathcal{P}} \subset B(\mathcal{P})$ is called the *Koszul dual* to $\mathcal{P}$.

The beauty of this notion is that $\mathcal{P}^{¡}$ is again quadratic and, moreover, can often be easily computed directly from generators and relations, E and $\mathcal{R}$, of $\mathcal{P}$.

5.2.2. Definition. A quadratic $\mathfrak{G}_c$-algebra $\mathcal{P}$ is called *Koszul*, if the associated inclusion of dg coproperads, $\imath\colon (\mathcal{P}^{¡}, 0) \to (B(\mathcal{P}), \partial_{\mathcal{P}})$, is a quasi-isomorphism.

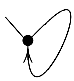
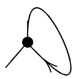
As the cobar construction functor B^c preserves quasi-isomorphisms between connected $\mathfrak{G}$-coalgebras, the composition

$$\pi : \mathcal{P}_\infty := B^c(\mathcal{P}^{\text{!`}}) \xrightarrow{B^c(\imath)} B^c(B(\mathcal{P})) \xrightarrow{\text{natural projection}} \mathcal{P}$$

is a quasi-isomorphism if and only if $\mathcal{P}$ is Koszul; in this case the dg free $\mathfrak{G}_c$-algebra $\mathcal{P}_\infty$ gives us a minimal resolution of the quadratic algebra $\mathcal{P}$. Almost all minimal resolutions listed in §2 have been obtained in this way.

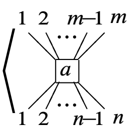
5.3. Homotopy transfer formulae. If $\mathcal{P}_\infty$ is a minimal resolution of some $\mathfrak{G}$-algebra $\mathcal{P}$, and (V, d) is a complex carrying a $\mathcal{P}$-structure, then one might expect that the associated cohomology space, $H(V, d)$, carries an induced structure of $\mathcal{P}_\infty$-algebra. In the case when $\mathcal{P}$ is an operad of associative algebras, existence of such induced $\mathcal{A}ss_\infty$-structures was proven by Kadeishvili in [Ka] and the first explicit formulae have been shown in [Me1]. Later Kontsevich and Soibelman [KoSo] have nicely rewritten these homotopy transfer formulae in terms of sums of decorated graphs. In fact, it is a general phenomenon that the homotopy transfer formulae can be represented as sums of graphs. The required graphs are precisely the ones which describe the image of the natural inclusion $\imath : (\mathcal{P}^{\text{!`}}, 0) \to (B(\mathcal{P}), \partial_{\mathcal{P}})$, and apply to any quadratic $\mathfrak{G}$-algebra, not necessarily the Koszul one [Me7].

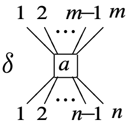
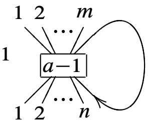
5.4. Example: unimodular Lie 1-bialgebras versus quantum BV manifolds. The wheeled prop, $\mathcal{ULie}^1\mathcal{B}$, of *unimodular* Lie 1-bialgebras was defined in [Me7] (cf. §3.4) as the quotient $\mathcal{F}_c^{\circlearrowright}\langle B\rangle/\langle\mathcal{R}\rangle$ of the free wheeled properad generated by the $\mathbb{S}$-bimodule (9) modulo the ideal generated by relations (7), (10) and the following ones,

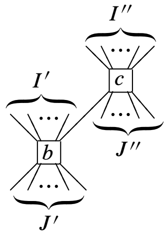
 $= 0,$  $= 0,$

expressing unimodularity of both binary operations. This is a quadratic wheeled properad so that one can apply the above general machinery to compute its Koszul dual coproperad, $\mathcal{ULie}^1\mathcal{B}^{\text{!`}}$, and then the dg properad $\mathcal{P}_\infty := B^c(\mathcal{ULie}^1\mathcal{B}^{\text{!`}})$ which turns out to be a free wheeled properad, $\mathcal{F}_c^{\circlearrowright}\langle Z\rangle$, generated by an $\mathbb{S}$-bimodule

$$Z(m,n) := \bigoplus_{a\geq 0}^{\infty} \mathrm{sgn}_m \otimes \mathbb{1}_n[m-2-2a]$$

$= \mathrm{span}\Big\langle$  $\Big\rangle_{\substack{m+n+2a\geq 3\\ m+a\geq 1, n+a\geq 1}}$, and equipped with the following differential:

δ  $= (-1)^{m-1}$ 

$+ \sum_{\substack{a=b+c\\ b,c\geq 0}} \sum_{\substack{m=I'\sqcup I''\\ [n]=J'\sqcup J''}} (-1)^{\sigma(I_1\sqcup I_2)+|I_1|(|I_2|+1)}$  .

It is not known at present whether or not $\mathcal{ULie}^1\mathcal{B}$ is Koszul, i.e., whether or not the above free properad is a (minimal) resolution of the latter. In any case, $\mathcal{ULie}^1\mathcal{B}_\infty$ gives us an approximation to that minimal resolution, and has, in fact, a geometrically meaningful set, $\{\mathcal{ULie}^1\mathcal{B}_\infty \to \mathcal{E}nd_V\}$, of all possible representations. To describe this set let us recall a few notions from the Schwarz model [Sc] of the Batalin–Vilkovisky quantization formalism [BaVi].

5.5. Formal quantum BV manifolds. Let $\{x^a, \psi_a, \hbar\}_{1\leq a\leq n}$, $n \in \mathbb{N}$, be a set of formal homogeneous variables of degrees $|x^a| + |\psi_a| = 1$ and $|\hbar| = 2$, and let $\mathcal{O}^{\hbar}_{x,\psi} := \mathbb{K}[[x^a, \psi_a, \hbar]]$ be the associated free graded commutative ring which we view from now on as a $\mathbb{K}[[\hbar]]$-algebra. The degree -1 Lie bracket,

$$\{f \bullet g\} := (-1)^{|f|}\Delta(fg) - (-1)^{|f|}\Delta(f)g - f\Delta(g) \quad \text{for all } f, g \in \mathcal{O}^{\hbar}_{x,\psi},$$

makes $\mathcal{O}^{\hbar}_{x,\psi}$ into a Gerstenhaber $\mathbb{K}[[\hbar]]$-algebra (see §3.6). Here and elsewhere $\Delta := \sum_{a=1}^n (-1)^{|x^a|}\frac{\partial^2}{\partial x^a\partial\psi_a}$. A *quantum master function* is, by definition, a degree 2 element $\Gamma \in \mathcal{O}^{\hbar}_{x,\psi}$ satisfying a so called *quantum master equation*

$$\hbar\Delta\Gamma + \frac{1}{2}\{\Gamma \bullet \Gamma\} = 0. \tag{13}$$

Such an element makes the $\mathbb{K}[[\hbar]]$-module $\mathcal{O}^{\hbar}_{x,\psi}$ *differential* with the differential $\Delta_\Gamma := \hbar\Delta + \{\Gamma\bullet\ \}$. Note that this differential does *not* respect the algebra structure in $\mathcal{O}^{\hbar}_{x,\psi}$ but respects the Poisson brackets.

Consider a group of $\mathbb{K}[[\hbar]]$-algebra automorphisms, $F\colon \mathcal{O}^{\hbar}_{x,\psi} \to \mathcal{O}^{\hbar}_{x,\psi}$, preserving the Lie brackets, $F(\{f \bullet g\}) = \{F(f) \bullet F(g)\}$ (but not necessarily the operator Δ); this group is uniquely determined by a collection, $\mathcal{N} := \{|x^a|, |\psi_a|\}_{1\leq a\leq n}$, of $2n$ integers and is denoted by $\mathrm{Symp}_{\mathcal{N}}$. It is often called a group of *symplectomorphsims* of the Gerstenhaber algebra $(\mathcal{O}^{\hbar}_{x,\psi}, \{\bullet\})$. A remarkable fact [Kh] is that $\mathrm{Symp}_{\mathcal{N}}$ acts on the set of quantum master functions by the formula

$$e^{\frac{F(\Gamma)}{\hbar}} := \left[\mathrm{Ber}\begin{pmatrix} \frac{\partial F(x^a)}{\partial x^b} & \frac{\partial F(x^a)}{\partial \psi_b} \\ \frac{\partial F(\psi_a)}{\partial x^b} & \frac{\partial F(\psi_a)}{\partial \psi_b}\end{pmatrix}\right]^{-\frac{1}{2}} e^{\frac{\Gamma(x,\psi,\hbar)}{\hbar}}. \tag{14}$$

5.5.1. Definition. An equivalence class of pairs, $(\mathcal{O}^{\hbar}_{x,\psi}, \Gamma)$, under the action of the group $\mathrm{Symp}_{\mathcal{N}}$ is called a formal *quantum BV manifold* $\mathcal{M}$ of dimension $\mathcal{N}$. A particular representative, $(\mathcal{O}^{\hbar}_{x,\psi}, \Gamma)$, of $\mathcal{M}$ is called a *Darboux coordinate chart* on $\mathcal{M}$.

In geometric terms, $\mathcal{M}$ is a formal odd symplectic manifold equipped with a special type semidensity [Kh], [Sc]. We need an extra structure on $\mathcal{M}$ which we again define with the help of a Darboux coordinate chart. Notice that the ideals, I_x and I_ψ, in the $\mathbb{K}[[\hbar]]$-algebra $\mathcal{O}^{\hbar}_{x,\psi}$ generated, respectively, by $\{x^a\}_{1\le a\le n}$ and $\{\psi_a\}_{1\le a\le n}$, are also Lie ideals; geometrically, they define a pair of transversally intersecting Lagrangian submanifolds of $\mathcal{M}$. A quantum BV manifold $\mathcal{M}$ is said to have *split quasi-classical limit* (or, slightly shorter, $\mathcal{M}$ is *quasi-classically split*) if it admits a Darboux coordinate chart in which the master function $\Gamma(x,\psi,\hbar) = \sum_{n\ge 0}\Gamma_n(x,\psi)\hbar^n$ satisfies the following two boundary conditions:

$$\Gamma_0 \in I_x I_y, \quad \Gamma_1 \in I_x + I_y.$$

In plain terms, these conditions mean that $\Gamma(x,\psi,\hbar)$ is given by a formal power series of the form

$$\Gamma(x,\psi,\hbar) = \underbrace{\sum_{a,b}\Gamma_{(0)}{}^{a}_{b}\, x^b\psi_a}_{\Gamma_0} + \underbrace{\sum_{\substack{p+q+2n\ge 3\\ p+n\ge 1\\ q+n\ge 1}} \frac{1}{p!q!}\Gamma_{(n)}{}^{b_1\dots b_q}_{a_1\dots a_p}\, x^{a_1}\dots x^{a_p}\psi_{b_1}\dots\psi_{b_q}\hbar^n}_{\mathbf{\Gamma}} \tag{15}$$

for some $\Gamma_{(n)}{}^{b_1\dots b_q}_{a_1\dots a_p} \in \mathbb{K}$. The quantum master equation (13) immediately implies that $\{\Gamma_0, \Gamma_0\} = 0$ so that $\eth := \{\Gamma_0 \bullet\}$ is a differential in the Gerstenhaber algebra $\mathcal{O}^{\hbar}_{x,\psi}$. Then the master equation (13) for a quasi-classically split quantum master function can be equivalently rewritten in the form

$$\eth\mathbf{\Gamma} + \hbar\Delta\mathbf{\Gamma} + \frac{1}{2}\{\mathbf{\Gamma}\bullet\mathbf{\Gamma}\} = 0,$$

where $\mathbf{\Gamma}$ is an element of $\mathcal{O}^{\hbar}_{x,\psi}$ of polynomial order at least 3 (here we set, by definition, the polynomial order of the generators x and ψ equal to 1 and the polynomial order of $\hbar$ equal to 2). The differential $\eth$ induces a differential on the tangent space $\mathcal{T}_*\mathcal{M}$ to $\mathcal{M}$ at the distinguished point; we denote it by the same letter $\eth$. Such a quantum

BV manifold is called *minimal* if $\eth = 0$ and *contractible* if there exists a Darboux coordinate chart in which $\mathbf{\Gamma} = 0$ (i.e., $\Gamma = \eth$) and the tangent complex $(\mathcal{T}_*\mathcal{M}, \eth)$ is acyclic (cf. §3.2.4).

An important class of so called *BF* field theories (see, e.g., [CaRo] and references cited there) have associated quantum BV manifolds which do satisfy the split quasi-classical limit condition.

5.5.2. Proposition. *For any dg vector space V, there is a one-to-one correspondence between representations, $\mathcal{U}\mathcal{L}ie^1\mathcal{B}_\infty \to \mathcal{E}nd_V$, and structures of formal quasi-classically split quantum BV manifold on $\mathcal{M}_{V\oplus V^*[1]}$, the formal manifold associated to $V \oplus V^*[1]$.*

5.6. Morphisms of quantum BV manifolds ([Me7]). The above proposition together with the Koszul duality theory approach to the homotopy transfer outlined in §5.3 provide us with highly non-trivial formulae for constructing quantum BV manifold structures out of dg unimodular Lie 1-bialgebras. We would like to have a *category* of quantum BV manifolds in which such homotopy transfer formulae can be interpreted as morphisms. This can be achieved via the following

5.6.1. Definitions. (i) A *morphism* of quasi-classically split quantum BV manifolds $F: \mathcal{M} \to \widehat{\mathcal{M}}$ is, by definition, a morphism of dg $\mathbb{K}[[\hbar]]$-modules,

$$F: \left(\mathcal{O}_{\widehat{\mathcal{M}}} \simeq \mathcal{O}^\hbar_{\hat{x},\hat{\psi}}, \Delta_{\hat{\Gamma}}\right) \longrightarrow \left(\mathcal{O}_{\mathcal{M}} \simeq \mathcal{O}^\hbar_{x,\psi}, \Delta_\Gamma\right),$$

inducing in the classical limit $\hbar \to 0$ a morphism of algebras, $F|_{\hbar=0}\colon \mathcal{O}_{\hat{x},\hat{\psi}} \to \mathcal{O}_{x,\psi}$ which preserves the ideals, $F|_{\hbar=0}(\langle\hat{x}\rangle) \subset \langle x\rangle$ and $F|_{\hbar=0}(\langle\hat{\psi}\rangle) \subset \langle\psi\rangle$, of the distinguished Lagrangian submanifolds in $\widehat{\mathcal{M}}|_{\hbar=0}$ and $\mathcal{M}|_{\hbar=0}$.

(ii) If $F: \mathcal{M} \to \widehat{\mathcal{M}}$ is a morphism of quantum BV manifolds, then $dF|_{\hbar=0}$ induces in fact a morphism of dg vector spaces, $(\mathcal{T}_*\mathcal{M}, \eth) \to (\mathcal{T}_*\widehat{\mathcal{M}}, \hat{\eth})$; F is called a *quasi-isomorphism* if the latter map induces an isomorphism of the associated cohomology groups.

5.6.2. Theorem ([Me7]). *Every quantum quasi-classically split BV manifold is isomorphic to the product of a minimal one and of a contractible one. In particular, every such manifold is quasi-isomorphic to a minimal one.*

5.7. Homotopy transfer of quantum BV-structures via Feynman integral. Homotopy transfer formulae of $\mathcal{P}_\infty$-structures given by Koszul duality theory are given by sums of decorated graphs which resemble Feynman diagrams in quantum field theory. This resemblance was made a rigorous fact in [Mn] for the case of the wheeled operad of unimodular Lie algebras (see §3.3).

Given any complex V and a dg Lie 1-bialgebra structure on V with degree 0 Lie cobrackets $\Delta^{CoLie}\colon V \to \wedge^2 V$ and degree 1 Lie brackets $[\bullet]\colon \odot^2 V \to V[1]$, the associated by Koszul duality theory homotopy formulae transfer this rather trivial quantum BV manifold structure on V to a highly non-trivial quantum master function on its cohomology $H(V)$; the same formulae can also be described [Me7] by a standard Batalin–Vilkovisky quantization [BaVi] of a BF-type field theory on the space $V \oplus V^*[1]$ with the action given by

$$\begin{aligned} S\colon V \oplus V^*[1] &\longrightarrow \mathbb{K}, \\ p \oplus \omega \longmapsto S(p,\omega) &:= \langle p, d\omega\rangle + \frac{1}{2}\langle p, [\omega,\omega]\rangle + \frac{1}{2}\langle [p \bullet p], \omega\rangle, \end{aligned}$$

where $\langle\ ,\ \rangle$ stands for the natural pairing, and $[\ ,\ ]\colon \odot^2 (V^*[1]) \to V^*[2]$ for the dualization of Δ^{CoLie}. Thus at least in some cases the Koszul duality technique for homotopy transfer of ∞-structures is identical to the Feynman diagram technique in theoretical physics. The beauty of the latter lies in its combinatorial simplicity (due to the Wick theorem), while the power of the former lies in its generality: the Koszul duality theory applies to any (non-commutative case including) quadratic $\mathfrak{G}$-algebras.

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5ECM Amsterdam 2008

Positive definite functions in distance geometry

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Abstract. I. J. Schoenberg proved that a function is positive definite in the unit sphere if and only if this function is a nonnegative linear combination of Gegenbauer polynomials. This fact plays a crucial role in Delsarte's method for finding bounds for the density of sphere packings on spheres and Euclidean spaces.

One of the most exciting applications of Delsarte's method is a solution of the kissing number problem in dimensions 8 and 24. However, 8 and 24 are the only dimensions in which this method gives a precise result. For other dimensions (for instance, three and four) the upper bounds exceed the lower. We have found an extension of Delsarte's method that allows to solve the kissing number problem (as well as the one-sided kissing number problem) in dimensions three and four.

In this paper we also will discuss the maximal cardinalities of spherical two-distance sets. With help of the so-called polynomial method and Delsarte's method these cardinalities can be determined for all dimensions $n < 40$.

Recently, extensions were found of Schoenberg's theorem for multivariate positive-definite functions. With help of these extensions and semidefinite programming some upper bounds for spherical codes can be improved.

Mathematics Subject Classification (2000). 42A85, 52C17, 52C35.

Keywords. Positive definite functions, Delsarte's method, kissing numbers, 2-distance set, semidefinite programming.

1. Introduction

Let M be a metric space with a distance function τ. A real continuous function $f(t)$ is said to be positive definite (p.d.) in M if for arbitrary points $p_1, \dots, p_r$ in M, real variables $x_1, \dots, x_r$, and arbitrary r, we have

$$\sum_{i,j=1}^{r} f(t_{ij})\, x_i x_j \geq 0, \quad t_{ij} = \tau(p_i, p_j),$$

*Research supported in part by NSF grant DMS0807640 and NSA grant MSPF-08G-201.

or, equivalently, the matrix $\big(f(t_{ij})\big) \succeq 0$, where the sign $\succeq 0$ stands for "is positive semidefinite".

Let $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$. Schoenberg [34] proved that:

$$f(\cos\varphi) \textit{ is p.d. in } \mathbb{S}^{n-1} \textit{ if and only if } f(t) = \textstyle\sum_{k=0}^{\infty} f_k G_k^{(n)}(t) \textit{ with all } f_k \geq 0.$$

Here $G_k^{(n)}(t)$ are the Gegenbauer polynomials.

Schoenberg's theorem has been generalized by Bochner [8] to more general spaces. Namely, the following fact holds: *f is p.d. in a 2-point-homogenous space M if and only if $f(t)$ is a nonnegative linear combination of the zonal spherical functions* $\Phi_k(t)$ (see details in [16, Theorem 2], [10, Chapter 9]).

Note that the Bochner–Schoenberg theorem is widely used in coding theory and discrete geometry for finding bounds for error-correcting codes, constant weight codes, spherical codes, sphere packings and other packing problems in 2-point-homogeneous spaces (see [10], [16], [26], [25], [27], [32] and many others).

The paper is organized as follows:

Section 2 recalls definitions of Gegenbauer polynomials and considers Delsarte's method for spherical codes.

Section 3 discusses applications of Delsarte's method to the kissing number problem. One of the most exciting applications of Delsarte's method is a solution of the kissing number problem in dimensions 8 and 24. However, 8 and 24 are the only dimensions in which this method gives a precise result. For other dimensions (for instance, three and four) the upper bounds exceed the lower. We have found an extension of the Delsarte method that allows to solve the kissing number problem (as well as the one-sided kissing number problem) in dimensions three and four.

Section 4 discusses maximal cardinalities of spherical two-distance sets. With help of the so-called polynomial method and Delsarte's method these cardinalities can be determined for all dimensions $n < 40$.

Section 5 considers Sylvester's theorem and semidefinite programming (SDP) bounds for codes. Delsarte's method and its extensions allow to consider the upper bound problem for codes in 2-point-homogeneous spaces as a linear programming problem with perhaps infinitely many variables. We show that by using power sums of distances as variables this problem can be considered as a finite semidefinite programming problem. This method allows to improve some linear programming upper bounds.

Section 6 discusses an application of the extended Schoenberg's theorem to multivariate Gegenbauer polynomials. This extension derives new positive semidefinite constraints for the distance distribution which can be applied to spherical codes.

2. Delsarte's method

2-A. The Gegenbauer polynomials. We recall the definition of Gegenbauer polynomials. Let the polynomials $C_k^{(n)}(t)$ be defined by the expansion

$$(1-2rt+r^2)^{(2-n)/2} = \sum_{k=0}^{\infty} r^k C_k^{(n)}(t).$$

Then the polynomials $G_k^{(n)}(t) := C_k^{(n)}(t)/C_k^{(n)}(1)$ are called *Gegenbauer* or *ultraspherical* polynomials. (So the normalization of $G_k^{(n)}$ is determined by the condition $G_k^{(n)}(1) = 1$.) The Gegenbauer polynomials $G_k^{(n)}$ can also be defined by the recurrence formula

$$G_0^{(n)} = 1, \quad G_1^{(n)} = t, \quad \ldots, \quad G_k^{(n)} = \frac{(2k+n-4)\, t\, G_{k-1}^{(n)} - (k-1)\, G_{k-2}^{(n)}}{k+n-3}.$$

Note that for any even $k \geq 0$ (resp. odd) $G_k^{(n)}(t)$ is even (resp. odd). Therefore, $G_{2\ell}$ and $G_{2\ell+1}$ are orthogonal on $[-1,1]$. Moreover, all polynomials $G_k^{(n)}$ are orthogonal on $[-1,1]$ with respect to the weight function $(1-t^2)^{(n-3)/2}$:

$$\int_{-1}^{1} G_k^{(n)}(t)\, G_\ell^{(n)}(t)\, (1-t^2)^{(n-3)/2}\, dt = 0, \quad k \neq \ell.$$

Recall the addition theorem for Gegenbauer polynomials:

$$\begin{aligned} &G_k^{(n)}(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\varphi) \\ &\quad = \sum_{s=0}^{k} c_{nks}\, G_{k-s}^{(n+2s)}(\cos\theta_1)\, G_{k-s}^{(n+2s)}(\cos\theta_2)\, (\sin\theta_1)^s\, (\sin\theta_2)^s\, G_s^{(n-1)}(\cos\varphi), \end{aligned}$$

where c_{nks} are *positive* coefficients whose values are of no concern here (see [9], [14]).

2-B. Schoenberg's theorem. Using the addition theorem for Gegenbauer polynomials, Schoenberg [34] proved the following theorem:

Theorem 2.1. *A continuous real function $f(\cos\varphi)$ is positive definite in $\mathbb{S}^{n-1}$ if and only if f is a nonnegative linear combination of the Gegenbauer polynomials, i.e., $f(t) = \sum_{k=0}^{\infty} f_k G_k^{(n)}(t)$ with all $f_k \geq 0$.*

This theorem can also be proved with help of the addition theorem for harmonic polynomials (see details in [32]).

2-C. Delsarte's inequality. Let $\{p_1, p_2, \ldots, p_M\}$ be any finite subset of the unit sphere $\mathbb{S}^{n-1}$. By $\varphi_{ij} = \operatorname{dist}(p_i, p_j)$ we denote the spherical (angular) distance between p_i, p_j. Clearly, $\cos\varphi_{ij} = \langle p_i, p_j\rangle$.

If a symmetric matrix is positive semidefinite, then the sum of all its entries is nonnegative. Schoenberg's theorem implies that the matrix $\big(G_k^{(n)}(t_{ij})\big)$ is positive semidefinite, where $t_{ij} := \cos\varphi_{ij}$, i.e.,

$$\sum_{i=1}^{M}\sum_{j=1}^{M} G_k^{(n)}(t_{ij}) \geq 0. \tag{2.1}$$

Suppose a continuous functions $f : [-1, 1] \to \mathbb{R}$ is p.d. in $\mathbb{S}^{n-1}$. Then

$$f(t) = \sum_{k=0}^{\infty} f_k G_k^{(n)}(t)$$

with all $f_k \geq 0$. Let

$$S(X) = S_f(X) := \sum_{i=1}^{M}\sum_{j=1}^{M} f(t_{ij}).$$

Using (2.1), we get

$$S(X) = \sum_{k=0}^{\infty} f_k\Big(\sum_{i=1}^{M}\sum_{j=1}^{M} G_k^{(n)}(t_{ij})\Big) \geq \sum_{i=1}^{M}\sum_{j=1}^{M} f_0 G_0^{(n)}(t_{ij}) = f_0 M^2.$$

Thus

$$S_f(X) \geq f_0 M^2. \tag{2.2}$$

2-D. Delsarte's bound. We say that $X = \{p_1, \ldots, p_M\} \subset \mathbb{S}^{n-1}$ is a *spherical ψ-code,* where $0 < \psi < \pi$, if for all $i \neq j$, $t_{ij} = \cos\phi_{ij} \leq z := \cos\psi$, i.e., $t_{ij} \in [-1, z]$. In other words, the angular separation between distinct points from X is at least ψ. Denote by $A(n, \psi)$ the maximal size of a ψ-code in $\mathbb{S}^{n-1}$.

Theorem 2.2 ([11], [12], [16]). *Let a continuous function $f : [-1, 1] \to \mathbb{R}$ be p.d. in $\mathbb{S}^{n-1}$. Let $f(t) \leq 0$ for all $t \in [-1, \cos\psi]$. Then*

$$A(n, \psi) \leq \frac{f(1)}{f_0}.$$

Proof. Let $X = \{p_1, \ldots, p_M\} \subset \mathbb{S}^{n-1}$ be a spherical ψ-code. Clearly, $f(t_{ii}) = f(1)$. By assumptions we have $f(t_{ij}) \leq 0$ for all $i \neq j$. Therefore

$$S_f(X) = Mf(1) + 2f(t_{12}) + \cdots + 2f(t_{M-1,M}) \leq Mf(1).$$

If we combine this with (2.2), then we get $M \leq f(1)/f_0$. $\square$

3. The kissing problem

3-A. The kissing number problem. The kissing number problem asks for the maximal number $k(n)$ of nonoverlapping spheres of equal size in n-dimensional space that can touch another sphere of the same size. In other words, $k(n) = A(n, \pi/3)$, i.e., $k(n)$ is the maximal size of a spherical $\pi/3$-code of length (dimension) n.

This problem in dimension three was the subject of a famous discussion between Isaac Newton and David Gregory in 1694. In three dimensions the problem was finally solved only in 1953 by Schütte and van der Waerden [36].

In 1979 Levenshtein [19], and, independently, Odlyzko and Sloane [31] (= [10, Chapter 13]), using Delsarte's method, have proved that $k(8) = 240$, and $k(24) = 196560$. Moreover, Bannai and Sloane [5] (= [10, Chapter 14]) proved that the maximal kissing arrangements in these dimensions are unique up to isometry. However, $n = 8, 24$ are the only dimensions in which this method gives a precise result. For other dimensions (for instance, $n = 3, 4$) the upper bounds exceed the lower.

3-B. The kissing problem in dimensions 8 and 24. The proofs in [19], [31] that $k(8) = 240$ and $k(24) = 196560$ are surprisingly short, clean, and technically easier than all known proofs in three dimensions. Indeed, let

$$f_8(t) = (t - 1/2)\, t^2 (t + 1/2)^2 (t + 1) = \sum_{k=0}^{6} f_k^{(8)} G_k^{(8)}(t),$$

and

$$f_{24}(t) = (t - 1/2)(t - 1/4)^2\, t^2 (t + 1/4)^2 (t + 1/2)^2 (t + 1) = \sum_{k=0}^{10} f_k^{(24)} G_k^{(24)}(t).$$

Since all $f_k^{(8)} \geq 0$, $f_k^{(24)} \geq 0$ and $f_8(t) \leq 0$, $f_{24}(t) \leq 0$ for all $t \in [-1, 1/2]$, Theorem 2.2 yields

$$k(8) \leq \frac{f_8(1)}{f_0^{(8)}} = 240, \quad k(24) \leq \frac{f_{24}(1)}{f_0^{(24)}} = 196560.$$

For $n = 8, 24$ the minimal vectors in sphere packings E_8 and Leech lattices give these kissing numbers. Thus $k(8) = 240$, and $k(24) = 196560$.

3-C. The kissing problem in four dimensions. It is not hard to see that $k(4) \geq 24$. Indeed, the unit sphere in $\mathbb{R}^4$ centered at $(0, 0, 0, 0)$ has 24 unit spheres around it, centered at the points $(\pm\sqrt{2}, \pm\sqrt{2}, 0, 0)$, with any choice of signs and any ordering of the coordinates. The convex hull of these 24 points yields a famous 4-dimensional regular polytope – the "24-cell". Its facets are 24 regular octahedra.

Let $f_{OS}(t) = f_0 + f_1 G_1^{(4)}(t) + \cdots + f_9 G_9^{(4)}(t)$, where $f_0 = 1$, $f_1 = 3.6181$, $f_2 = 6.1156$, $f_3 = 7.0393$, $f_4 = 5.0199$, $f_5 = 2.313$, $f_6 = f_7 = f_8 = 0$, $f_9 = 0.4525$. This polynomial was applied by Odlyzko and Sloane [31] to prove that $k(4) \le 25$. Since $f_{OS}(t) \le 0$ for $t \in [-1, 1/2]$, Delsarte's bound gives

$$k(4) = A(4, \pi/3) \le f_{OS}(1)/f_0 = f_{OS}(1) \approx 25.5584.$$

Thus, $k(4) \le 25$.

Note that Arestov and Babenko [1] proved that the bound $k(4) \le 25$ cannot be improved using Delsarte's method.

Let

$$f_4(t) := \frac{1344}{25}t^9 - \frac{2688}{25}t^7 + \frac{1764}{25}t^5 + \frac{2048}{125}t^4 - \frac{1229}{125}t^3 - \frac{516}{125}t^2 - \frac{217}{500}t - \frac{2}{125}.$$

In [26] we proved that $k(4) = 24$. This proof is based on the following two lemmas:

Lemma 3.1. *Let $X = \{x_1, \dots, x_M\}$ be points in the unit sphere $\mathbb{S}^3$. Then*

$$S(X) = \sum_{i=1}^{M}\sum_{j=1}^{M} f_4(\langle x_i, x_j \rangle) \ge M^2.$$

Proof. The expansion of f_4 in terms of $U_k = G_k^{(4)}$ is

$$f_4 = \sum_{i=0}^{9} f_i^{(4)} U_i = U_0 + 2\,U_1 + \frac{153}{25}U_2 + \frac{871}{250}U_3 + \frac{128}{25}U_4 + \frac{21}{20}U_9.$$

We see that all $f_i^{(4)} \ge 0$ and $f_0^{(4)} = 1$. So Lemma 3.1 follows from (2.2). ☐

Lemma 3.2. *Suppose $X = \{x_1, \dots, x_M\}$ is a subset of $\mathbb{S}^3$ such that the angular separation between any two distinct points x_i, x_j is at least $\pi/3$. Then*

$$S(X) = \sum_{i=1}^{M}\sum_{j=1}^{M} f_4(\langle x_i, x_j \rangle) < 25M.$$

It is not easy to prove this lemma. A proof is given in [26, Sections 4, 5, 6].

Theorem 3.1. $k(4) = 24$.

Proof. Let X be a spherical $\pi/3$-code in $\mathbb{S}^3$ with $M = k(4)$ points. Then X satisfies the assumptions in Lemmas 3.1, 3.2. Therefore, $M^2 \le S(X) < 25M$. From this $M < 25$ follows, i.e., $M \le 24$. From the other side we have $k(4) \ge 24$, showing that $M = k(4) = 24$. ☐

3-D. The kissing problem in three dimensions. Our extension of Delsarte's method can be applied to other dimensions and spherical ψ-codes. The most interesting application is a new proof for the Newton–Gregory problem, $k(3) < 13$. In dimension three all computations are technically much easier than for $n = 4$ (see [24]).

Let

$$f_3(t) = \frac{2431}{80}t^9 - \frac{1287}{20}t^7 + \frac{18333}{400}t^5 + \frac{343}{40}t^4 - \frac{83}{10}t^3 - \frac{213}{100}t^2 + \frac{t}{10} - \frac{1}{200}.$$

Then for any M-point kissing arrangement X we have $S_{f_3}(X) \leq 12.88M$ (see details in [23], [24]). The expansion of f_3 in terms of Legendre polynomials $P_k = G_k^{(3)}$ is

$$f_3 = P_0 + 1.6P_1 + 3.48P_2 + 1.65P_3 + 1.96P_4 + 0.1P_5 + 0.32P_9.$$

Since $f_0^{(3)} = 1,\ f_i^{(3)} \geq 0$, we have $S_{f_3}(X) \geq M^2$. Thus, $k(3) \leq 12.88 < 13$.

3-E. The one-sided kissing problem in four dimensions. Let H be a closed half-space of $\mathbb{R}^n$. Suppose S is a unit sphere in H that touches the supporting hyperplane of H. The one-sided kissing number $B(n)$ is the maximal number of unit nonoverlapping spheres in H that can touch S.

If nonoverlapping unit spheres kiss (touch) the unit sphere S in $H \subset \mathbb{R}^n$, then the set of kissing points is an arrangement on the closed hemisphere S_+ of S such that the (Euclidean) distance between any two points is at least 1. So the one-sided kissing number problem can be stated in another way: How many points can be placed on the surface of S_+ so that the angular separation between any two points is at least $\pi/3$? In other words, $B(n)$ is the maximal cardinality of a $\pi/3$-code on the hemisphere S_+.

Clearly, $B(2) = 4$. It is not hard to prove that $B(3) = 9$. Using extensions of Delsarte's method we proved that $B(4) = 18$ (see [25] for a proof and references). Recently several new upper bounds have been obtained for the one-sided kissing numbers [6], [27], [3].

4. Spherical two-distance sets

4-A. Two-distance sets. A set S in Euclidean space $\mathbb{R}^n$ is called a *two-distance set*, if there are two distances c and d, and the distances between pairs of points of S are either c or d. If a two-distance set S lies in the unit sphere $\mathbb{S}^{n-1}$, then S is called a *spherical two-distance set.* In other words, S is a set of unit vectors, there are two real numbers a and b, $-1 \leq a, b < 1$, and inner products of distinct vectors of S are either a or b.

The ratios of distances of two-distance sets are quite restrictive. Namely, Larman, Rogers, and Seidel [17] have proved the following fact: if the cardinality of a two-distance set S in $\mathbb{R}^n$, with distances c and d, $c < d$, is greater than $2n + 3$, then the ratio c^2/d^2 equals $(k-1)/k$ for an integer k with

$$2 \leq k \leq \frac{1+\sqrt{2n}}{2}.$$

Einhorn and Schoenberg [13] proved that there are finitely many two-distance sets S in $\mathbb{R}^n$ with cardinality $|S| \geq n+2$. Delsarte, Goethals, and Seidel [12] proved that the largest cardinality of spherical two-distance sets in $\mathbb{R}^n$ (we denote it by $g(n)$) is bounded by $n(n+3)/2$, i.e.,

$$g(n) \leq \frac{n(n+3)}{2}.$$

Moreover, they give examples of spherical two-distance sets with $n(n+3)/2$ points for $n=2,6,22$. (Therefore, in these dimensions we have equality $g(n)=n(n+3)/2$.) Blockhuis [7] showed that the cardinality of (Euclidean) two-distance sets in $\mathbb{R}^n$ does not exceed $(n+1)(n+2)/2$.

The standard unit vectors $e_1,\dots,e_{n+1}$ form an orthogonal basis of $\mathbb{R}^{n+1}$. Denote by Δ_n the regular simplex with vertices $2e_1,\dots,2e_{n+1}$. Let Λ_n be the set of points $e_i+e_j,\ 1\leq i<j\leq n+1$. Since Λ_n lies in the hyperplane $\sum x_k=2$, we see that Λ_n represents a spherical two-distance set in $\mathbb{R}^n$. On the other hand, Λ_n is the set of mid-points of the edges of Δ_n. Thus,

$$g(n) \geq |\Lambda_n| = \frac{n(n+1)}{2}.$$

For $n<7$ the largest cardinality of Euclidean two-distance sets is $g(n)$, where $g(2)=5,\ g(3)=6,\ g(4)=10,\ g(5)=16$, and $g(6)=27$ (see [21]). However, for $n=7,8$ Lisoněk [21] discovered non-spherical maximal two-distance sets of cardinality 29 and 45 respectively.

4-B. Spherical two-distance sets with $a+b\geq 0$. In [29], using the polynomial method, we proved the following fact:

Theorem 4.1. *Let S be a spherical two-distance set in $\mathbb{R}^n$ with inner products a and b. If $a+b\geq 0$, then*

$$|S| \leq \frac{n(n+1)}{2}.$$

Recently, Nozaki [30] extended this theorem for spherical d-distance sets.

Theorem 4.2. *Let S be a spherical d-distance set in $\mathbb{R}^n$ with inner products $a_1,\dots,a_d$. Let*

$$\prod_{k=1}^{d}(t-a_k) = \sum_{k=0}^{d} f_k G_k^{(n)}(t).$$

Then

$$|S| \leq \sum_{k:f_k>0} h_k,$$

where

$$h_k = \binom{n+k-2}{k} + \binom{n+k-3}{k-1}.$$

4-C. Delsarte's method for spherical two-distance sets. Let S be a spherical two-distance set in $\mathbb{R}^n$ with inner products a and b, where $a > b$. Let $c = \sqrt{2-2a}$, $d = \sqrt{2-2b}$. Then c and d are the Euclidean distances of S.

Let

$$b_k(a) = \frac{ka-1}{k-1}.$$

If k is defined by the equation: $b_k(a) = b$, then $(k-1)/k = c^2/d^2$. Therefore, if $|S| > 2n+3$, then k is an integer number and $k \in \{2, \dots, K(n)\}$ [17]. Here, $K(n) = \lfloor \frac{1+\sqrt{2n}}{2} \rfloor$.

Consider the case $a + b_k(a) < 0$. Since $b_k(a) \geq -1$, we have

$$a \in I_k := \left[\frac{2-k}{k}, \frac{1}{2k-1}\right).$$

Therefore, for fixed n, $k \in \{2, \dots, K(n)\}$, and $a \in I_k$ we have spherical codes with two inner products a and $b_k(a)$. The maximal cardinality of these codes can be bounded by Delsarte's method (see details in [29]). Since for $6 < n < 40, n \neq 22, 23$, Delsarte's bounds are not greater than $n(n+1)/2$, we have that $g(n) = n(n+1)/2$. For $n = 23$ we obtain $g(23) \leq 277$. But $g(23) \geq 276$. This proves the following theorem:

Theorem 4.3. *If* $6 < n < 22$ *or* $23 < n < 40$, *then*

$$g(n) = \frac{n(n+1)}{2}.$$

For $n = 23$ *we have*

$$g(23) = 276 \quad \textit{or} \quad 277.$$

The case $n = 23$ is very interesting. In this dimension the maximal number of equiangular lines (or equivalently, the maximal cardinality of a two-distance set with $a + b = 0$) is 276 [18]. On the other hand, $|\Lambda_{23}| = 276$. So in 23 dimensions we have two very different two-distance sets with 276 points.

Note that for $n = 23$ Delsarte's bound is ≈ 277.095. So this numerical bound is not far from 277. Perhaps stronger tools, such as semidefinite programming bounds, are needed here to prove that $g(23) = 276$.

Our numerical calculations show that the barrier $n = 40$ is in fact fundamental: Delsarte's bounds are incapable of resolving the $n \geq 40$, $k = 2$ case. That means: a new idea is required to deal with $n \geq 40$.

5. Sylvester's theorem and SDP bounds for codes

5-A. 2-point-homogeneous spaces. We say that a G-space $\boldsymbol{M}$ is a *2-point-homogeneous space* if (i) $\boldsymbol{M}$ is a metric space with a distance ρ defined on it; (ii) $\boldsymbol{M}$ is a set on which a group G acts; (iii) ρ is strongly invariant under G, i.e., for $x, x', y, y' \in \boldsymbol{M}$ with $\rho(x, y) = \rho(x', y')$ there is an element $g \in G$ such that $g(x) = x'$ and $g(y) = y'$.

These assumptions are quite restrictive. In fact, if G is infinite and $\boldsymbol{M}$ is a compact space, then Wang [39] has proved that $\boldsymbol{M}$ is a sphere; or a real, complex or quaternionic projective space; or the Cayley projective plane. However, the finite 2-point-homogeneous spaces have note yet been completely classified (for the most important examples and references, see [10]).

With any 2-point-homogeneous space $\boldsymbol{M}$ and an integer number $k \geq 0$ are associated the *zonal spherical functions* $\Phi_k(t)$ such that $\{\Phi_k(t)\}_{k=0,1,2,\ldots}$ are orthogonal on $T := \{\tau(x, y) : x, y \in \boldsymbol{M}\}$, where τ is the certain function in ρ (i.e., $\tau(x, y) = F(\rho(x, y))$) defined by $\boldsymbol{M}$. For all continuous compact $\boldsymbol{M}$ and for all currently known finite cases, $\Phi_k(t)$ *is a polynomial of degree* k. The normalization is given by the rule: $\Phi_k(\tau_0) = 1$, where $\tau_0 := \tau(x, x)$. Then $\Phi_0(t) = 1$.

Note that if $\boldsymbol{M}$ is a Hamming space $\boldsymbol{F}_2^n$ with $\tau(x, y) = \rho(x, y) =$ Hamming distance, then $\tau_0 = 0$. Here $\Phi_k(t)$ is the Krawtchouk polynomial $K_k(t, n)$. Consider the case $\boldsymbol{M}$ = unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ with $\tau(x, y) = \cos \rho(x, y) = \langle x, y \rangle$, where $\rho(x, y)$ is the angular distance between x and y. Then the corresponding zonal spherical function $\Phi_k(t)$ is the Gegenbauer polynomial $G_k^{(n)}(t)$.

5-B. The Bochner–Schoenberg theorem. The main property for zonal spherical functions is called "positive-definite degenerate kernels" or "p.d.k." [10]. This property first was discovered by Bochner [8] (general spaces) and independently for spherical spaces by Schoenberg [34].

Now we explain what the p.d.k. property means for finite subsets in $\boldsymbol{M}$.

Theorem 5.1 ([8], [34], [16]). *Let $\boldsymbol{M}$ be a 2-point-homogeneous space. Then for any integer $k \geq 0$ and for any finite $C = \{x_i\} \subset \boldsymbol{M}$ the matrix $\big(\Phi_k(\tau(x_i, x_j))\big)$ is positive semidefinite.*

This theorem implies the fact that plays a crucial role for the linear programming bounds. For any positive semidefinite matrix the sum of its entries is nonnegative. Then

Theorem 5.2 ([11], [12], [16], [31]). *For any finite $C = \{x_i\} \subset \boldsymbol{M}$ we have*

$$\sum_{i=1}^{|C|}\sum_{j=1}^{|C|} \Phi_k(\tau(x_i, x_j)) \geq 0.$$

Since $\Phi_k(\tau(x_i, x_i)) = \Phi_k(\tau_0) = 1$, this theorem implies

$$\frac{1}{|C|}\sum_{i,j:i\neq j} \Phi_k(\tau(x_i, x_j)) \geq -1, \quad k = 0, 1, 2, \ldots. \tag{5.1}$$

5-C. The linear programming bounds. Let S be a fixed subset of T. We say that a finite subset $C \subset \boldsymbol{M}$ is an *S-code* if $\tau(x, y) \in S$ for all $x, y \in C, x \neq y$. The largest cardinality $|C|$ of an S-code will be denoted by $A(\boldsymbol{M}, S)$.

The *distance distribution* $\{\alpha_t\}$ of C is defined by

$$\alpha_t := \frac{1}{|C|}(\text{number of ordered pairs } x, y \in C \text{ with } \tau(x, y) = t).$$

We obviously have

$$\alpha_{\tau_0} = 1, \quad \alpha_t \geq 0,\ t \in T, \quad \sum_{t\in T}\alpha_t = |C|. \tag{5.2}$$

(5.1) and (5.2) make it possible to regard the problem of bounding $A(\boldsymbol{M}, S)$ as a linear programming problem:

Primal problem (*LPP*): Choose a natural number s, a subset $\{\tau_1, \ldots, \tau_s\}$ of S, and real numbers $\alpha_{\tau_1}, \ldots, \alpha_{\tau_s}$ so as to

$$\text{maximize } \alpha_{\tau_1} + \cdots + \alpha_{\tau_s}$$

subject to

$$\alpha_{\tau_i} \geq 0,\ i = 1, \ldots, s, \quad \sum_{i=1}^{s}\alpha_{\tau_i}\Phi_k(\tau_i) \geq -1,\ k = 0, 1, \ldots.$$

This is a linear programming problem with perhaps infinitely many unknowns α_t and constraints (5.1), (5.2). If C is an S-code then its distance distribution certainly satisfies the constraints (5.1), (5.2). So if the maximal value of the sum $\alpha_{\tau_1} + \cdots + \alpha_{\tau_s}$

that can be attained is A^*, then $A(\boldsymbol{M}, S) \le 1 + A^*$. (The extra 1 arises because the term $\alpha_{\tau_0} = 1$ does not occur in this sum.)

Dual problem (*LPD*): Choose a natural number N and real numbers $f_1, \dots, f_N$ so as to

$$\text{minimize} \;\; f_1 + \cdots + f_N$$

subject to

$$f_k \ge 0, \;\; k = 1, \dots, N, \quad \sum_{k=1}^{N} f_k \,\Phi_k(t) \le -1, \;\; t \in S.$$

Thus, we have

Theorem 5.3. *If A^* is the optimal solution to either of the primal or dual problems, then $A(\boldsymbol{M}, S) \le 1 + A^*$.*

5-D. Sylvester's theorem. Sylvester's theorem (see [33], [27]) gives an answer to the following question: Suppose we know for complex numbers $t_1, \dots, t_n$ only its power sums s_k, where s_k are real numbers. How to determine the number of real t_i in an interval $[a, b]$?

Let

$$R_m := \begin{pmatrix} s_0 & s_1 & \dots & s_{m-1} \\ s_1 & s_2 & \dots & s_m \\ \vdots & \vdots & \ddots & \vdots \\ s_{m-1} & s_m & \dots & s_{2m-2} \end{pmatrix},$$

$$F_m^+(a) := \begin{pmatrix} s_1 - a s_0 & s_2 - a s_1 & \dots & s_m - a s_{m-1} \\ s_2 - a s_1 & s_3 - a s_2 & \dots & s_{m+1} - a s_m \\ \vdots & \vdots & \ddots & \vdots \\ s_m - a s_{m-1} & s_{m+1} - a s_m & \dots & s_{2m-1} - a s_{2m-2} \end{pmatrix},$$

$$F_m^-(b) := \begin{pmatrix} b s_0 - s_1 & b s_1 - s_2 & \dots & b s_{m-1} - s_m \\ b s_1 - s_2 & b s_2 - s_3 & \dots & b s_m - s_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ b s_{m-1} - s_m & b s_m - s_{m+1} & \dots & b s_{2m-2} - s_{2m-1} \end{pmatrix},$$

and

$$H_m(a,b) = H_m(s_0, s_1 \dots, s_{2m-1}, [a,b]) := \begin{pmatrix} R_m & 0 & 0 \\ 0 & F_m^+(a) & 0 \\ 0 & 0 & F_m^-(b) \end{pmatrix}.$$

Theorem 5.4. *Consider real numbers $t_1, \dots, t_n$ in $[a, b]$. Then for any natural number m the matrix $H_m(a, b)$ is positive semidefinite. Moreover, for $m \geq \operatorname{rank}(R_n)$ the converse holds, i.e., if $H_m(a, b) \succeq 0$, then complex numbers $t_1, \dots, t_n$ with real s_k are real numbers in $[a, b]$.*

In fact, the constraint $H_m(a, b) \succeq 0$ does not depend on $n = s_0$. Indeed, let

$$\bar{s}_k := \frac{t_1^k + \cdots + t_n^k}{n} = \frac{s_k}{n}, \qquad \bar{H}_m(a, b) = \frac{1}{n} H_m(a, b).$$

In other words, $\bar{H}_m(a, b)$ can be obtained by substituting $\bar{s}_k$ for s_k in $H_m(a, b)$:

$$\bar{H}_m(a, b) = \bar{H}_m(\bar{s}_1, \dots, \bar{s}_{2m-1}, [a, b]) := H_m(1, \bar{s}_1, \dots, \bar{s}_{2m-1}, [a, b]).$$

Thus $H_m(a, b) \succeq 0$ if and only if $\bar{H}_m(a, b) \succeq 0$.

5-E. Semidefinite programming. The standard form of the SDP problem is the following [37], [38]:

Primal problem:

$$\text{minimize} \quad c_1 x_1 + \cdots + c_\ell x_\ell$$

subject to

$$X \succeq 0, \quad \text{where} \quad X = T_1 x_1 + \cdots + T_\ell x_\ell - T_0.$$

Dual problem:

$$\text{maximize} \quad \langle T_0, Y \rangle$$

subject to

$$\langle T_i, Y \rangle = c_i, \quad i = 1, \dots \ell,$$

$$Y \succeq 0.$$

Here $T_0, T_1, \dots, T_\ell, X$, and Y are real $N \times N$ symmetric matrices, $(c_1, \dots, c_\ell)$ is a cost vector, $(x_1, \dots, x_\ell)$ is a variable vector, and by $\langle A, B \rangle$ we denote the inner product, i.e., $\langle A, B \rangle = \operatorname{Tr}(AB) = \sum a_{ij} b_{ij}$.

5-F. The SDP bounds. From here on we assume that $\Phi_k(t)$ is a polynomial of degree k, $\Phi_k(\tau_0) = 1$, and $S = T \cap [a, b]$ (the most interesting case for coding theory and sphere packings).

In fact, the optimal solution A^* of the LPP and LPD problems in Section 5-C depends only on the family of polynomials $\Phi := \{\Phi_k(t)\}_{k=0,1,\dots}$ and $[a, b]$. We denote $1 + A^*$ by $\mathrm{LP}(\Phi, [a, b])$.

Since

$$\Phi_k(t) = p_{k0} + p_{k1} t + \cdots + p_{kk} t^k,$$

we have

$$\Phi_k(t_1) + \cdots + \Phi_k(t_\ell) = \sum_{d=0}^{k} p_{kd} s_d, \quad s_d = t_1^d + \cdots + t_\ell^d.$$

Let $C = \{v_i\}$ be an S-code on $\boldsymbol{M}$, and let $\tau_{i,j} = \tau(v_i, v_j)$. Note that the number of ordered pairs (v_i, v_j), $i \neq j$, equals $n = |C|(|C|-1)$. Then (5.1) can be written in the form

$$y + p_{k0} + \sum_{d=1}^{k} p_{kd} x_d \geq 0, \tag{5.3}$$

where

$$y = \frac{1}{|C|-1}, \quad x_d = \bar{s}_d = \frac{s_d}{n}, \quad s_d = \sum_{i,j,i \neq j} \tau_{i,j}^d.$$

From Theorem 5.4 we have for any m that

$$\bar{H}_m(x_1, \ldots, x_{2m-1}, [a,b]) \succeq 0. \tag{5.4}$$

Now we introduce the simplest SDP bound.

SDP$_0$ problem: Choose a natural number m and real numbers $y, x_1, \ldots, x_{2m-1}$ so as to

$$\text{minimize} \quad y$$

subject to

$$y + \sum_{i=1}^{k} p_{ki} x_i \geq -p_{k0}, \quad k = 1, \ldots, 2m-1,$$

$$\bar{H}_m(x_1, \ldots, x_{2m-1}, [a,b]) \succeq 0.$$

Note that in (5.3), $|C| = (1+y)/y$. Thus

Theorem 5.5. *If y^* is the optimal solution of the SDP$_0$ problem, then*

$$A(\boldsymbol{M}, S) \leq \mathrm{SDP}_0(\Phi, [a,b], m) := \frac{1+y^*}{y^*}.$$

Since we just substituted $H_m \succeq 0$ for $t \in S$ in the LPP problem, we can expect that $\mathrm{SDP}_0(\Phi, [a,b]) = \mathrm{LP}(\Phi, [a,b])$ (see details in [27]).

In fact, for a continuous $\boldsymbol{M}$ the LPP and LPD problems are not finite linear programming problems. These problems can be solved only via discretization. For instance, Odlyzko and Sloane [31] (≡ [10, Chapter 13]) applied LPD for upper bounds on kissing numbers, where they replaced S by 1001 equidistant points in S. For the LPP problem it is not clear how to do a discretization of $\{\alpha_\tau\}$. On the other hand, for a given m the SDP_0 is a finite primal SDP problem. As a by-product of solutions of this problem we have bounds on $|C|$ and power sums s_k (see [27, Section 5]).

In [27, Section 6] it is shown that some recent extensions of Delsarte's method can be reformulated as SDP problems (SDPA). Section 7 extends the SDPA bounds to subsets of a 2-point-homogeneous space and shows that some upper bounds for codes can be improved. In particular we obtain new bounds of one-sided kissing numbers.

6. Multivariate positive definite functions

6-A. Schrijver's method. Recently, Schrijver [35] using semidefinite programming (SDP) improved some upper bounds on binary codes. Even more recently, Schrijver's method has been adapted for non-binary codes (Gijswijt, Schrijver, and Tanaka [15]), and for spherical codes (Bachoc and Vallentin [2], [3], [4]). In fact, this method using the stabilizer subgroup of the isometry group derives new positive semidefinite constraints which are stronger than linear inequalities in the Delsarte linear programming method. We consider and extend this method for spherical codes in [28]. Note that this approach is different from the method considered in Section 5.

6-B. Multivariate positive definite functions on spheres. Let us consider the following problem: *For given points $Q = \{q_1, \dots, q_m\}$ in M to describe the class of continuous functions $F(t, \boldsymbol{u}, \boldsymbol{v})$ in $2m + 1$ variables with $t \in \mathbb{R}$, $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^m$, $F(t, \boldsymbol{u}, \boldsymbol{v}) = F(t, \boldsymbol{v}, \boldsymbol{u})$ such that for arbitrary points $p_1, \dots, p_r$ in M the matrix $\big(F(t_{ij}, \boldsymbol{u}_i, \boldsymbol{u}_j)\big) \succeq 0$, where $t_{ij} = \tau(p_i, p_j)$, $\boldsymbol{u}_i = (\tau(p_i, q_1), \dots, \tau(p_i, q_m))$.*

Denote this class by $\mathrm{PD}(M, Q)$. If $Q = \emptyset$, then $\mathrm{PD}(M, Q)$ is the class of p.d. functions in M. In this case an answer is given by the Bochner–Schoenberg theorem.

Let $0 \le m \le n - 2$, $t \in \mathbb{R}$, $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^m$ for $m > 0$, and $\boldsymbol{u} = \boldsymbol{v} = 0$ for $m = 0$. Then the following polynomial in $2m + 1$ variables of degree k in t is well defined:

$$G_k^{(n,m)}(t, \boldsymbol{u}, \boldsymbol{v}) := (1 - |\boldsymbol{u}|^2)^{k/2} (1 - |\boldsymbol{v}|^2)^{k/2} G_k^{(n-m)} \left(\frac{t - \langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\sqrt{(1 - |\boldsymbol{u}|^2)(1 - |\boldsymbol{v}|^2)}} \right).$$

In [28] we proved the following theorem:

Theorem 6.1. *Let $0 \le m \le n - 2$. Let $Q = \{q_1, \dots, q_m\} \subset \mathbb{S}^{n-1}$ with $\operatorname{rank}(Q) = m$. Let $e_1, \dots, e_m$ be an orthonormal basis of the linear space with the basis $q_1, \dots, q_m$, and let L_Q denotes the linear transformation of coordinates. Then*

$F \in \mathrm{PD}(\mathbb{S}^{n-1}, Q)$ *if and only if*

$$F(t, \boldsymbol{u}, \boldsymbol{v}) = \sum_{k=0}^{\infty} f_k(\boldsymbol{u}, \boldsymbol{v})\, G_k^{(n,m)}(t, L_Q(\boldsymbol{u}), L_Q(\boldsymbol{v})),$$

where $f_k(\boldsymbol{u}, \boldsymbol{v}) \succeq 0$ *(i.e.,* $f_k(\boldsymbol{u}, \boldsymbol{v}) = h_1(\boldsymbol{u})h_1(\boldsymbol{v}) + \cdots + h_\ell(\boldsymbol{u})h_\ell(\boldsymbol{v})$*) for all* $k \geq 0$.

Note that for the case $m = 0$ this is Schoenberg's theorem [34], and for $m = 1$ it is the Bachoc–Vallentin theorem [3].

6-C. Upper bounds for spherical codes. In this subsection we set up upper bounds for spherical codes which are based on multivariate p.d. functions. These bounds extend the famous bound of Delsarte. Note that for the case $m = 1$ this bound is the Bachoc–Vallentin bound [4].

Definition 6.1. Consider a vector $J = (j_1, \ldots, j_d)$. Split the set of numbers $\{j_1, \ldots, j_d\}$ into maximal subsets $I_1, \ldots, I_k$ with equal elements. That means, if $I_r = \{j_{r_1}, \ldots, j_{r_s}\}$, then $j_{r_1} = \cdots = j_{r_s} = a_r$ and all other $j_\ell \neq a_r$. Without loss of generality it can be assumed that $i_1 = |I_1| \geq \cdots \geq i_k = |I_k| > 0$. (Note that we have $i_1 + \cdots + i_k = d$.) Denote by $\psi(J)$ the vector $\omega = (i_1, \ldots, i_k)$.

Let

$$W_d := \{\omega = (i_1, \ldots, i_k) : i_1 + \cdots + i_k = d,\ i_1 \geq \cdots \geq i_k > 0,\ i_1, \ldots, i_k \in \mathbb{Z}\}.$$

Let $\omega \in W_d$. Denote

$$\tilde{q}_\omega(N) := \#\{J = (j_1, \ldots, j_d) \in \{1, \ldots, N\}^d : \psi(J) = \omega\},$$

$$q_\omega(N) := \frac{\tilde{q}_\omega(N)}{N}.$$

It is not hard to see that if $\omega \in W_d$, then $q_\omega(N)$ is a polynomial of degree $d - 1$, and

$$\sum_{\omega \in W_d} q_\omega(N) = N^{d-1}.$$

Definition 6.2. For any vector $\boldsymbol{x} = \{x_{ij}\}$ with $1 \leq i < j \leq d$ denote by $A(\boldsymbol{x})$ a symmetric $d \times d$ matrix $\left(a_{ij}\right)$ with all $a_{ii} = 1$ and $a_{ji} = a_{ij} = x_{ij}$, $i < j$.

Let $0 < \theta < \pi$ and

$$X(\theta) := \{\boldsymbol{x} = \{x_{ij}\} : x_{ij} \in [-1, \cos\theta] \text{ or } x_{ij} = 1,\ 1 \leq i < j \leq d\}.$$

Now for any $\boldsymbol{x} = \{x_{ij}\} \in X(\theta)$ we define a vector $J(\boldsymbol{x}) = (j_1, \ldots, j_d)$ such that $j_k = k$ if there are no $i < k$ with $x_{ik} = 1$, otherwise $j_k = i$, where i is the minimum index with $x_{ik} = 1$.

Let $\omega \in W_d$. Denote

$$D_\omega(\theta) := \{\boldsymbol{x} \in X(\theta) : \psi(J(\boldsymbol{x})) = \omega \text{ and } A(\boldsymbol{x}) \succeq 0\}.$$

Let $f(\boldsymbol{x})$ be a real function in $\boldsymbol{x}$, and let

$$B_\omega(\theta, f) := \sup_{\boldsymbol{x} \in D_\omega(\theta)} f(\boldsymbol{x}).$$

Note that the assumption $A(\boldsymbol{x}) \succeq 0$ implies existence of unit vectors $p_1, \dots, p_d$ such that $A(\boldsymbol{x})$ is the Gram matrix of these vectors, i.e., $x_{ij} = \langle p_i, p_j \rangle$. Moreover, if $x_{ij} = 1$, then $p_i = p_j$. In particular, $D_{(d)}(\theta) = \{(1, \dots, 1)\}$ and therefore $B_{(d)}(\theta, f) = f(1, \dots, 1)$.

Definition 6.3. Let $\boldsymbol{x} = \{x_{ij}\}$, where $1 \le i < j \le m + 2 \le n$, and let $A(\boldsymbol{x}) \succeq 0$. Then there exist $P = \{p_1, \dots, p_{m+2}\} \subset \mathbb{S}^{n-1}$ such that $x_{ij} = \langle p_i, p_j \rangle$. Let $F(\boldsymbol{x})$ be a continuous function in $\boldsymbol{x}$ with $F(\tilde{\mathbf{x}}_{k\ell}) = F(\boldsymbol{x})$ for all $\tilde{\mathbf{x}}_{k\ell}$ that can be obtained by interchanging two points p_k and p_ℓ in P. We say that $F(\boldsymbol{x}) \in \mathrm{PD}^n_m$ if for all $\boldsymbol{x}$ with $A(\boldsymbol{x}) \succeq 0$ we have $\tilde{F}(x_{12}, \boldsymbol{u}_1, \boldsymbol{u}_2) \in \mathrm{PD}(\mathbb{S}^{n-1}, Q(\boldsymbol{x}))$, where $\boldsymbol{u}_i = (x_{i3}, \dots, x_{i,m+2})$, $Q(\boldsymbol{x}) = \{p_3, \dots, p_{m+2}\}$, and $\tilde{F}(x_{12}, \boldsymbol{u}_1, \boldsymbol{u}_2) = F(\boldsymbol{x})$.

For the classical case $m = 0$ Schoenberg's theorem says that $f \in \mathrm{PD}^n_0$ if and only if

$$f(t) = \sum_k f_k G_k^{(n)}(t)$$

with all $f_k \ge 0$. Using Theorem 6.1 it is not hard to describe the class of functions in PD^n_m for all $m \le n - 2$.

Let C be an N-element subset of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. It is called an (n, N, θ) *spherical code* if every pair of distinct points (c, c') of C have inner product $\langle c, c' \rangle$ at most $\cos\theta$.

In [28] we proved the following theorem.

Theorem 6.2. *Let* $f_0 > 0$, $0 \le m \le n - 2$, *and* $F(\boldsymbol{x}) = f(\boldsymbol{x}) - f_0 \in \mathrm{PD}^n_m$. *Then an* (n, N, θ) *spherical code satisfies*

$$f_0 N^{m+1} \le \sum_{\omega \in W_{m+2}} B_\omega(\theta, f)\, q_\omega(N).$$

It is easy to see for $m = 0$ that $q_{(2)}(N) = 1$, $q_{(1,1)}(N) = N - 1$, and $B_{(2)}(\theta, f) = f(1)$. Therefore, from Theorem 6.2 we have

$$f_0 N \le f(1) + B_{(1,1)}(\theta, f)(N - 1).$$

Suppose $B_{(1,1)}(\theta, f) \le 0$, i.e., $f(t) \le 0$ for all $t \in [-1, \cos\theta]$. Thus for (n, N, θ) spherical code we obtain Delsarte's bound

$$N \le \frac{f(1)}{f_0}.$$

The Bachoc–Vallentin bound [4, Theorem 4.1] is the bound in Theorem 6.2 for $m = 1$ and $B_{(1,1,1)}(\theta, f) \le 0$. Indeed, let $B_{(2,1)}(\theta, f) \le B$. Since $q_{(3)}(N) = 1$, $q_{(2,1)}(N) = 3(N-1)$, and $B_{(3)}(\theta, f) = f(1,1,1)$, we have

$$f_0 N^2 \le f(1,1,1) + 3(N-1)B.$$

Let us consider Theorem 6.2 also for the case $m = 2$ with $B_{(1,1,1,1)}(\theta, f) \le 0$. Let $B_{(3,1)}(\theta, f) \le B_1$, $B_{(2,2)}(\theta, f) \le B_2$, and $B_{(2,1,1)}(\theta, f) \le B_3$. Then

$$f_0 N^3 \le f(1,1,1,1,1,1) + 4(N-1)B_1 + 3(N-1)B_2 + 6(N-1)(N-2)B_3.$$

Let $f(\boldsymbol{x})$ be a polynomial of degree d. Then the assumptions in Theorem 6.2 can be written as positive semidefinite constraints for the coefficients of F (see for details [2], [3], [4], [15], [35]). Actually, the bound given by Theorem 6.1 can be obtained as a solution of an SDP (semidefinite programming) optimization problem. In [2], [3], Bachoc and Vallentin, using numerical solutions of the SDP problem for the case $m = 1$, have obtained new upper bounds for the kissing numbers and for the one-sided kissing numbers in several dimensions $n \le 10$.

However, the dimension of the corresponding SDP problem grows so fast whenever d and m are increasing that this problem can be treated numerically only for relatively small d and small m. It is an interesting problem to find (explicitly) suitable polynomials F for Theorem 6.2 and using them to obtain new bounds for spherical codes.

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5ECM Amsterdam 2008

From sparse graphs to nowhere dense structures: decompositions, independence, dualities and limits

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Abstract. This paper surveys the recent research related to the structural properties of sparse graphs and general finite relational structures. We provide a classification of classes of finite relational structures which is defined by means of a sequence of derived classes (defined by local changes). The resulting classification is surprisingly robust and it leads to many equivalent formulations which play an important role in applications in model theory, algorithmic graph theory and structural theory. One of these equivalent formulations guarantees a quick test for a finite type decomposition (called low tree depth decomposition). We give numerous examples from geometry and extremal graph theory and apply the result to dualities for special classes of graphs. Finally we characterize the existence of all restricted dualities in terms of limit objects induced by the convergence defined on the homomorphism order of graphs.

Mathematics Subject Classification (2000). Primary 05C75; Secondary 05C15, 05C83, 05C85

Keywords. Homomorphism, finite structure, duality theorem, decomposition, algorithm, shallow minor, graph classes, extremal graph theory.

Introduction

Dense graphs have been extensively studied in the context of extremal graph theory. The outstanding Szemerédi Regularity Lemma [51] states that any dense network has properties which are close to the ones of a random graph. In particular, a large dense network cannot be too irregular. This structural result is one of the cornerstones of contemporary combinatorics (and one would like to say of mathematics in general). It also led to manifold applications and generalizations, see e.g. [22], [21], [24], [52], [13]. The research covered by this paper is related to the recent development which is based on the study of *homomorphisms* of graphs (and structures). (It is perhaps of interest to note in how many different areas and what variety of contexts the notion of a homomorphism recently appeared, see e.g. [19].) The main idea is to study the local structure of a large graph G by counting the homomorphisms from various small graphs F into G (this relates to the area called *property testing*), and to study

*Supported by grant 1M0021620808 of the Czech Ministry of Education and AEOLUS.

the global structure of G by *counting* its homomorphisms into various small graphs H (sometimes interpreted as *templates*). Regularity is viewed here as a structural approximation in a proper metric and also as a convergence. For a survey of this development see [3]. Very schematically, this may be outlined by the diagram

$$\begin{tikzcd}
F_1 \arrow[dr] & & H_1 \\
\cdots \arrow[r] & G \arrow[ur] \arrow[r] \arrow[dr] & \cdots \\
F_p \arrow[ur] & & H_q.
\end{tikzcd}$$

This approach proved to be very fruitful and relates (among others) to the notion of *quasi-random* graphs, see e.g. [5], and to the full characterizations of testable graph properties, see e.g. [1], [3].

Nevertheless, such an approach becomes difficult when the considered structures become sparse (see [2], [14] for results extending this approach to sparse graphs). In particular, Szemerédi's regularity lemma concerns graphs which have (at least locally) m edges which is quadratic with respect to the number n of vertices, or at least as large as $n^{1+\epsilon}$ if one consider extensions and generalizations of this lemma to the sparse context, see e.g. [21]. This area is a subject of an intensive research see e.g. [15].

In this paper we take a different approach via the homomorphism order. We shall see that in this setting, at a proper level of generality, some of the results for dense graphs can be extended to the world of sufficiently sparse classes of graphs. This leads to a new classification of classes of structures (nowhere dense vs somewhere dense classes) which is very robust with respect to standard graph operations (such as contraction of edges and cloning of vertices). We display a variety of equivalent definitions of this classification which in turn leads to applications in extremal combinatorics, complexity of algorithms and model theory. This classification also extends our earlier work on classes with bounded expansion [31], [32]. In the second part of this paper we deal with dualities, i.e., with those classes of structures which can be alternatively described either by finitely many forbidden substructures or as a Constraint Satisfaction Problem. Such instances are called finite dualities and we characterize classes which have all restricted dualities (Theorem 27). To do so we develop the continuous version of our classification. This part is very recent and we include several proofs. Although in most of this paper we deal with graphs, the results can be extended to oriented graphs, hypergraphs and to general relational structures by means of an appropriate construction (such as Gaifman graph, 2-section, bigraph of incidence).

Part I Classification of graph classes

1. The nowhere dense/somewhere dense dichotomy

1.1. Classification by shallow minors. In the following we consider finite simple undirected graphs, except when explicitly stated otherwise, and we denote by $\mathcal{G}$ the class of all such graphs.

We use standard graph theory terminology, however, we find it useful to introduce the following: for a graph $G = (V, E)$, we denote by $|G|$ the *order* of G (that is: $|V|$) and by $\|G\|$ the *size* of a graph of G (that is: $|E|$).

The *distance* in a graph G between two vertices x and y is the minimum length of a path linking x and y (or ∞ if x and y do not belong to the same connected component of G) and is denoted by $\mathrm{dist}_G(x, y)$. Let $G = (V, E)$ be a graph and let d be an integer. The *d-neighborhood* $N_d^G(u)$ of a vertex $u \in V$ is the subset of vertices of G at distance at most d from u in G: $N_d^G(u) = \{v \in V : \mathrm{dist}_G(u, v) \leq d\}$.

A class $\mathcal{C}$ of graphs is *hereditary* if every induced subgraph of a graph in $\mathcal{C}$ belongs to $\mathcal{C}$, and it is *monotone* of every subgraph of a graph in $\mathcal{C}$ belongs to $\mathcal{C}$.

For any graphs H and G and any integer d, the graph H is said to be a *shallow minor* of G at *depth d* ([48] attributes this notion, then called *low depth minor*, to Ch. Leiserson and S. Toledo) if there exists a subset $\{x_1, \dots, x_p\}$ of G and a collection $\mathcal{P}$ of disjoint subsets $V_1 \subseteq N_d^G(x_1), \dots, V_p \subseteq N_d^G(x_p)$ such that H is a subgraph of the graph $G/\mathcal{P}$ obtained from G by contracting each V_i into x_i and removing loops and multiple edges (see Figure 1).

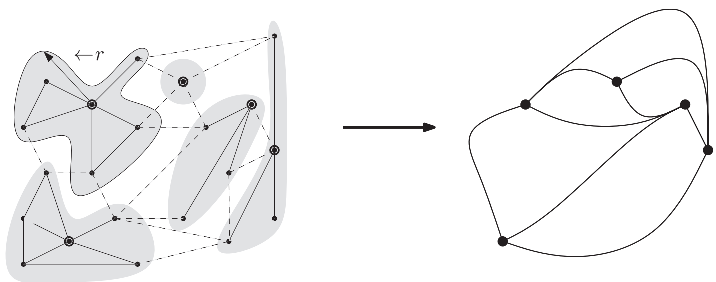


Figure 1. A shallow minor of depth r of a graph G is a simple subgraph of a minor of G obtained by contracting vertex disjoints subgraphs with radius at most r.

The set of all shallow minors of G at depth d is denoted by $G \triangledown d$. In particular, $G \triangledown 0$ is the set of all subgraphs of G. Hence we have the following non decreasing sequence of classes:

$$G \in G \triangledown 0 \subseteq G \triangledown 1 \subseteq \cdots \subseteq G \triangledown d \subseteq \cdots G \triangledown \infty.$$

We extend this definition to arbitrary class of graphs $\mathcal{C}$ by

$$\mathcal{C} \triangledown d = \bigcup_{G \in \mathcal{C}} G \triangledown d.$$

We have the following hierarchy of classes:

$$\mathcal{C} \subseteq \mathcal{C} \triangledown 0 \subseteq \mathcal{C} \triangledown 1 \subseteq \cdots \subseteq \mathcal{C} \triangledown d \subseteq \cdots \mathcal{C} \triangledown \infty.$$

We call this sequence *minor resolution* of the class $\mathcal{C}$. Note that $\mathcal{C} \triangledown 0$ is the monotone closure of $\mathcal{C}$ and that $\mathcal{C} \triangledown \infty$ is the minor closed class generated by $\mathcal{C}$. The minor resolution of a class leads to a classification of classes and to their interesting properties. The following is the key definition of this paper:

Definition 1. A class of graphs $\mathcal{C}$ is *somewhere dense* if there exists an integer d such that $\mathcal{C} \triangledown d = \mathcal{G}$. In other words: $\mathcal{C}$ is somewhere dense if every graph is a bounded depth shallow minor of a graph in $\mathcal{C}$.

If an infinite class is not somewhere dense, it is *nowhere dense*.

For relational structures we can define analogous notions using incidence graph (or Gaifman graphs) and for oriented graphs we can consider corresponding symmetrization. For the sake of simplicity in this paper we illustrate our results mostly on classes of undirected graphs.

Although this definition may seem to be quite arbitrary, it is very robust and more stable than expected. For instance, as we shall show now, this classification is consistent with a classification theorem based on the logarithmic asymptotic densities of graphs densities of graphs. We then list five other characterizations. In Section 3 we give several examples and in Section 4 we give a summary of these equivalent definitions for the case of classes with bounded expansion.

1.2. Classification by edge densities. Let $\mathcal{C}$ be an infinite class of graphs and let $f : \mathcal{C} \to \mathbb{R}$ be a graph invariant. Let $\mathrm{Inj}(\mathbb{N}, \mathcal{C})$ be the set of all injective mappings from $\mathbb{N}$ to $\mathcal{C}$. Then we define

$$\limsup_{G \in \mathcal{C}} f(G) = \sup_{\phi \in \mathrm{Inj}(\mathbb{N}, \mathcal{C})} \limsup_{i \to \infty} f(\phi(i)).$$

Notice that $\limsup_{G \in \mathcal{C}} f(G)$ always exist and is either a real number or $\pm\infty$.

Theorem 1 (Trichotomy theorem). *Let $\mathcal{C}$ be an infinite class of graphs* (*asymptotically not all edgeless*)*. Then the limit*

$$\ell\mathrm{dens}(\mathcal{C}) = \lim_{i \to \infty} \limsup_{G \in \mathcal{C} \triangledown i} \frac{\log \|G\|}{\log |G|}$$

may take only three values, namely 0, 1 *and* 2*. Moreover, we have*

$$\ell\mathrm{dens}(\mathcal{C}) = \begin{cases} 0 & \textit{iff } \sup_{G\in\mathcal{C}} \|G\| < \infty, \\ 0 \textit{ or } 1 & \textit{iff } \mathcal{C} \textit{ is nowhere dense,} \\ 2 & \textit{iff } \mathcal{C} \textit{ is somewhere dense.} \end{cases}$$

For a proof see [34]. It can be seen easily that $\ell\mathrm{dens}\mathcal{C} = 0$ if and only if the class $\mathcal{C}$ contains only graphs with at most k_0 edges. These *essentially finite* classes are interesting. A prime example is the class of all core graphs with tree depth bounded (see Section 1.5 for the definition of tree depth).

1.3. Classification by topological resolution. A graph G' is a *subdivision* of a graph G if G' arises from G by adding vertices (of degree 2) on the edges of G. Thus in the topological sense we have the same graph: all edges of G are replaced by simple openly disjoint paths. If all these paths have length $\leq 2d+1$ we say that G' is a p-shallow subdivision of G. Conversely, we say that H is *topological shallow minor at depth d* of a graph G if there exists a subgraph H' of G such that H' is a shallow subdivision of H at depth d.

Having defined this we can proceed similarly as for the shallow minors and define the notion of topological minor resolution:

The set of all topological shallow minors of G at depth d is denoted by $G \mathrel{\widetilde{\nabla}} d$. In particular, $G \mathrel{\widetilde{\nabla}} 1$ is the set of all subgraphs of G. Hence we have the following non decreasing sequence of classes:

$$G \in G \mathrel{\widetilde{\nabla}} 1 \subseteq G \mathrel{\widetilde{\nabla}} 2 \subseteq \cdots \subseteq G \mathrel{\widetilde{\nabla}} d \subseteq \cdots G \mathrel{\widetilde{\nabla}} \infty.$$

We extend this definition to an arbitrary graph class $\mathcal{C}$ by

$$\mathcal{C} \mathrel{\widetilde{\nabla}} d = \bigcup_{G\in\mathcal{C}} G \mathrel{\widetilde{\nabla}} d.$$

Now we have the following hierarchy of graph classes:

$$\mathcal{C} \subseteq \mathcal{C} \mathrel{\widetilde{\nabla}} 1 \subseteq \mathcal{C} \mathrel{\widetilde{\nabla}} 2 \subseteq \cdots \subseteq \mathcal{C} \mathrel{\widetilde{\nabla}} d \subseteq \cdots \mathcal{C} \mathrel{\widetilde{\nabla}} \infty.$$

We call this sequence *topological minor resolution* of class $\mathcal{C}$. Note that $\mathcal{C} \mathrel{\widetilde{\nabla}} 1$ is the monotone closure of $\mathcal{C}$ and that $C \mathrel{\widetilde{\nabla}} \infty$ is the topological minor closed class generated by $\mathcal{C}$.

Let $\mathcal{C}$ be an infinite class of graphs and let $f : \mathcal{C} \to \mathbb{R}$ be a graph invariant. Let $\mathrm{Inj}(\mathbb{N}, \mathcal{C})$ be the set of all injective mappings from $\mathbb{N}$ to $\mathcal{C}$. Then we define

$$\limsup_{G\in\mathcal{C}} f(G) = \sup_{\phi\in\mathrm{Inj}(\mathbb{N},\mathcal{C})} \limsup_{i\to\infty} f(\phi(i)).$$

Notice that $\limsup_{G\in\mathcal{C}} f(G)$ always exists and either is a real number or $\pm\infty$.

Theorem 2 (Trichotomy theorem). *Let $\mathcal{C}$ be an infinite class of graphs (asymptotically not all edgeless). Then the limit*

$$\ell\mathrm{dens}(\mathcal{C}) = \lim_{i\to\infty} \limsup_{G\in\mathcal{C}\,\widetilde{\nabla}\, i} \frac{\log \|G\|}{\log |G|}$$

may only take three values, namely 0, 1 *and* 2. *Moreover, we have*

$$\ell\mathrm{dens}(\mathcal{C}) = \begin{cases} 0 & \text{iff } \sup_{G\in\mathcal{C}} \|G\| < \infty, \\ 0 \text{ or } 1 & \text{iff } \mathcal{C} \text{ is nowhere dense,} \\ 2 & \text{iff } \mathcal{C} \text{ is somewhere dense.} \end{cases}$$

For a proof see [34]. (This extends work of Zdeněk Dvořák [10], [11].)

Why do we state this topological variant of shallow minors, when we then claim just analogous results? The main reason is that this connection is surprising and non-trivial. The fact that minors and topological minors lead to the same classification of classes is interesting in the context of graph-minor theory where minors and topological minors lead often to very different results (as demonstrated for example by Hajós and Hadwiger's conjectures), see [34], [35] for more details.

It follows directly from the definition of the minor resolution that a class $\mathcal{C}$ is nowhere dense iff for every i the supremum of $\omega(G)$ for $G \in \mathcal{C} \,\nabla\, i$ is finite (here $\omega(G)$ is the maximal complete graph in G). It is perhaps surprising that nowhere dense classes may be defined by their independence number as well.

1.4. Classification by independence. In the context of relativizations of first-order homomorphism preservation theorems to specific classes of structures Anush Dawar [6] introduced the following notion of quasi-wideness.

Let $r \geq 1$ be an integer. A subset A of vertices of a graph G is *r-independent* if the distance between any two distinct elements of A is strictly greater than r. Note that if we denote by $\alpha_r(G)$ the maximum size of an r-independent set of G, then $\alpha_1(G)$ is the usual independence number $\alpha(G)$ of graph G.

A graph G is *quasi-wide* if there is a function $s\colon \mathbb{N} \to \mathbb{N}$ such that for every integers d and m, every sufficiently big graph $G \in \mathcal{C}$ (i.e., of order at least $F(d,m)$) contains a subset S of size at most $s = s(d)$ so that the graph $G - S$ contains a d-independent set of vertices of size at least m (see Figure 2)

The quasi-wide property is not hereditary. Thus we introduce the following, stronger version:

A graph G is *uniformly quasi-wide* if there is a function $s\colon \mathbb{N} \to \mathbb{N}$ such that for every integers d and m, every sufficiently big subset A of vertices of a graph $G \in \mathcal{C}$ (i.e., such that $|A| \geq F(d,m)$) is such that G contains a subset S of size at most $s = s(d)$ so that $G - S$ contains a d-independent set of vertices of size at least m included in A (see Figure 3)

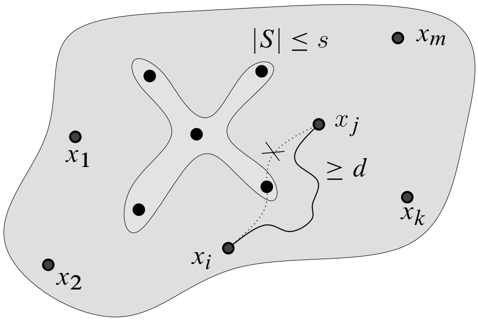


Figure 2. For every m, every sufficiently large $G \in \mathcal{C}$ contains a subset S of size at most $s(d)$ so that $G - S$ has a d-independent set of size m.

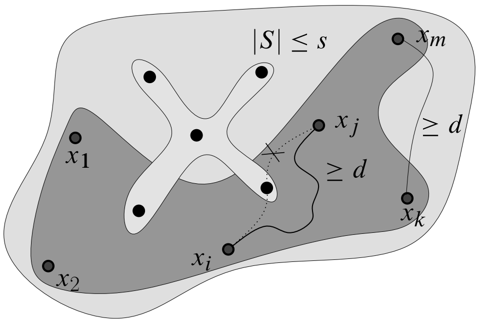


Figure 3. For every m, every sufficiently large subset A of a graph $G \in \mathcal{C}$ includes a d-independent set of size m after the deletion of at most $s(d)$ vertices of G.

It appears that uniform quasi-wideness is strongly related to our classification:

Theorem 3. *Let $\mathcal{C}$ be an infinite class of graphs. Then the following conditions are equivalent:*

- *$\mathcal{C}$ is nowhere dense,*
- *the hereditary closure of $\mathcal{C}$ is quasi-wide,*
- *$\mathcal{C}$ is uniformly quasi-wide.*

This is a non-trivial result with several consequences, see [37].

1.5. Classification by decomposition. A *rooted forest* is a disjoint union of rooted trees. The *height* of a vertex x in a rooted forest F is the number of vertices of the path from the root (of the tree to which x belongs to) to x and is noted height(x, F). The *height* of F is the maximum height of the vertices of F. Let x, y be vertices of F. The vertex x is an *ancestor* of y in F if x belongs to the path linking y and the root of the tree of F to which y belongs to. The *closure* clos(F) of a rooted forest F is the graph with vertex set $V(F)$ and edge set $\{\{x, y\} : x$ is an ancestor of y in $F, x \neq y\}$. A rooted forest F defines a partial order on its set of vertices: $x \leq_F y$ if x is an ancestor of y in F. The comparability graph of this partial order is obviously clos(F).

The *tree-depth* td(G) of a graph G is the minimum height of a rooted forest F such that $G \subseteq \text{clos}(F)$ [30] (see Figure 4).

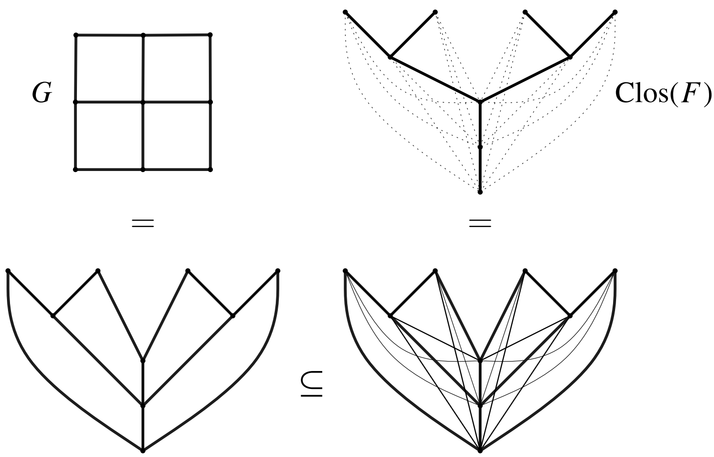


Figure 4. The tree-depth of the 3×3 grid is 4.

This definition is analogous to the definition of *rank function* of a graph which has been recently used for analysis of countable graphs, see e.g. [44] as well as to the quantifier rank (as used e.g. by Ben Rossman [49]) and to other concepts in the algorithmic graph theory such as *vertex ranking, ordered coloring, centered coloring*, see e.g. [7], [50].

A principal property of the class of all graphs with td$(G) \leq k$ is that this class is finite when restricted to *core* graphs (or core structures, see e.g. [19]). This holds more generally for colored graphs and for relational structures in general. This has also a number of consequences. For example the class of all graphs with td$(G) \leq k$ is well quasi ordered with respect to induced subgraph ordering. Nevertheless one should remark that the number of core graphs with td$(G) \leq k$ has an Ackermann growth.

In [30] we introduced the following parametrized generalization of the chromatic number: for any integer p, $\chi_p(G)$ denotes the minimum number of colors one shall use to color the vertices of G in such a way that for every subset I of at most p

colors, the subgraph G_I of G induced by the vertices with color in I has tree-depth at most $|I|$. Thus χ_1 is the usual chromatic number of a graph (i.e., no edge is monochromatic) and χ_2 is minimal coloring with the property that no path with 4 vertices gets less than 3 colors.

There is a *fraternal orientation augmentation algorithm* (not to be described here, see [29], [32]) which we will call "Algorithm A", which computes for any pair (G, p) a vertex coloring of G using $\leq N_p(G)$ colors with the additional property that any subset of $i \leq p$ colors induce a subgraph of tree-depth at most i. This algorithm runs (for a graph with n vertices) in the worst time $O(n^2 p \log pN_p^2(G))$. Moreover the number $N_p(G))$ which this algorithm uses is a good bound on $\chi_p(G)$. More formally this is expressed by the following result which yields yet another characterization of nowhere dense classes ([32], [34]):

Theorem 4. *Let $\mathcal{C}$ be an infinite class of graphs. Then the following conditions are equivalent:*

- *$\mathcal{C}$ is nowhere dense,*
- *for every integer p,* $\limsup_{G\in\mathcal{C}} \dfrac{\log \chi_p(G)}{\log |G|} = 0$*,*
- *for every integer p,* $\limsup_{G\in\mathcal{C}} \dfrac{\log N_p(G)}{\log |G|} = 0$*, where $N_p(G)$ is the number of colors used by "Algorithm A" when run on the pair* (G, p).

Notice that the running time of "Algorithm A" is at most $|G|^{o(1)}$ for fixed p and for G restricted to a nowhere dense class. Thus any graph G in a (fixed) nowhere dense class $\mathcal{C}$ can be decomposed into a small number of classes such that the subgraphs induced by any $\leq p$ classes of the partition have components of only finitely many (homomorphism) types. Thus p is then parameter expressing the precision of such decomposition. Moreover such decomposition can be found in almost linear number of steps. This has a number of algorithmic consequences ([29], [32]. Such a decomposition is called *Low Tree Depth Decomposition*.

1.6. Classification by vertex ordering. As a generalization of both arrangeability and coloring number Kierstead and Yang introduced in [20] two new series of invariants col_k and wcol_k, that is: the *coloring number* of rank k and the *weak coloring number* of rank k.

Let L be a linear order on the vertex set of a graph G, and let x, y be vertices of G. We say y is *weakly k-accessible from x* if $y <_L x$ and there exists an x-y-path P of length at most k (i.e., with at most k edges) with minimum vertex y with respect to $<_L$ (see Figure 5). The vertex is *k-accessible* from x if $y <_L x$ and there exists an x-y-path P of length at most k with minimum vertex y and second minimum vertex x with respect to $<_L$.

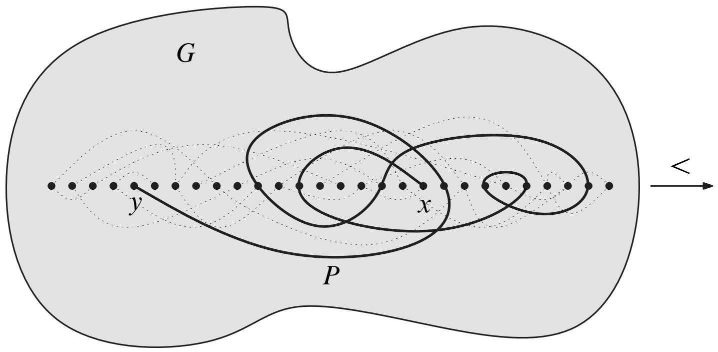


Figure 5. The vertex y is weakly 8-accessible from x.

Let $Q_k(G,L,x)$ and $R_k(G,L,x)$ be the sets of vertices that are respectively weakly k-accessible and k-accessible from x:

$$Q_k(G,L,x) = \{y : \exists x\text{-}y \text{ path } P \text{ such that } \min P = y\}$$
$$R_k(G,L,x) = \{y : \exists x\text{-}y \text{ path } P \text{ such that } \min P = y \text{ and } \min(P-y) = x\}$$

The *weak k-coloring number* $\mathrm{wcol}_k(G)$ and the *k-coloring number* $\mathrm{col}_k(G)$ of G are defined by

$$\mathrm{wcol}_k(G) = 1 + \min_L \max_{v \in V(G)} |Q_k(G,L,v)|,$$
$$\mathrm{col}_k(G) = 1 + \min_L \max_{v \in V(G)} |R_k(G,L,v)|.$$

It is easy to see ([20]) that these two graph invariants are polynomially dependent:

$$\mathrm{col}_k(G) \le \mathrm{wcol}_k(G) \le (\mathrm{col}_k(G))^k$$

These parameters form two non-decreasing sequences. The sequence of weak-coloring numbers has the tree-depth as its maximum:

$$\mathrm{col}(G) = \mathrm{wcol}_1(G) \le \mathrm{wcol}_2(G) \le \cdots \le \mathrm{wcol}_k(G) \le \cdots \le \mathrm{wcol}_\infty(G) = \mathrm{td}(G).$$

Generalized coloring numbers are strongly related to the maximum density of shallow minors. It has been proved by X. Zhu that there exist polynomials F_k such that the following holds:

Theorem 5 ([53]). *For every half integer*[1] *k and every graph G:*

$$1 + \sup_{G \in \mathcal{C} \,\nabla\, k} \frac{\|G\|}{|G|} \le \mathrm{wcol}_{2k+1}(G) \le F_{2k+1}\left(\sup_{G \in \mathcal{C} \,\nabla\, k} \frac{\|G\|}{|G|}\right)$$

[1] When k is not an integer, an appropriate generalization of $G \,\nabla\, k$ has to be used [39], [53].

From this follows the following characterization of nowhere dense classes:

Theorem 6. *Let $\mathcal{C}$ be an infinite class of graphs. Then the following conditions are equivalent:*

- *$\mathcal{C}$ is nowhere dense,*
- *for every integer p,* $\limsup_{G\in\mathcal{C}} \dfrac{\log \mathrm{col}_p(G)}{\log |G|} = 0$,
- *for every integer p,* $\limsup_{G\in\mathcal{C}} \dfrac{\log \mathrm{wcol}_p(G)}{\log |G|} = 0$.

1.7. Classification by counting. The trichotomy theorem (Theorem 1) is related to counting the numbers of copies of K_2 in a graph. This may be extended (using the decomposition theorem) if we consider homomorphism or induced copies of any non-trivial graph F. (Recall that $\hom(F, G)$ denotes the number of homomorphisms from F to G and that $\#F \subseteq G$ denotes the number of induced subgraphs of G which are isomorphic to F.)

Theorem 7. *Let F be a (connected) non trivial graph (i.e., with at least one edge). Then the following limits*

$$\lim_{i\to\infty} \limsup_{G\in\mathcal{C}\,\nabla\, i} \frac{\log \hom(F, G)}{\log |G|}$$

and

$$\lim_{i\to\infty} \limsup_{G\in\mathcal{C}\,\nabla\, i} \frac{\log \#F \subseteq G}{\log |G|}$$

can only take the values $-\infty, 0, 1, \dots, \alpha(F)$ and $|F|$, where $\alpha(F)$ stands for the independence number of F. Moreover, $\mathcal{C}$ is somewhere dense if and only if the limit is $|F|$.

For a proof, see [36].

1.8. Examples

1.8.1. Simplicial graphs. A *k-dimensional simplex*, or *k-simplex*, is the convex hull of $k + 1$ affinely independent points in $\mathbb{R}^d$ space. A *d-dimensional simplicial complex* is a collection of k-simplexes, $k \le d$, closed under sub-simplex and intersection. For example, a 3-dimensional simplicial complex is a collection of cells (3-simplexes), faces (2-simplexes), edges (1-simplexes) and vertices (0-simplexes). A *d-dimensional simplicial graph* is the collection of edges and vertices of a d-dimensional simplicial complex. The *aspect ratio* of a body is its diameter divided dth root of its volume [25]. The volume of a regular d-simplex, d-cube, and d-ball of unit diameter are respectively $2^{-d/2}\sqrt{d+1}/d!$, $d^{-d/2}$ and $2^{-d}\pi^{d/2}/(d/2)!$. Hence the aspect ratios of a d-simplex, d-cube, and d-ball are respectively $\alpha_s = 2^{1/2}(d!)^{1/d}(d+1)^{-1/(2d)} \sim \sqrt{2}d/e, \alpha_c = \sqrt{d}$, and $\alpha_b = 2\pi^{-1/2}(d/2)!^{1/d} \sim \sqrt{2d/(e\pi)}$. A simplicial graph of aspect ratio α means a simplicial graph coming from a complex in which every d-simplex has aspect ratio at most α.

Classes of simplicial graphs with bounded aspect ratio exclude big shallow complete minors as proved by Plotkin, Rao and Smith [48]. It follows that such classes are nowhere dense.

1.8.2. High girth graphs. A standard example of a monotone nowhere dense class of graphs is the class of the graphs whose maximum degree does not exceed some function of the girth, i.e., $\mathcal{B}_\phi = \{G : \Delta(G) \leq \phi(\mathrm{girth}(G))\}$.

Such classes may have average degree as big as $n^{o(1)}$ as a consequence (see for instance [4]): For every positive integer n and an "expected degree" k (where $k < n/3$), there exists a graph G of order n, size $\lfloor nk/2 \rfloor$, vertex degrees in $\{k-1, k, k+1\}$ and whose girth g is such that $g > \log_k(n) + O(1)$. Hence, for any decreasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{x\to\infty} f(x) = 0$ there exists a constant C such that the class $\mathcal{B}_\phi$ defined by $\phi(x) = (f^{-1}(1/x) + C)^{1/x}$ contains graphs with order n, girth at least $1/f(n)$ and degrees $k \pm 1$ with $k \approx n^{f(n)}$.

1.8.3. Classical sparse classes. Figure 6 shows the inclusion map of some important hereditary nowhere dense classes which were studied in combinatorial as well as algorithmic context.

2. Bounded expansion classes

A specific example of classes which are nowhere dense are classes with bounded expansion. These classes have been introduced in [29]. A class $\mathcal{C}$ has *bounded expansion* if there exists a function $f : \mathbb{N} \to \mathbb{R}$ (called *expansion function*) such that

$$\forall d \in \mathbb{N} : \quad \sup_{G \in \mathcal{C} \,\nabla\, d} \frac{\|G\|}{|G|} \leq f(d).$$

The value $\sup_{G\in\mathcal{C}\,\nabla\, d} \frac{\|G\|}{|G|}$ is denoted by $\nabla_d(\mathcal{C})$ and, in the particular case of a single element class $\{G\}$, $\nabla_d(G)$ is called the *greatest reduced average density* (grad) of G of rank d.

For an extensive study of bounded expansion classes we refer the reader to [31], [32], [33], [10], [11], [40].

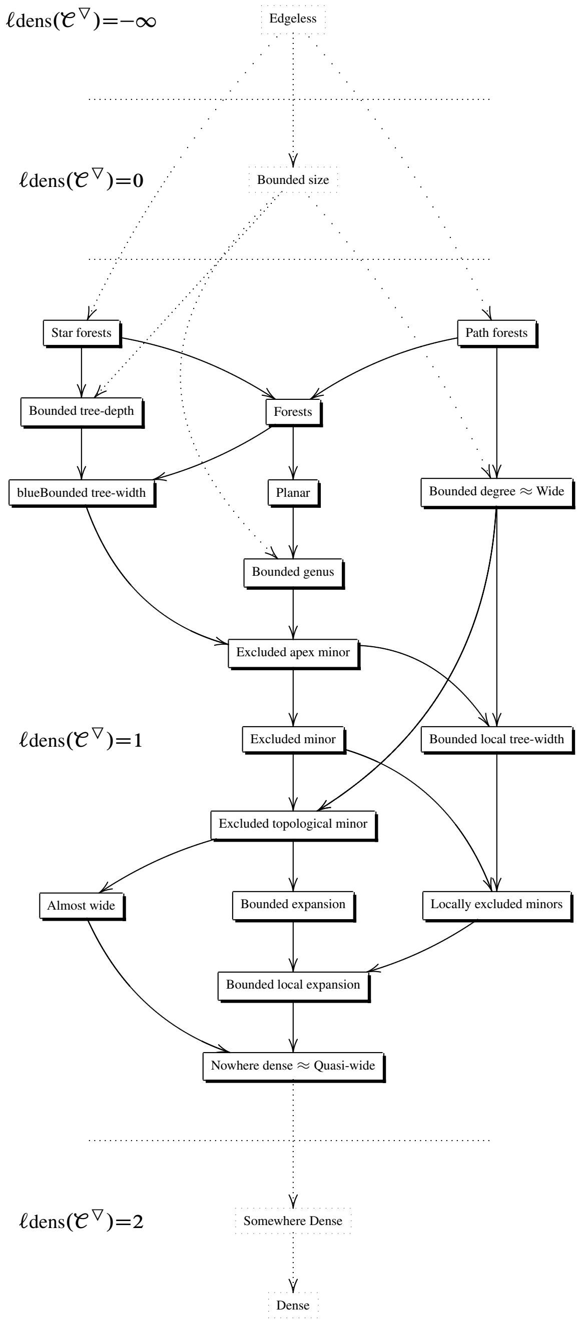


Figure 6. Inclusion map of some important hereditary nowhere dense classes.

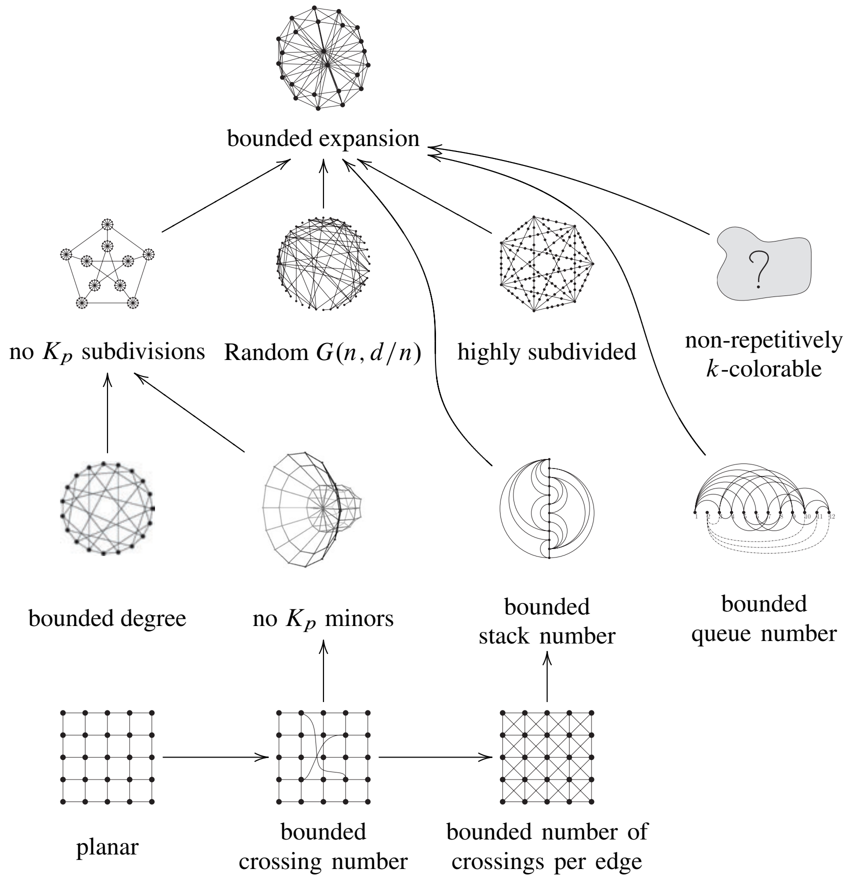


Figure 7. Classes with bounded expansion.

(See [40] for the definition of stack and queue numbers. This paper contains further examples of bounded expansion classes.)

As for nowhere dense classes, several equivalent characterizations exist for classes with bounded expansion:

Theorem 8. *Let $\mathcal{C}$ be a class of graphs. The following properties are equivalent:*

- *$\mathcal{C}$ has bounded expansion,*
- *for every integer p, $\sup_{G\in\mathcal{C}} \chi_p(G) < \infty$,*
- *for every integer p, $\sup_{G\in\mathcal{C}} N_p(G) < \infty$, where $N_p(G)$ is the number of colors used by "Algorithm A" when run on the pair (G, p).*

Notice that the running time of “Algorithm A” is at most $O(|G|)$ for fixed p and for G restricted to a bounded expansion class. Thus low tree depth decompositions can be found in a linear time.

Thus any graph G in a (fixed) bounded expansion class $\mathcal{C}$ can be decomposed into a fixed number $N_p(G)$ of classes such that the subgraphs induced by any $\leq p$ classes of the partition have components of only finitely many (homomorphism) types. Thus p is then parameter expressing the precision of such decomposition. Moreover such decomposition can be found in a linear number of steps. Not surprisingly, this has a number of algorithmic consequences ([29], [32]. Such a decomposition is called *Low Tree Depth Decomposition*.

As nowhere dense classes do, the classes with bounded expansion have several characterizations based on different aspects (density, orientation, decomposition, ordering, etc.). For instance, it has been proved in [10], [11] that bounded expansion classes are characterized by not containing shallow subdivisions of graphs with high minimum degree.

Part II Distances and dualities

3. Algebra of classes

For a graph G we denote by $[G]$ the set of all graphs H which are homomorphically equivalent to G. $[G]$ is called *homomorphism equivalence class* of G. It is a well-known that for a finite graph (structure) G there exists up to an isomorphism unique $G' \in [G]$ with the smallest number of vertices. Such an H is called the *core* of G ([19]). For homomorphism equivalence classes $[G]$ and $[H]$ we put $[G] \leq [H]$ and $[G] \to [H]$ iff $G' \to H'$ for every $G' \in [G]$ and $H' \in [H]$. The set of all homomorphism equivalence classes will be denoted by $[\mathcal{G}]$ and $\leq$ is a partial order defined on $[\mathcal{G}]$. $[\mathcal{G}]$ is a partial order with remarkable properties, most notably it is (countable) universal partial order which is also dense, see [19]. We shall add to these facts some properties of an interesting completion of $[\mathcal{G}]$. Towards this end we define $(\to G)$ as the class of all the graphs having a homomorphism to G. Similarly we denote by $(G \to)$ the class of all graphs which admit a homomorphism from G. Hence

$$[G] = (\to G) \cap (G \to) \tag{1}$$

and by $(\to [G])$ and $([G] \to)$ we shall understand analogously defined subclasses of $[\mathcal{G}]$. These classes express usual graph constructions related to homomorphism order:

- the sum

$$(\to [G+H]) = \{[A+B] \colon [A] \in (\to [G]), [B] \in (\to [H])\}$$
$$([G+H] \to) = ([G] \to) \cap ([H] \to);$$

- the categorical product

$$(\to [G \times H]) = (\to [G]) \cap (\to [H])$$
$$([G \times H] \to) = \{[A \times B] \colon [A] \in ([G] \to), [B] \in ([H] \to)\}.$$

(The last equation follows from the standard trick that graphs $(F + G) \times (F + H)$ and F are homomorphism equivalent whenever $G \times H \to F$.)

A (*graph*) *ideal* is a subset $\mathcal{I} \subseteq \mathcal{G}$ such that:

- $\forall G \in \mathcal{I}, \forall H \in \mathcal{G},\ H \to G\ \Rightarrow H \in \mathcal{I}$,
- $\forall G_1, G_2 \in \mathcal{I},\ G_1 + G_2 \in \mathcal{I}$.

An ideal $\mathcal{I}$ is *principal* if there exists a graph G (the *principal element* of $\mathcal{I}$) such that $\mathcal{I} = (\to G)$. An ideal $\mathcal{I}$ is *prime* if $G \times H \in \mathcal{I}$ implies $G \in \mathcal{I}$ or $H \in \mathcal{I}$. If the principal ideal $(\to G)$ generated by a graph G is prime, G is said to be *multiplicative*. In other words; G is multiplicative if $H_1 \times H_2 \to G$ implies $H_1 \to G$ or $H_2 \to G$.

A (*graph*) *filter* is a subset $\mathcal{F} \subseteq \mathcal{G}$ such that:

- $\forall G \in \mathcal{F}, \forall H \in \mathcal{G},\ H \leftarrow G\ \Rightarrow H \in \mathcal{F}$,
- $\forall G_1, G_2 \in \mathcal{F},\ G_1 \times G_2 \in \mathcal{F}$.

A filter $\mathcal{F}$ is *principal* if there exists a graph G (the *principal element* of $\mathcal{F}$) such that $\mathcal{F} = (G \to)$. A filter $\mathcal{F}$ is *prime* if $G + H \in \mathcal{F}$ implies $G \in \mathcal{F}$ or $H \in \mathcal{F}$. Notice that if F is a connected graph the principal filter $(F \to)$ is prime.

For an ideal $\mathcal{I}$ and a filter $\mathcal{F}$ we define

$$\mathcal{I}^\star = \{G,\ \forall H \in \mathcal{I} \colon H \to G\}, \tag{2}$$
$$\mathcal{F}^\star = \{G,\ \forall H \in \mathcal{F} \colon H \leftarrow G\}. \tag{3}$$

Notice that $\mathcal{I}^\star$ is a filter and that $\mathcal{F}^\star$ is an ideal and that for every graph G we have $(\to G)^\star = (G \to)$ and $(G \to)^\star = (\to G)$. These relations provided a motivation for the duality notion, see [41].

For the proofs of the results stated here and for a more extensive study of this topic we refer the reader to [38].

4. Distances

4.1. Distance between graph sets. Recall that $\mathcal{G}$ denotes the class of all (finite simple) graphs. For an integer t, we denote by $\mathcal{G}^t$ the subset of $\mathcal{G}$ with graphs of order at most t.

For a subset $\mathcal{A} \subseteq \mathcal{G}$, we define the *weight* $w(\mathcal{A})$ of $\mathcal{A}$ by

$$w(\mathcal{A}) = \begin{cases} 0 & \text{if } \mathcal{A} = \emptyset, \\ 2^{-\min\{|G|,\ G \in \mathcal{A}\}} & \text{otherwise.} \end{cases} \tag{4}$$

This weight allows to define an ultrametric on the power set of $\mathcal{G}$ by

$$\mathrm{D}(\mathcal{A}, \mathcal{B}) = w\big((\mathcal{A} \setminus \mathcal{B}) \cup (\mathcal{B} \setminus \mathcal{A})\big). \tag{5}$$

Also, we define a non-symmetric version which intuitively "measures" how far $\mathcal{A}$ is from being included in $\mathcal{B}$:

$$\iota(\mathcal{A}, \mathcal{B}) = w(\mathcal{A} \setminus \mathcal{B}), \tag{6}$$

so that

$$\mathrm{D}(\mathcal{A}, \mathcal{B}) = \max(\iota(\mathcal{A}, \mathcal{B}) + \iota(\mathcal{B}, \mathcal{A})). \tag{7}$$

4.2. The left distance. The *left distance* dist_L between homomorphism equivalence classes is defined by

$$\mathrm{dist}_L([G], [H]) = D((\to G), (\to H)). \tag{8}$$

Notice that this definition is valid, as $(\to G)$ does not depend on the particular choice of G in $[G]$. The left distance gives $[\mathcal{G}]$ the structure of an ultrametric space as $\mathrm{dist}_L([G_1], [G_3]) \leq \max(\mathrm{dist}_L([G_1], [G_2]), \mathrm{dist}_L([G_2], [G_3]))$. To simplify the notation we extend the definition of dist to $\mathcal{G}$ (it is of course $\mathrm{dist}_L(G, H) = \mathrm{dist}_L([G], [H])$).

This distance has a strong relationship with homomorphism invariant first order properties:

Theorem 9 (Rossman [49]). *For every integer n, there exists $\epsilon > 0$ and a function $\Re_n \colon \mathcal{G} \to \mathcal{G}$ such that*

- $\forall G \in \mathcal{G},\ [\Re_n(G)] = [G]$;
- $\forall G_1, G_2 \in \mathcal{G},\ \mathrm{dist}_L(G_1, G_2) < \epsilon \implies \Re_n(G_1) \equiv^n \Re_n(G_2)$.

Here $G \equiv^n H$ means that G and H satisfy exactly the same first-order properties of quantifier rank at most n.

A sequence (G_i) of graphs is a *Left Cauchy Sequence* (LCS) if it is a Cauchy sequence according to the distance dist_L. The completion $\overline{\mathcal{G}}_L$ of $\mathcal{G}$ (with distance dist_L) is a compact space (according to the Heine–Borel theorem). If (G_i) is a LCS we will denote its limit by left $\lim_{i\to\infty} G_i$ an shortly by left $\lim G_i$.

For instance, the sequence of odd cycles converges to K_2, that is:

$$\text{left} \lim C_{2i+1} = K_2.$$

We extend the homomorphism order to $\overline{[\mathcal{G}]}_L$ by

$$\text{left} \lim G_i \leq_L \text{left} \lim H_i \iff \lim_{i\to\infty} \iota((\to G_i), (\to H_i)) = 0. \tag{9}$$

Notice that the relation $\leq_L$ does not depend of the converging sequences and that it extends the homomorphism relation on finite graphs (identified with constant sequences). Also, we define

$$(\to \text{left lim } G_i) = \{H \in \mathcal{G} :\ H \to \text{left lim } G_i\}. \tag{10}$$

Theorem 10. *The mapping* left lim $G_i \mapsto (\to$ left lim $G_i)$ *is a bijection between left limits and graph ideals.*

Corollary 11. *The left homomorphism relation* $\leq_L$ *on* $\overline{[\mathcal{G}]}_L$ *has the following alternative characterizations:*

$$\begin{aligned} \text{left lim } G_i \leq_L \text{left lim } H_i &\iff (\to \text{left lim } G_i) \subseteq (\to \text{left lim } H_i) &(11)\\ &\iff (\text{left lim } G_i \to) \supseteq (\text{left lim } H_i \to). &(12)\end{aligned}$$

4.3. The right distance. Similarly to the left distance, the *right distance* dist_R between homomorphism equivalence classes is defined by

$$\text{dist}_R([G],[H]) = D((G \to),(H \to)). \tag{13}$$

The right distance gives $[\mathcal{G}]$ the structure of an ultrametric space since it holds that $\text{dist}_R([G_1],[G_3]) \leq \max(\text{dist}_R([G_1],[G_2]), \text{dist}_R([G_2],[G_3]))$. We again write $\text{dist}_R(G,H) = \text{dist}_R([G],[H])$

A sequence (G_i) of graphs is a *Right Cauchy Sequence* (RCS) if it is a Cauchy sequence according to the distance dist_R. The completion $\overline{[\mathcal{G}]}_R$ of $[\mathcal{G}]$ (with distance dist_R) is again a compact space. If (G_i) is a RCS we will denote its limit by right $\lim_{i\to\infty} G_i$ and shortly by right lim G_i.

For instance, the sequence of odd cycles converges, but the limit is not a graph.

We extend the homomorphism relation to $\overline{[\mathcal{G}]}_R$ by

$$\text{right lim } G_i \leq_R \text{left lim } H_i \iff \lim_{i\to\infty} \iota((H_i \to),(G_i \to)) = 0. \tag{14}$$

Notice that the relation $\leq_R$ does not depend of the converging sequences and that it extends the homomorphism relation on finite graphs (identified with constant sequences). Also, we define

$$(\text{right lim } G_i \to) = \{H \in \mathcal{G} :\ \text{right lim } G_i \to H\}. \tag{15}$$

Theorem 12. *The mapping* right lim $G_i \mapsto$ (right lim $G_i \to$) *is a bijection between right limits and graph filters.*

Corollary 13. *The right homomorphism relation* $\leq_R$ *on* $\overline{[\mathcal{G}]}_R$ *has the following alternative characterizations:*

$$\begin{aligned} \text{left lim } G_i \leq_R \text{left lim } H_i &\iff (\to \text{right lim } G_i) \subseteq (\to \text{right lim } H_i) &(16)\\ &\iff (\text{right lim } G_i \to) \supseteq (\text{right lim } H_i \to). &(17)\end{aligned}$$

Note that this characterization is similar to the one given for the definition of $\leq_L$.

A basic result about the right distance is implied by the following result (which is a culmination of intensive combinatorial research which goes back to [16], [42]):

Theorem 14 (Nešetřil and Zhu [47]). *For every graph H and every choice of positive integers k and l there exists a graph G together with a surjective homomorphism $G \to H$ with the following properties:*

(1) $\mathrm{girth}(G) > l$*;*

(2) *For every graph H' with at most k vertices, there exists a homomorphism from G to H' if and only if there exists a homomorphism from H to H'.*

This lemma (sometimes called Sparse Incomparability Lemma, see e.g. [19]) takes the following form:

Lemma 15 (Ambivalence Lemma). *For every $K_2 \to H_1 \to H_2$ and every $\epsilon > 0$ there exists a graph G such that*

- $H_1 \to G \to H_2$*,*
- $\mathrm{dist}_L([H_1], [G]) < \epsilon$*,*
- $\mathrm{dist}_R([G], [H_2]) < \epsilon$*.*

Corollary 16. *Let (G_i) be an LCS and let (H_i) be a RCS. Assume that for every integer i holds $K_2 \to G_i \to H_i$. Then there exists a sequence (M_i) which is both an LCS and a RCS, such that*

$$\text{left lim } M_i = \text{left lim } G_i \tag{18}$$

$$\text{right lim } M_i = \text{right lim } H_i \tag{19}$$

This sequence (M_i) has then the limit both in $\overline{\mathcal{G}}_L$ and $\overline{\mathcal{G}}_R$ which we call *chimera* (see Corollary 19). This double limit leads to the notions in the following section.

4.4. Full distance. Let dist be the distance between homomorphism equivalence classes defined by:

$$\mathrm{dist}([G], [H]) = \max(\mathrm{dist}_L([G], [H]), \mathrm{dist}_R([G], [H]))$$

Note that $\mathrm{dist}([G_1], [G_3]) \leq \max(\mathrm{dist}([G_1], [G_2]), \mathrm{dist}([G_2], [G_3]))$ hence $([\mathcal{G}], \mathrm{dist})$ is an ultrametric space. Again we write $\mathrm{dist}(G, H) = \mathrm{dist}([G], [H])$.

The completion $\overline{\mathcal{G}}$ is again a compact space. If (G_i) is a Cauchy sequence for dist, we will denote by its limit by $\lim_{i\to\infty} G_i$. For a sequence (G_i) to convergence it is necessary and sufficient that the sequence converges for both the distances dist_L and dist_R.

We extend the homomorphism relation to limits by

$$\lim_i G_i \to \lim_i H_i \iff \lim_{i\to\infty} \max(\iota((\to G_i), (\to H_i)), \iota((H_i \to), (G_i \to))) = 0. \tag{20}$$

Notice that this relation does not depend of the converging sequences and that it extends the homomorphism relation on finite graphs (identified with constant sequences),as well as the extensions we defined for left limits and right limits. Also, we define

$$(\to \lim_i G_i) = \{H \in \mathcal{G} :\ H \to \lim_i G_i\}, \tag{21}$$

$$(\lim_i G_i \to) = \{H \in \mathcal{G} :\ \lim_i G_i \to H\}. \tag{22}$$

Theorem 17. *The mapping* $\mathbb{L} \mapsto ((\to \mathbb{L}), (\mathbb{L} \to))$ *if a bijection between the limits* $\mathbb{L} \in [\overline{\mathcal{G}}]$ *such that* $K_2 \to \mathbb{L}$ *and the pairs* $(\mathcal{I}, \mathcal{F})$ *such that:*

- *the set* $\mathcal{I}$ *is a graph ideal which contains* K_2*,*
- *the set* $\mathcal{F}$ *is a graph filter,*
- *for every* $G \in \mathcal{I}$ *and every* $H \in \mathcal{F}$ *holds* $G \to H$*.*

Corollary 18. *Let* $\mathbb{A}, \mathbb{B} \in [\overline{\mathcal{G}}]$*. Then*

$$\mathbb{A} \to \mathbb{B} \iff (\to \mathbb{A}) \subseteq (\to \mathbb{B}) \text{ and } (\mathbb{A} \to) \supseteq (\mathbb{B} \to). \tag{23}$$

This characterization allows the definition of a continuous homomorphism indicator η:

$$\eta(\mathbb{A}, \mathbb{B}) = \max(\iota((\to \mathbb{A}), (\to \mathbb{B})), \iota((\mathbb{B} \to), (\mathbb{A} \to))), \tag{24}$$

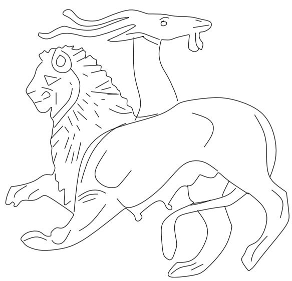
so that we have

$$\mathbb{A} \to \mathbb{B} \iff \eta(\mathbb{A}, \mathbb{B}) = 0. \tag{25}$$

Also, we have the following extension of Corollary 16.

Corollary 19 (Chimera existence). *For every* $\mathbb{A} \to \mathbb{B}$ *in* $[\overline{\mathcal{G}}]$ *such that* $K_2 \to \mathbb{A}$*, there exists* $\mathbb{L}$*, called* $(\mathbb{A}, \mathbb{B})$*-**chimera**, so that*

- $\mathbb{A} \to \mathbb{L} \to \mathbb{B}$,
- $(\to \mathbb{A}) = (\to \mathbb{L})$,
- $(\mathbb{L} \to) = (\mathbb{B} \to)$.



*That is: a limit between $\mathbb{A}$ and $\mathbb{B}$ which is equivalent to $\mathbb{A}$ from the left (for homomorphisms **from** finite graphs), and equivalent to $\mathbb{B}$ from the right (for homomorphisms **to** finite graphs).*

4.5. Directed graphs and structures. In order to extend our results to directed graphs, we have to characterize what the limits of Cauchy sequences of directed graphs are. What has been proved for left and the right distances remains unchanged, except for the ambivalence Lemma 15. For directed graphs we need a new version:

Lemma 20 (Ambivalence Lemma for Directed Graphs). *Let $\vec{G}_1 \to \vec{G}_2$ and let $\epsilon > 0$. Assume there exists no oriented tree $\vec{T}$ with $2^{-\max(|\vec{T}|,|D(\vec{T})|)} < \epsilon$ such that $\vec{T} \to \vec{G}_2$ but $\vec{T} \nrightarrow \vec{G}_1$.*

Then there exists a directed graph $\vec{H}$ such that

- $\vec{G}_1 \to \vec{H} \to \vec{G}_2$,
- $\mathrm{dist}_R(\vec{H}, \vec{G}_2) < \epsilon$,
- $\mathrm{dist}_L(\vec{G}_1, \vec{H}) < \epsilon$.

We deduce the following characterization of limits of directed graphs.

Theorem 21. *The mapping $\vec{\mathbb{L}} \mapsto ((\to \vec{\mathbb{L}}), (\vec{\mathbb{L}} \to))$ if a bijection between the limits $\vec{\mathbb{L}} \in \overline{\vec{\mathcal{G}}}$ and the pairs $(\vec{\mathcal{I}}, \vec{\mathcal{F}})$ such that:*

- *the set $\vec{\mathcal{I}}$ is a directed graph ideal,*
- *the set $\vec{\mathcal{F}}$ is a directed graph filter,*
- $\vec{\mathcal{F}}^\star \cap \overrightarrow{\mathrm{Tree}} \subseteq \vec{\mathcal{I}} \subseteq \vec{\mathcal{F}}^\star$.

As in the undirected case (Corollary 18) we have a simple expression for homomorphisms between limits:

$$\vec{\mathbb{A}} \to \vec{\mathbb{B}} \iff (\to \vec{\mathbb{A}}) \subseteq (\to \vec{\mathbb{B}}) \text{ and } (\vec{\mathbb{A}} \to) \supseteq (\vec{\mathbb{B}} \to) \tag{26}$$

For structures we can derive a similar ambivalence theorem. This is again in terms of relational trees and thus it is related to the results in the next section.

5. Dualities

Suppose we want to find a decomposition of a graph with special properties. Such a task appears in the theory of scheduling and other applied areas (particularly in the area of Constraint Satisfaction Problems). As typically the problem is hereditary we can look for obstacles to our decomposition, for minimal subgraphs without our decomposition. In this sense we are considering the dual problem: instead of looking for a global decomposition we want to look for locally forbidden subgraphs. Can it happen that there is essentially only one obstacle? In this extremal case we speak about a *singleton duality*. Such situation can be formalized by means of homomorphisms between graphs. In this setting duality is captured by the following scheme:

$$F \not\longrightarrow G \iff G \longrightarrow H.$$

Here $G \to H$, denoting the existence of a homomorphism from G to H, means that a desired decomposition exists, while $F \not\to G$ denoting the non-existence of a homomorphism from F to G, means that no obstacle $f(G)$, $f : F \to G$, exists in G.

Unfortunately, for undirected graphs there exists only one duality, namely the pair (K_2, K_1) (as has been observed already in [41] where the notion of duality was defined). However, if we consider limits, and if we naturally extend the notion of duality for limit objects, then we obtain many more examples. More precisely this can be done as follows:

A *full duality* in $[\overline{\mathcal{G}}]$ is a pair $(\mathbb{F}, \mathbb{D})$ of elements of $[\overline{\mathcal{G}}]$ such that

$$\forall \mathbb{L} \in \bar{\mathcal{G}} : \quad \mathbb{F} \not\longrightarrow \mathbb{L} \iff \mathbb{L} \longrightarrow \mathbb{D}.$$

Lemma 22. *Let* $(\mathbb{F}, \mathbb{D})$ *be a full duality. Then one of* $\mathbb{F}, \mathbb{D}$ *is equivalent to a graph.*

Proof. As $\mathbb{D} \to \mathbb{D}$ we have $\mathbb{F} \nrightarrow \mathbb{D}$. This means that there exists a graph T such that either $T \to \mathbb{F}$ and $T \nrightarrow \mathbb{D}$ – hence $\mathbb{F} \to T$ according to duality thus $T \rightleftarrows \mathbb{F}$, or $\mathbb{D} \to T$ and $\mathbb{F} \nrightarrow T$ – hence $T \to \mathbb{D}$ by duality thus $T \rightleftarrows \mathbb{D}$. □

This may be seen as a solution of a weaker form of a conjecture formulated in [43]: It has been conjectured there that for every maximal antichain G_1, G_2 in the homomorphism order of countable graphs one of the graphs G_i is finite. As obviously every duality pair is a maximal antichain the Lemma 22 verifies this conjecture for antichains corresponding to duality pairs.

Theorem 23. *Every connected graph has a full dual, every multiplicative graph is a full dual, and these are the only full dualities.*

Sketch of the proof. We already proved that every duality pair contains a graph. Moreover, it is easily checked that if one of $\mathbb{F}$, $\mathbb{D}$ is a graph and if duality holds for graphs, then it also holds for limits:

- Let $(F, \mathbb{D})$ be a duality which holds for graphs. Let $\mathbb{L}$ be the limit of a Cauchy sequence (L_i). Then $F \nrightarrow \mathbb{L}$ is equivalent to $\exists i_0 \forall i \geq i_0,\ F \nrightarrow L_i$, i.e., $\exists i_0 \forall i \geq i_0,\ L_i \to \mathbb{D}$, which is equivalent to $\mathbb{L} \to \mathbb{D}$;

- Similarly, let $(\mathbb{F}, D)$ be a duality which holds for graphs. Let $\mathbb{L}$ be the limit of a Cauchy sequence (L_i). Then $\mathbb{L} \nrightarrow D$ is equivalent to $\exists i_0 \forall i \geq i_0,\ L_i \nrightarrow D$, i.e., $\exists i_0 \forall i \geq i_0,\ \mathbb{F} \to L_i$, which is equivalent to $\mathbb{D} \to \mathbb{L}$.

It can be seen easily that every graph has a full dual. Hence it suffices to characterize those full duality pairs $(\mathbb{F}, D)$ such that D is a graph. If D is multiplicative, let $\mathcal{S} = \{G : G \nrightarrow D\}$ and let $\mathcal{I}$ be the class of all graphs having a homomorphism to every graph in $\mathcal{S}$. The set $\mathcal{I}$ is an ideal by construction and, as D is multiplicative, $\mathcal{S}$ is a filter hence the pair $(\mathcal{I}, \mathcal{S})$ defines a limit object $\mathbb{F}$. For every graph G, we have $\mathbb{F} \to G$ if and only if $G \in \mathcal{S}$, i.e., if and only if $G \nrightarrow D$. It follows that every multiplicative graph is a full dual. Conversely, let $(\mathbb{F}, D)$ be a full duality. Let $\mathcal{I} = \mathcal{I}(\mathbb{F})$ and $\mathcal{S} = \mathcal{S}(\mathbb{F})$. According to the duality, $\mathbb{F} \to G$ if and only if $G \nrightarrow D$ thus $\mathcal{S} = \{G : G \nrightarrow D\}$ is a filter, which means that D is multiplicative. $\square$

5.1. Duality of directed graphs. One should stress that the absence of finite dualities for undirected graphs is a singular fact. Many more dualities exists for richer structures and they were characterized in [45]. Already for oriented graphs the situation changes and more dualities exist. Precisely, Nešetřil and Tardif [45] (see also [19]) proved that every oriented tree F has a dual (see Figure 8). The difficult problem of recognizing duals of directed trees is addressed by Nešetřil and Tardif [46] and Larose, Lotten and Tardif [23].

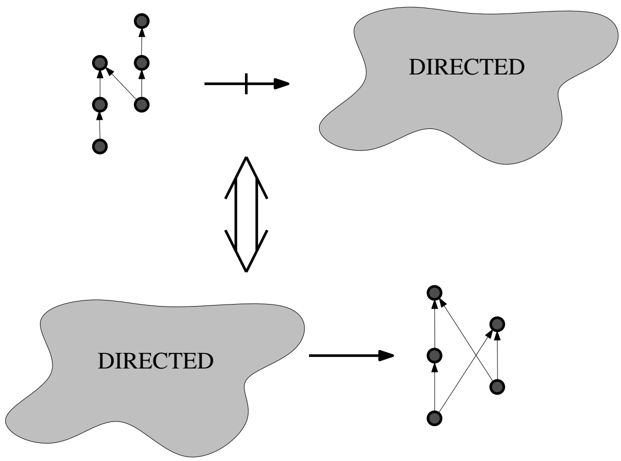


Figure 8. A duality pair for directed graphs.

The assumption of Lemma 20 that no small directed tree $\vec{T}$ is such that $\vec{T} \to \vec{G}_2$ but $\vec{T} \nrightarrow \vec{G}_1$ finds here its explanation and can be shown to be a necessary condition: Assume for contradiction that there exists a directed tree $\vec{T}$ such that $\vec{T} \to \vec{G}_2$ and

$\vec{T} \nrightarrow \vec{G}_1$, although some $\vec{H}$ exists such that $\vec{G}_1 \to \vec{H} \to \vec{G}_2$, $\mathrm{dist}_R(\vec{H}, \vec{G}_2) < \epsilon$, and $\mathrm{dist}_L(\vec{G}_1, \vec{H}) < \epsilon$, with $\epsilon < 2^{-\max|\vec{T}|,|\vec{D}_T|}$ (where $\vec{D}_T$ is the dual of $\vec{T}$). As $\vec{T} \to \vec{G}_2$ we have $\vec{G}_2 \nrightarrow \vec{D}_T$. As $\mathrm{dist}_R(\vec{H}, \vec{G}_2) < \epsilon$ and $\mathrm{dist}_L(\vec{G}_1, \vec{H}) < \epsilon$ we get $\vec{T} \nrightarrow \vec{H}$ (because $\vec{T} \nrightarrow \vec{G}_1$) and $\vec{H} \nrightarrow \vec{D}_T$ (because $\vec{G}_2 \nrightarrow \vec{D}_T$), contradicting the duality of $(\vec{T}, \vec{D}_T)$.

Our results on duality of undirected graphs easily extend to the directed case:

Lemma 24. *Let* $(\vec{\mathbb{F}}, \vec{\mathbb{D}})$ *be a full duality. Then* (*at least*) *one of* $\vec{\mathbb{F}}, \vec{\mathbb{D}}$ *is equivalent to a directed graph.* □

Theorem 25. *Every connected directed graph has a full dual, every multiplicative directed graph is a full dual, and these are the only full dualities.*

Proof. The proof of the directed case is similar to the one of the undirected one.

That one of $\vec{\mathbb{F}}, \vec{\mathbb{D}}$ is a directed graph is Lemma 24. If $\vec{F}$ is a connected directed graph, the set $\mathcal{F} = \{\vec{G} : \vec{F} \nrightarrow \vec{G}$ is a filter and, according to Theorem 21, there exists $\vec{\mathbb{D}} \in \bar{\bar{\vec{\mathcal{G}}}}$ such that $(\to \vec{\mathbb{D}}) = \mathcal{F}$ and $(\vec{\mathbb{D}} \to) = \mathcal{F}^\star$. Then we get $\vec{F} \nrightarrow \vec{\mathbb{L}} \iff \vec{\mathbb{L}} \to \vec{\mathbb{D}}$.

If $\vec{D}$ is a multiplicative directed graph, the set $\mathcal{I} = \{\vec{G} : \vec{G} \nrightarrow \vec{D}$ is an ideal. Of course $\mathcal{I}^{\star\star} \supseteq \mathcal{I}$. Also, if $\vec{T}$ is a directed tree with dual $\vec{D}_T$, then we have

$$\begin{aligned} \vec{T} \notin \mathcal{I} &\iff \forall \vec{G} \in \mathcal{I},\ \vec{T} \nrightarrow \vec{G} && \text{(as } \mathcal{I} \text{ is an ideal)} \\ &\iff \forall \vec{G} \in \mathcal{I},\ \vec{G} \to \vec{D}_T && \text{(by duality)} \\ &\iff \vec{D}_T \in \mathcal{I}^\star. \end{aligned}$$

Also:

$$\begin{aligned} \vec{T} \in \mathcal{I}^{\star\star} &\iff \forall \vec{G} \in \mathcal{I}^\star,\ \vec{T} \to \vec{G} \\ &\iff \forall \vec{G} \in \mathcal{I}^\star,\ \vec{G} \nrightarrow \vec{D}_T && \text{(by duality)} \\ &\iff \vec{D}_T \notin \mathcal{I}^\star && \text{(as } \mathcal{I}^\star \text{ is a filter).} \end{aligned}$$

Hence $\vec{T} \in \mathcal{I}$ if and only if $\vec{T} \in \mathcal{I}^{\star\star}$ and, according to Theorem 21, there exists $\vec{\mathbb{F}} \in \bar{\bar{\vec{\mathcal{G}}}}$ such that $(\vec{\mathbb{F}} \to) = \mathcal{I}$ and $(\to \vec{\mathbb{F}}) = \mathcal{I}^\star$. Then we get $\vec{\mathbb{F}} \nrightarrow \vec{\mathbb{L}} \iff \vec{\mathbb{L}} \to \vec{D}$. Conversely, if $(\vec{\mathbb{F}}, \vec{D})$ is a duality then $\vec{D}$ is multiplicative (same proof as for undirected case). □

5.2. Restricted dualities. If we restrict the universe of the considered graphs G then we can expect more "dual phenomena". In such cases we speak about *restricted* *dualities*. Explicitly, a (singleton) $\mathcal{C}$- restricted duality is formed by a pair F, D such that for every graph $G \in \mathcal{C}$ holds:

$$F \not\longrightarrow G \iff G \longrightarrow H.$$

In the extremal case that for every connected $F \in \mathcal{C}$ there exists D_F such that F, D_F form a $\mathcal{C}$-restricted duality we say that $\mathcal{C}$ has *all restricted dualities* [33]. Before stating the main result we give two motivating examples.

5.2.1. Planar graphs. For instance, Figure 9 displays a restricted dualities for planar graphs: A planar graph has a homomorphism to the so-called Clebsch graph if and only if it does not contain a triangle. As triangle free planar graphs are 3-colorable, this generalizes the celebrated theorem of Grötzsch [17]. Of course, this characterization by a single obstacle leads to a fast algorithm (actually a linear-time one) to decide whether a decomposition exists. This has some interesting applications in the context of mathematical logic.

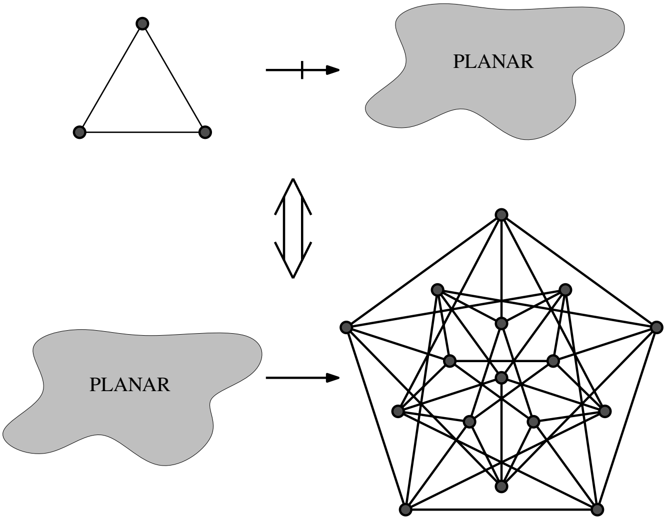


Figure 9. A restricted duality for planar graphs.

This particular result has been proven by Naserasr [26] and uses the property that every planar graph G may be colored by 16 colors in such a way that every cycle of length 5 gets at least 4 colors. Actually, even proving that a fixed number of colors would be sufficient for coloring every planar graphs in such a way every cycle of length 5 gets at least 4 colors is already non trivial.

5.2.2. Bounded degree graphs. A graph is *sub-cubic* if the degrees of all its vertices are ≤ 3. By Brooks theorem (see e.g. [8]) every sub-cubic connected graph is 3-colorable with the single exception of K_4. What about the class of all sub-cubic *triangle-free* graphs? Does there exists a triangle free 3-colorable bound? This question has been positively answered by Dreyer et al. [9] and Häggkvist and Hell [18]. In fact for every finite set $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ of connected graphs there exists a graph H with the following property:

$$G \longrightarrow H \text{ for every sub-cubic graph } G \in \mathrm{Forb_h}(\mathcal{F}).$$

(Here $\mathrm{Forb_h}(\mathcal{F})$ is the class of all graphs G which satisfy $F_i \nrightarrow G$ for every $i = 1, 2, \ldots, t$. Thus $\mathrm{Forb_h}(K_3)$ is the class of all triangle free graphs.) We can briefly say that the class of all sub-cubic graphs has all restricted dualities.

Note that while sub-cubic graphs, and more generally graphs with bounded degrees, have all restricted dualities, this is not true for classes of degenerate graphs [27], [28].

5.2.3. Bounded expansion classes. The two preceding examples actually fit in a more a general setting which has been proved by [33]:

Theorem 26. *Every class with bounded expansion has all restricted dualities.*

Explicitly: For every bounded expansion class $\mathcal{C}$ and for any finite set $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ of connected graphs there exists a graph $\mathcal{D}_{\mathcal{F}}$ such that $\mathcal{D}_{\mathcal{F}} \in$ $\mathrm{Forb_h}(\mathcal{F}$ and $G \to D_{\mathcal{F}}$ for every $G \in \mathcal{C}$ and $G \in \mathrm{Forb_h}(\mathcal{F}$.

5.3. Characterization of graph classes with restricted dualities. For restricted dualities such a characterization was not known and this has been left open as a problem (see e.g.[35]) . Using limit structures one can deduce such a characterization using the following notion: For a graph G and a real $\epsilon > 0$, define $\phi^{\epsilon}(G)$ as a minimum order of a graph H such that $G \to H$ and $\mathrm{dist}(G, H) \leq \epsilon$ (we arbitrarily choose between those graphs which have these properties, by using, for instance, some arbitrary linear order on $\mathcal{G}$; such a graph H we can call ϵ*-retract* of G).

Theorem 27. *Let $\mathcal{C}$ be a class of graphs. Then $\mathcal{C}$ has all restricted dualities if and only if for every $\epsilon > 0$ we have $\sup_{G \in \mathcal{C}} \phi^{\epsilon}(G) < \infty$.*

Moreover, for every connected graph F, there is a sequence

$$D_t(F) \leftarrow D_{t+1}(F) \leftarrow \cdots$$

of duals of F relative to $\mathcal{C}$ which converges to $\sup(\mathcal{C}^+ \cap \mathrm{Forb}(F))$, where $\mathcal{C}^+$ denotes the closure of $\mathcal{C}$ be finite disjoint unions.

The class of all perfect graphs (and equivalently, the class of all complete graphs) has all restricted dualities while it clearly fails to be a bounded expansion class. Yet this class is covered by Theorem 27.

6. Concluding remarks

(1) Low tree depth decomposition together with the finiteness of bounded tree depth graphs imply the following corollary:

> For every p and a bounded expansion class $\mathcal{C}$ there exists a positive integer $N = N(p, \mathcal{C})$ with the following property: every graph $G \in \mathcal{C}$ has a partition $V_1, \dots, V_N$ of its vertices such that the graph induced by any $p' \leq p$ parts has its core of size at most $f(p')$ where $f(p')$ depend neither on G nor on $\mathcal{C}$.

(2) If we want to bound $\chi_p(G)$ then we do not have to assume that G comes from a bounded expansion class. Instead it suffices that grads $\nabla_i(G)$ are all bounded by a constant for $i \leq p^p$. The flowchart of dependency of parameters is indicated in the following diagram.

$$\begin{tikzcd}
\text{bounded } \chi(G) & \text{bounded} \nabla_0(G) \arrow[l] & \text{(degenerate)} \\
\text{bounded } \chi_2(G) \arrow[u] \arrow[ur] & \text{bounded} \nabla_1(G) \arrow[l] \arrow[u] & \\
\text{bounded } \chi_p(G) \arrow[u] & \text{bounded} \nabla_p(G) \arrow[u] & \\
\text{bounded } \chi_{2p+2}(G) \arrow[u] \arrow[ur] & \text{bounded} \nabla_{p^p}(G) \arrow[u] \arrow[ul] & \\
\chi_r(G) < g(r) \arrow[u] \arrow[r, leftrightarrow] & \nabla_r(G) < f(r) \arrow[u] & \text{(bounded expansion)} \\
\text{bounded td}(G) \arrow[u] \arrow[r] & \text{bounded } \nabla(G) \arrow[u] & \text{(proper minor closed)}
\end{tikzcd}$$

(3) Bounded expansion classes with low-tree depth decompositions (which can be found linearly) include many particular problems which were studied individually. For example this yields a linear algorithm for the decision whether a given graph is an induced subgraph of a graph (in a fixed bounded expansion graph). This problem was previously only known to be linearly decidable for minor closed classes. Note also that already the chromatic number χ_1 was already studied extensively (as star chromatic number – a strengthening of acyclic chromatic number).

(4) As remarked earlier it is routine to convert most of the above results to results about oriented graphs, hypergraphs and relational structures. This can be done by use of incidence graphs (see e.g. [45], [35]), by means of Gaifman graphs, or directly. However in some cases the best results are obtained by ad hoc constructions and it seems that in this direction a proper setting is not yet found.

(5) Sub-linear separators need sub-exponential growth ([29]). A very small growth can still guarantee that a bounded expansion class is small [12]. In many such cases this is probably far from optimal.

(6) In the background of low tree depth decomposition lies the following particular (non-trivial) result:

> Let $\mathcal{C}$ be a bounded expansion class (resp. a nowhere dense class), let d be a positive integer. Denote by $\mathcal{C} \bullet K_d$ the class of all graphs $G \bullet K_d$ which arise from a $G \in \mathcal{C}$ by replacing every vertex of G by a complete graph K_d (this is sometimes called the *lexicographic product*). Then the class $\mathcal{C} \bullet K_d$ has bounded expansion (resp. is nowhere dense).

Note that if $\mathcal{C}$ is the class of planar graphs then already the class $\mathcal{C} \bullet K_2$ is so rich that every graph is a minor of a graph in $\mathcal{C} \bullet K_2$ (and yet $\mathcal{C} \bullet K_2$ has a bounded expansion).

Finally let us remark that the geometric approach (distances and limits) is further developed in [38].

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5ECM Amsterdam 2008

Bundle gerbes and surface holonomy

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Abstract. Hermitian bundle gerbes with connection are geometric objects for which a notion of surface holonomy can be defined for closed oriented surfaces. We systematically introduce bundle gerbes by closing the pre-stack of trivial bundle gerbes under descent.

Inspired by structures arising in a representation theoretic approach to rational conformal field theories, we introduce geometric structure that is appropriate to define surface holonomy in more general situations: Jandl gerbes for unoriented surfaces, D-branes for surfaces with boundaries, and bi-branes for surfaces with defect lines.

Mathematics Subject Classification (2000). 53C08, 53C80.

Keywords. Bundle gerbes, conformal field theory, D-branes, bi-branes.

1. Introduction

Two-dimensional quantum field theories have been a rich source of relations between different mathematical disciplines. A prominent class of examples of such theories are the two-dimensional rational conformal field theories, which admit a mathematically precise description (see [SFR] for a summary of recent progress). A large subclass of these also have a classical description in terms of an action, in which a term given by a surface holonomy enters.

The appropriate geometric object for the definition of surface holonomies for oriented surfaces with empty boundary are hermitian bundle gerbes. We systematically introduce bundle gerbes by first defining a pre-stack of trivial bundle gerbes, in such a way that surface holonomy can be defined, and then closing this pre-stack under descent. This construction constitutes in fact a generalization of the geometry of line bundles, their holonomy and their applications to classical particle mechanics.

Inspired by results in a representation theoretic approach to rational conformal field theories, we then introduce in the same spirit geometric structure that allows to define surface holonomy in more general situations: Jandl gerbes for unoriented surfaces, D-branes for surfaces with boundaries, and bi-branes for surfaces with defect lines.

Acknowledgements. K.W. gratefully acknowledges support by a Feodor-Lynen scholarship, granted by the Alexander von Humboldt foundation.

2. Hermitian line bundles and holonomy

Before discussing bundle gerbes, it is appropriate to summarize some pertinent aspects of line bundles.

One of the basic features of a (complex) line bundle L over a smooth manifold M is that it is locally trivializable. This means that M can be covered by open sets U_α such that there exist isomorphisms $\phi_\alpha\colon L|_{U_\alpha} \longrightarrow \mathbf{1}_{U_\alpha}$, where $\mathbf{1}_{U_\alpha}$ denotes the trivial line bundle $\mathbb{C} \times U_\alpha$. A choice of such maps ϕ_α defines gluing isomorphisms

$$g_{\alpha\beta}\colon \mathbf{1}_{U_\alpha}\big|_{U_\alpha\cap U_\beta} \longrightarrow \mathbf{1}_{U_\beta}\big|_{U_\alpha\cap U_\beta} \quad \text{with } g_{\beta\gamma}\circ g_{\alpha\beta} = g_{\alpha\gamma} \text{ on } U_\alpha\cap U_\beta\cap U_\gamma. \tag{2.1}$$

Isomorphisms between trivial line bundles are just smooth functions. Given a set of gluing isomorphisms one can obtain as additional structure the total space as the manifold

$$L := \bigsqcup_\alpha \mathbf{1}_{U_\alpha} /\!\sim\,, \tag{2.2}$$

with the relation $\sim$ identifying an element ℓ of $\mathbf{1}_{U_\alpha}$ with $g_{\alpha\beta}(\ell)$ of $\mathbf{1}_{U_\beta}$. In short, every bundle is glued together from trivial bundles.

In the following all line bundles will be equipped with a hermitian metric, and all isomorphisms are supposed to be isometries. Such line bundles form categories, denoted $\mathcal{B}un(M)$. The trivial bundle $\mathbf{1}_M$ defines a full, one-object subcategory $\mathcal{B}untriv(M)$ whose endomorphism set is the monoid of U(1)-valued functions on M. Denoting by $\pi_0(\mathcal{C})$ the set of isomorphism classes of a category $\mathcal{C}$ and by $H^\bullet(M,\underline{\mathrm{U}(1)})$ the sheaf cohomology of M with coefficients in the sheaf of U(1)-valued functions, we have the bijection

$$\pi_0(\mathcal{B}un(M)) \cong H^1(M,\underline{\mathrm{U}(1)}) \cong H^2(M,\mathbb{Z}), \tag{2.3}$$

under which the isomorphism class of the trivial bundle is mapped to zero.

Another basic feature of line bundles is that they pull back along smooth maps: for L a line bundle over M and $f\colon M' \longrightarrow M$ a smooth map, the pullback f^*L is a line bundle over M', and this pullback f^* extends to a functor

$$f^*\colon \mathcal{B}un(M) \longrightarrow \mathcal{B}un(M'). \tag{2.4}$$

Furthermore, there is a unique isomorphism $g^*(f^*L) \longrightarrow (f\circ g)^*L$ for composable maps f and g.

As our aim is to discuss holonomies, we should in fact consider a different category, namely line bundles equipped with (metric) connections. These form again a category, denoted by $\mathcal{B}un^{\nabla}(M)$, and this has again a full subcategory $\mathcal{B}untriv^{\nabla}(M)$ of trivial line bundles with connection. But now this subcategory has more than one object: every 1-form $\omega \in \Omega^1(M)$ can serve as a connection on a trivial line bundle $\mathbf{1}$ over M; the so obtained objects are denoted by $\mathbf{1}_\omega$. The set $\mathrm{Hom}(\mathbf{1}_\omega, \mathbf{1}_{\omega'})$ of connection-preserving isomorphisms $\eta\colon \mathbf{1}_\omega \longrightarrow \mathbf{1}_{\omega'}$ is the set of smooth functions $g\colon M \longrightarrow \mathrm{U}(1)$ satisfying

$$\omega' - \omega = -\mathrm{i}\,\mathrm{dlog}\, g. \tag{2.5}$$

Just like in (2.2), every line bundle L with connection can be glued together from line bundles $\mathbf{1}_{\omega_\alpha}$ along connection-preserving gluing isomorphisms $\eta_{\alpha\beta}$.

The curvature of a trivial line bundle $\mathbf{1}_\omega$ is $\mathrm{curv}(\mathbf{1}_\omega) := \mathrm{d}\omega \in \Omega^2(M)$, and is thus invariant under connection-preserving isomorphisms. It follows that the curvature of any line bundle with connection is a globally well-defined, closed 2-form. We recall that the cohomology class of this 2-form in real cohomology coincides with the characteristic class in (2.3).

In order to introduce the holonomy of line bundles with connection, we say that the holonomy of a trivial line bundle $\mathbf{1}_\omega$ over S^1 is

$$\mathrm{Hol}_{\mathbf{1}_\omega} := \exp\left(2\pi\mathrm{i} \int_{S^1} \omega\right) \in \mathrm{U}(1). \tag{2.6}$$

If $\mathbf{1}_\omega$ and $\mathbf{1}_{\omega'}$ are trivial line bundles over S^1, and if there exists a morphism η in $\mathrm{Hom}(\mathbf{1}_\omega, \mathbf{1}_{\omega'})$, we have $\mathrm{Hol}_{\mathbf{1}_\omega} = \mathrm{Hol}_{\mathbf{1}_{\omega'}}$ because

$$\int_{S^1} \omega' - \int_{S^1} \omega = \int_{S^1} -\mathrm{i}\,\mathrm{dlog}\,\eta \;\in \mathbb{Z}. \tag{2.7}$$

More generally, if L is any line bundle with connection over M, and $\Phi\colon S^1 \longrightarrow M$ is a smooth map, then the pullback bundle $\Phi^* L$ is trivial since $H^2(S^1, \mathbb{Z}) = 0$, and hence one can choose an isomorphism $\mathcal{T}\colon \Phi^* L \xrightarrow{\sim} \mathbf{1}_\omega$ for some $\omega \in \Omega^1(S^1)$. We then set

$$\mathrm{Hol}_L(\Phi) := \mathrm{Hol}_{\mathbf{1}_\omega}. \tag{2.8}$$

This is well-defined because any other trivialization $\mathcal{T}'\colon \Phi^* L \longrightarrow 1_{\omega'}$ provides a transition isomorphism $\eta := \mathcal{T}' \circ \mathcal{T}^{-1}$ in $\mathrm{Hom}(\mathbf{1}_\omega, \mathbf{1}_{\omega'})$. But as we have seen above, the holonomies of isomorphic trivial line bundles coincide.

Let us also mention an elementary example of a physical application of line bundles and their holonomies: the action functional S for a charged point particle. For (M, g) a (pseudo-)Riemannian manifold and $\Phi\colon \mathbb{R} \supset [t_1, t_2] \longrightarrow (M, g)$ the trajectory of a point particle of mass m and electric charge e, one commonly writes the action $S[\Phi]$ as the sum of the kinetic term

$$S_{\mathrm{kin}}[\Phi] = \frac{m}{2} \int_{t_1}^{t_2} g\left(\tfrac{\mathrm{d}\Phi}{\mathrm{d}t}, \tfrac{\mathrm{d}\Phi}{\mathrm{d}t}\right) \tag{2.9}$$

and a term

$$-\,e\int_{t_1}^{t_2}\Phi^*A, \tag{2.10}$$

with A the electromagnetic gauge potential. However, this formulation is inappropriate when the electromagnetic field strength F is not exact, so that a gauge potential A with $\mathrm{d}A = F$ exists only locally. As explained above, keeping track of such local 1-forms A_α and local 'gauge transformations', i.e., connection-preserving isomorphisms between those, leads to the notion of a line bundle L with connection. For a closed trajectory, i.e., $\Phi(t_1) = \Phi(t_2)$, the action should be defined as

$$\mathrm{e}^{\mathrm{i}S[\Phi]} = \mathrm{e}^{\mathrm{i}S_{\mathrm{kin}}[\Phi]}\,\mathrm{Hol}_L(\Phi). \tag{2.11}$$

An important feature of bundles in physical applications is the 'Dirac quantization' condition on the field strength F: the integral of F over any closed surface Σ in M gives an integer. This follows from the coincidence of the cohomology class of F with the characteristic class in (2.3). Another aspect is a neat explanation of the Aharonov–Bohm effect. A line bundle over a non-simply connected manifold can have vanishing curvature and yet non-trivial holonomies. In the quantum theory holonomies are observable, and thus the gauge potential A contains physically relevant information even if its field strength is zero. Both aspects, the quantization condition and the Aharonov–Bohm effect, persist in the generalization of line bundles to bundle gerbes, which we discuss next.

3. Gerbes and surface holonomy

In this section we formalize the procedure of Section 2 that has lead us from local 1-form gauge potentials to line bundles with connection: we will explain that it is the closure of the category of trivial bundles with connection under descent. We then apply the same principle to locally defined 2-forms, whereby we arrive straightforwardly at the notion of bundle gerbes with connection. We describe the notion of surface holonomy of such gerbes and their applications to physics analogously to Section 2.

3.1. Descent of bundles. As a framework for structures with a category assigned to every manifold and consistent pullback functors we consider presheaves of categories. Let $\mathcal{M}an$ be the category of smooth manifolds and smooth maps, and let $\mathcal{C}at$ be the 2-category of categories, with functors between categories as 1-morphisms and natural transformations between functors as 2-morphisms. Then a presheaf of categories is a lax functor

$$\mathcal{F} : \mathcal{M}an^{\mathrm{opp}} \longrightarrow \mathcal{C}at \tag{3.1}$$

It assigns to every manifold M a category $\mathcal{F}(M)$, and to every smooth map $f: M' \longrightarrow M$ a functor $\mathcal{F}(f): \mathcal{F}(M) \longrightarrow \mathcal{F}(M')$. By the qualification 'lax' we mean that the composition of maps must only be preserved up to coherent isomorphisms.

In Section 2 we have already encountered four examples of presheaves: the presheaf $\mathcal{B}un$ of line bundles, the presheaf $\mathcal{B}un^{\nabla}$ of line bundles with connection, and their sub-presheaves of trivial bundles.

To formulate a gluing condition for presheaves of categories we need to specify coverings. Here we choose *surjective submersions* $\pi: Y \longrightarrow M$. We remark that every cover of M by open sets U_α provides a surjective submersion with Y the disjoint union of the U_α; thus surjective submersions generalize open coverings. This generalization proves to be important for many examples of bundle gerbes, such as the lifting of bundle gerbes and the canonical bundle gerbes of compact simple Lie groups.

With hindsight, a choice of coverings endows the category $\mathcal{M}an$ with a Grothendieck topology. Both surjective submersions and open covers define a Grothendieck topology, and since every surjective submersion allows for local sections, the resulting two Grothendieck topologies are equivalent. And in fact the submersion topology is the maximal one equivalent to open coverings.

Along with a covering $\pi: Y \longrightarrow M$ there comes a simplicial manifold

$$\cdots \underset{\partial_3}{\overset{\partial_0}{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}}} Y^{[3]} \underset{\partial_2}{\overset{\partial_0}{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}}} Y^{[2]} \underset{\partial_1}{\overset{\partial_0}{\substack{\longrightarrow\\ \longrightarrow}}} Y \xrightarrow{\pi} M. \tag{3.2}$$

Here $Y^{[n]}$ denotes the n-fold fibre product of Y over M,

$$Y^{[n]} := \{(y_0, \dots, y_{n-1}) \in Y^n \mid \pi(y_0) = \cdots = \pi(y_{n-1})\}, \tag{3.3}$$

and the map $\partial_i: Y^{[n]} \longrightarrow Y^{[n-1]}$ omits the ith entry. In particular $\partial_0: Y^{[2]} \longrightarrow Y$ is the projection to the second factor and $\partial_1: Y^{[2]} \longrightarrow Y$ the one to the first. All fibre products $Y^{[k]}$ are smooth manifolds, and all maps ∂_i are smooth. Now let L be a line bundle over M. By pullback along π we obtain:

(BO1) An object $\tilde{L} := \pi^* L$ in $\mathcal{B}un(Y)$.

(BO2) A morphism

$$\phi: \partial_0^* \tilde{L} \cong \partial_0^* \pi^* L \xrightarrow{\sim} \partial_1^* \pi^* L \cong \partial_1^* \tilde{L} \tag{3.4}$$

in $\mathcal{B}un(Y^{[2]})$ induced from the identity $\pi \circ \partial_0 = \pi \circ \partial_1$.

(BO3) A commutative diagram

$$\begin{tikzcd} \partial_1^*\partial_0^*\tilde{L} = \partial_0^*\partial_0^*\tilde{L} \arrow[r, "\partial_0^*\phi"] \arrow[rr, bend right, "\partial_1^*\phi"'] & \partial_0^*\partial_1^*\tilde{L} = \partial_2^*\partial_0^*\tilde{L} \arrow[r, "\partial_2^*\phi"] & \partial_2^*\partial_1^*\tilde{L} = \partial_1^*\partial_1^*\tilde{L} \end{tikzcd} \tag{3.5}$$

of morphisms in $\mathcal{B}un(Y^{[3]})$; or in short, an equality $\partial_2^*\phi \circ \partial_0^*\phi = \partial_1^*\phi$.

We call a pair $(\tilde{L}, \phi)$ as in (BO1) and (BO2) which satisfies (BO3) a *descent object* in the presheaf $\mathcal{B}un$. Analogously we obtain for a morphism $f: L \longrightarrow L'$ of line bundles over M

(BM1) A morphism $\tilde{f} := \pi^* f : \tilde{L} \longrightarrow \tilde{L}'$ in $\mathcal{B}un(Y)$.

(BM2) A commutative diagram

$$\phi' \circ \partial_0^* \tilde{f} = \partial_1^* \tilde{f} \circ \phi \tag{3.6}$$

of morphisms in $\mathcal{B}un(Y^{[2]})$.

Such a morphism $\tilde{f}$ as in (BM1) obeying (BM2) is called a *descent morphism* in the presheaf $\mathcal{B}un$.

Descent objects and descent morphisms for a given covering π form a category $\mathcal{D}esc(\pi : Y \longrightarrow M)$ of descent data. What we described above is a functor

$$\iota_\pi : \mathcal{B}un(M) \longrightarrow \mathcal{D}esc(\pi : Y \longrightarrow M). \tag{3.7}$$

The question arises whether every 'local' descent object corresponds to a 'global' object on M, i.e., whether the functor ι_π is an equivalence of categories.

The construction generalizes straightforwardly to any presheaf of categories $\mathcal{F}$, and if the functor ι_π is an equivalence for all coverings $\pi : Y \longrightarrow M$, the presheaf $\mathcal{F}$ is called a *sheaf of categories* (or stack). Extending the gluing process from (2.2) to non-trivial bundles shows that the presheaves $\mathcal{B}un$ and $\mathcal{B}un^\nabla$ are sheaves. In contrast, the presheaves $\mathcal{B}untriv$ and $\mathcal{B}untriv^\nabla$ of trivial bundles are not sheaves, since gluing of trivial bundles does in general not result in a trivial bundle. In fact the gluing process (2.2) shows that every bundle can be obtained by gluing trivial ones. In short, the sheaf $\mathcal{B}un^\nabla$ of line bundles with connection is obtained by closing the presheaf $\mathcal{B}untriv^\nabla$ under descent.

3.2. Bundle gerbes. Our construction of line bundles started from trivial line bundles with connection which are just 1-forms on M, and the fact that 1-forms can be integrated along curves has lead us to the notion of holonomy. To arrive at a notion of *surface* holonomy, we now consider a category of 2-forms, or rather a 2-category:

- An object is a 2-form $\omega \in \Omega^2(M)$, called a *trivial bundle gerbe with connection* and denoted by $\mathcal{I}_\omega$.
- A 1-morphism $\eta\colon \omega \longrightarrow \omega'$ is a 1-form $\eta \in \Omega^1(M)$ such that $\mathrm{d}\eta = \omega' - \omega$.
- A 2-morphism $\phi\colon \eta \Longrightarrow \eta'$ is a smooth function $\phi : M \longrightarrow \mathrm{U}(1)$ such that $-\mathrm{i}\,\mathrm{dlog}(\phi) = \eta' - \eta$.

There is also a natural pullback operation along maps, induced by pullback on differential forms. The given data can be rewritten as a presheaf of 2-categories, as there is a 2-category attached to each manifold. This presheaf should now be closed under descent to obtain a sheaf of 2-categories. As a first step we complete the morphism categories under descent. Since these are categories of trivial line bundles with connections, we set

$$\mathrm{Hom}(\mathcal{I}_\omega, \mathcal{I}_{\omega'}) := \mathcal{B}un^{\nabla}_{\omega'-\omega}(M), \tag{3.8}$$

the category of hermitian line bundles with connection of fixed curvature $\omega' - \omega$. The horizontal composition is given by the tensor product in the category of bundles. Finally, completing the 2-category under descent, we get the definition of a bundle gerbe:

Definition 1. A bundle gerbe $\mathcal{G}$ (with connection) over M consists of the following data: a covering $\pi : Y \longrightarrow M$, and for the associated simplicial manifold

$$Y^{[4]} \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; Y^{[3]} \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; Y^{[2]} \;\underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}}\; Y \xrightarrow{\;\pi\;} M \tag{3.9}$$

(GO1) an object $\mathcal{I}_\omega$ of $\mathcal{G}rbtriv^{\nabla}(Y)$: a 2-form $\omega \in \Omega^2(Y)$;

(GO2) a 1-morphism
$$L\colon\; \partial_0^* \mathcal{I}_\omega \longrightarrow \partial_1^* \mathcal{I}_\omega \tag{3.10}$$
in $\mathcal{G}rbtriv^{\nabla}(Y^{[2]})$: a line bundle L with connection over $Y^{[2]}$;

(GO3) a 2-isomorphism
$$\mu\colon\; \partial_2^* L \otimes \partial_0^* L \Longrightarrow \partial_1^* L \tag{3.11}$$
in $\mathcal{G}rbtriv^{\nabla}(Y^{[3]})$: a connection-preserving morphism of line bundles over $Y^{[3]}$;

(GO4) an equality
$$\partial_2^*\mu \circ (\mathrm{id} \otimes \partial_0^*\mu) = \partial_1^*\mu \circ (\partial_3^*\mu \otimes \mathrm{id}) \tag{3.12}$$
of 2-morphisms in $\mathcal{G}rbtriv^{\nabla}(Y^{[4]})$.

For later applications it will be necessary to close the morphism categories under a second operation, namely direct sums. Closing the category of line bundles with connection under direct sums leads to the category of complex vector bundles with connection, i.e., we set

$$\mathrm{Hom}(\mathcal{I}_\omega, \mathcal{I}_{\omega'}) := \mathcal{V}ect\mathcal{B}un^{\nabla}_{\omega'-\omega}(M), \tag{3.13}$$

where the curvature of these vector bundles is constrained to satisfy

$$\frac{1}{n}\,\mathrm{tr}(\mathrm{curv}(L)) = \omega' - \omega, \tag{3.14}$$

with n the rank of the vector bundle. Notice that this does not affect the definition of a bundle gerbe, since the existence of the 2-isomorphism μ restricts the rank of L to be one.

As a next step, we need to introduce 1-morphisms and 2-morphisms between bundle gerbes. 1-morphisms have to compare two bundle gerbes $\mathcal{G}$ and $\mathcal{G}'$. We assume first that both bundle gerbes have the same covering $Y \longrightarrow M$.

Definition 2. i) A 1-morphism between two bundle gerbes $\mathcal{G} = (Y, \omega, L, \mu)$ and $\mathcal{G}' = (Y, \omega', L', \mu')$ over M with the same surjective submersion $Y \longrightarrow M$ consists of the following data on the associated simplicial manifold:

$$Y^{[4]} \;\substack{\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow}\; Y^{[3]} \;\substack{\longrightarrow\\[-4pt]\longrightarrow\\[-4pt]\longrightarrow}\; Y^{[2]} \;\underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}}\; Y \xrightarrow{\;\pi\;} M. \tag{3.15}$$

(G1M1) a 1-morphism $A\colon \mathcal{I}_\omega \longrightarrow \mathcal{I}_{\omega'}$ in $\mathcal{G}rbtriv^{\nabla}(Y)$: a rank-$n$ hermitian vector bundle A with connection of curvature $\frac{1}{n}\,\mathrm{tr}(\mathrm{curv}(L)) = \omega' - \omega$;

(G1M2) a 2-isomorphism $\alpha\colon L' \otimes \partial_0^* A \Longrightarrow \partial_1^* A \otimes L$ in $\mathcal{G}rbtriv^{\nabla}(Y^{[2]})$: a connection-preserving morphism of hermitian vector bundles;

(G1M3) a commutative diagram

$$(\mathrm{id} \otimes \mu') \circ (\partial_2^* \alpha \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \partial_0^* \alpha) = \partial_1^* \alpha \circ (\mu \otimes \mathrm{id}) \tag{3.16}$$

of 2-morphisms in $\mathcal{G}rbtriv^{\nabla}(Y^{[3]})$.

ii) A 2-morphism between two such 1-morphisms (A, α) and (A', α') consists of

(G2M1) a 2-morphism $\beta\colon A \Longrightarrow A'$ in $\mathcal{G}rbtriv^{\nabla}(Y)$: a connection-preserving morphism of vector bundles;

(G2M2) a commutative diagram

$$\alpha' \circ (\mathrm{id} \otimes \partial_0^* \beta) = (\partial_1^* \beta \otimes \mathrm{id}) \circ \alpha \tag{3.17}$$

of 2-morphisms in $\mathcal{G}rbtriv^{\nabla}(Y^{[2]})$.

Since 1-morphisms are composed by taking tensor products of vector bundles, a 1-morphism is invertible if and only if its vector bundle is of rank one.

In order to define 1-morphisms and 2-morphisms between bundle gerbes with possibly different coverings $\pi\colon Y \longrightarrow M$ and $\pi'\colon Y' \longrightarrow M$, we pull all the data back to a common refinement of these coverings and compare them there. We call a covering $\zeta\colon Z \longrightarrow M$ a common refinement of π and π' iff there exist maps $s\colon Z \longrightarrow Y$ and $s'\colon Z \longrightarrow Y'$ such that

$$\begin{tikzcd} Y \arrow[dr, "\pi"'] & Z \arrow[l, "s"'] \arrow[r, "s'"] \arrow[d, "\zeta"] & Y' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd} \tag{3.18}$$

commutes. An example of such a common refinement is the fibre product $Z := Y \times_M Y' \longrightarrow M$, with the maps $Z \longrightarrow Y$ and $Z \longrightarrow Y'$ given by the projections. The important point about a common refinement $Z \longrightarrow M$ is that the maps s and s' induce simplicial maps

$$Y^\bullet \longleftarrow Z^\bullet \longrightarrow Y'^\bullet . \tag{3.19}$$

For bundle gerbes $\mathcal{G} = (Y, \omega, L, \mu)$ and $\mathcal{G}' = (Y', \omega', L', \mu')$ we obtain new bundle gerbes with surjective submersion Z by pulling back all the data along the simplicial maps s and s'. Explicitly,

$$\mathcal{G}_Z := (Z, s_0^*\omega, s_1^*L, s_2^*\mu) \quad \text{and} \quad \mathcal{G}'_Z = (Z, s_0'^*\omega', s_1'^*L', s_2'^*\mu').$$

Also morphisms can be refined by pulling them back.

Definition 3. i) A 1-*morphism* between two bundle gerbes $\mathcal{G} = (Y, \omega, L, \mu)$ and $\mathcal{G}' = (Y', \omega', L', \mu')$ consists of a common refinement $Z \longrightarrow M$ of the coverings $Y \longrightarrow M$ and $Y' \longrightarrow M$ and a morphism (A, α) of the two refined gerbes $\mathcal{G}_Z$ and $\mathcal{G}'_Z$.

ii) A 2-morphism between 1-morphisms $\mathfrak{m} = (Z, A, \alpha)$ and $\mathfrak{m}' = (Z', A', \alpha')$ consists of a common refinement $W \longrightarrow M$ of the coverings $Z \longrightarrow M$ and $Z' \longrightarrow M$ (respecting the projections to Y and Y', respectively) and a 2-morphism β of the refined morphisms $\mathfrak{m}_W$ and $\mathfrak{m}'_W$. In addition two such 2-morphisms (W, β) and (W', β') must be identified iff there exists a further common refinement $V \longrightarrow M$ of $W \longrightarrow M$ and $W' \longrightarrow M$, compatible with the other projections, such that the refined 2-morphisms agree on V.

For a gerbe $\mathcal{G} = (Y, \omega, L, \mu)$ and a refinement $Z \longrightarrow M$ of Y the refined gerbe $\mathcal{G}_Z$ is isomorphic to $\mathcal{G}$. This implies that every gerbe is isomorphic to a gerbe defined over an open covering $Z := \bigsqcup_{i \in I} U_i$. Furthermore we can choose the covering in such a way that the line bundle over double intersections is trivial as well. When doing so we obtain the familiar description of gerbes in terms of local data, reproducing formulas by [Al], [Ga1]. Extending this description to morphisms it is straightforward to show that gerbes are classified by the so-called Deligne cohomology $H^k(M, \mathcal{D}(2))$ in degree two:

$$\pi_0(\mathcal{G}rb^{\nabla}(M)) \cong H^2(M, \mathcal{D}(2)). \tag{3.20}$$

Analogously we get the classification of gerbes without connection as

$$\pi_0(\mathcal{G}rb(M)) \cong H^2(M, \underline{\mathrm{U}(1)}) \cong H^3(M, \mathbb{Z}). \tag{3.21}$$

3.3. Surface holonomy. The holonomy of a trivial bundle gerbe $\mathcal{I}_\omega$ over a closed oriented surface Σ is by definition

$$\mathrm{Hol}_{\mathcal{I}_\omega} := \exp\Big(2\pi\mathrm{i} \int_\Sigma \omega\Big) \in \mathrm{U}(1). \tag{3.22}$$

If $\mathcal{I}_\omega$ and $\mathcal{I}_{\omega'}$ are two trivial bundle gerbes over Σ such that there exists a 1-isomorphism $\mathcal{I}_\omega \longrightarrow I_{\omega'}$, i.e., a vector bundle L of rank one, we have an equality $\mathrm{Hol}_{\mathcal{I}_\omega} = \mathrm{Hol}_{\mathcal{I}_{\omega'}}$ because

$$\int_\Sigma \omega' - \int_\Sigma \omega = \int_\Sigma \mathrm{curv}(L) \in \mathbb{Z}. \tag{3.23}$$

More generally, consider a bundle gerbe $\mathcal{G}$ with connection over a smooth manifold M, and a smooth map

$$\Phi \colon \Sigma \longrightarrow M \tag{3.24}$$

defined on a closed oriented surface Σ. Since $H^3(\Sigma, \mathbb{Z}) = 0$, the pullback $\Phi^*\mathcal{G}$ is isomorphic to a trivial bundle gerbe. Hence one can choose a trivialization, i.e., a 1-isomorphism

$$\mathcal{T} : \Phi^*\mathcal{G} \xrightarrow{\sim} \mathcal{I}_\omega \tag{3.25}$$

and define the holonomy of $\mathcal{G}$ around Φ by

$$\mathrm{Hol}_{\mathcal{G}}(\Phi) := \mathrm{Hol}_{\mathcal{I}_\omega}. \tag{3.26}$$

In the same way as for the holonomy of a line bundle with connection, this definition is independent of the choice of the 1-isomorphism $\mathcal{T}$. Namely, if $\mathcal{T}' \colon \Phi^*\mathcal{G} \xrightarrow{\sim} \mathcal{I}_{\omega'}$ is another trivialization, we have a transition isomorphism

$$L := \mathcal{T}' \circ \mathcal{T}^{-1} \colon \mathcal{I}_\omega \xrightarrow{\sim} \mathcal{I}_{\omega'}, \tag{3.27}$$

which shows the independence.

3.4. Wess–Zumino terms. As we have seen in Section 2, the holonomy of a line bundle with connection supplies a term in the action functional of a classical charged particle, describing the coupling to a gauge field whose field strength is the curvature of the line bundle. Analogously, the surface holonomy of a bundle gerbe with connection defines a term in the action of a classical charged string. Such a string is described in terms of a smooth map $\Phi\colon \Sigma \longrightarrow M$. The exponentiated action functional of the string is (compare (2.11))

$$\mathrm{e}^{\mathrm{i}S[\Phi]} = \mathrm{e}^{\mathrm{i}S_{\mathrm{kin}}[\Phi]}\,\mathrm{Hol}_{\mathcal{G}}(\Phi), \tag{3.28}$$

where $S_{\mathrm{kin}}[\Phi]$ is a kinetic term which involves a conformal structure on Σ. Physical models whose fields are maps defined on surfaces are called (non-linear) *sigma models*, and the holonomy term is called a *Wess–Zumino term*. Such terms are needed in certain models in order to obtain quantum field theories that are conformally invariant.

A particular class of sigma models with Wess–Zumino term is given by WZW (Wess–Zumino–Witten) models. For these the target space M is a connected compact simple Lie group G, and the curvature of the bundle gerbe $\mathcal{G}$ is an integral multiple of the canonical 3-form

$$H = \langle \theta \wedge [\theta \wedge \theta] \rangle \in \Omega^3(G)$$

(θ is the left-invariant Maurer–Cartan form on G, and $\langle \cdot\,,\cdot \rangle$ the Killing form of the Lie algebra $\mathfrak{g}$ of G). WZW models have been a distinguished arena for the interplay between Lie theory and the theory of bundle gerbes [Ga1], [GR]. This has lead to new insights both in the physical applications and in the underlying mathematical structures. Some of these will be discussed in the following sections.

Defining Wess–Zumino terms as the holonomy of a bundle gerbe with connection allows one in particular to explain the following two facts.

- *The Aharonov–Bohm effect*: This occurs when the bundle gerbe has a *flat* connection, i.e., its curvature $H \in \Omega^3(M)$ vanishes. This does not mean, though, that the bundle gerbe is trivial, since its class in $H^3(M,\mathbb{Z})$ may be pure torsion. In particular, it can still have non-constant holonomy, and thus a non-trivial Wess–Zumino term.

  An example for the Aharonov–Bohm effect is the sigma model on the 2-torus $T = S^1 \times S^1$. By dimensional reasons, the 3-form H vanishes. Nonetheless, since $H^2(T,\mathrm{U}(1)) = \mathrm{U}(1)$, there exists a whole family of Wess–Zumino terms parameterized by an angle, of which only the one with angle zero is trivial.

- *Discrete torsion*: The set of isomorphism classes of bundle gerbes with connection that have the same curvature H is parameterized by $H^2(M,\mathrm{U}(1))$ via the map

$$H^2(M,\mathrm{U}(1)) \longrightarrow \mathrm{Tors}(H^3(M,\mathbb{Z})). \tag{3.29}$$

If this group is non-trivial, there exist *different* Wess–Zumino terms for one and the same field strength H; their difference is called ‘discrete torsion’.

An example for discrete torsion is the level-k WZW model on the Lie group $\mathrm{PSO}(4n)$. Since $H^2(\mathrm{PSO}(4n), \mathrm{U}(1)) = \mathbb{Z}_2$, there exist two non-isomorphic bundle gerbes with connection having equal curvature.

4. The representation theoretic formulation of RCFT

4.1. Sigma models. Closely related to surface holonomies are novel geometric structures that have been introduced for unoriented surfaces, for surfaces with boundary, and for surfaces with defect lines. These structures constitute the second theme of this contribution, extending the construction of gerbes and surface holonomy via descent; they will be discussed in Sections 5, 6 and 7.

These geometric developments were in fact strongly inspired by algebraic and representation theoretic results in two-dimensional quantum field theories. To appreciate this connection we briefly review in this section the relation between spaces of maps $\Phi\colon \Sigma \longrightarrow M$, as they appear in the treatment of holonomies, and quantum field theories.

As already indicated in Section 3.4, a classical field theory, the (non-linear) *sigma model*, on a two-dimensional surface Σ, called the *world sheet*, can be associated to the space of smooth maps Φ from Σ to some smooth manifold M, called the *target space*. Appropriate structure on the target space determines a Lagrangian for the field theory on Σ. Geometric structure on M, e.g. a (pseudo-) Riemannian metric G, becomes, from this point of view, for any given map Φ a background function $G(\Phi(x))$ for the field theory on Σ.

Three main issues will then lead us to a richer structure related to surface holonomies:

- In string theory (where the world sheet Σ arises as the surface swept out by a string moving in M) and in other applications as well, one also encounters sigma models on world sheets Σ that have *non-empty boundary*. We will explain how the geometric data relevant for encoding boundary conditions – so called D-branes – can be derived from geometric principles.

- String theories of type I, which form an integral part of string dualities, involve *unoriented* world sheets. In string theory it is therefore a fundamental problem to exhibit geometric structure on the target space that provides a notion of holonomy for unoriented surfaces.

- An equally natural structure present in quantum field theory are *topological defect lines*, along which correlation functions of bulk fields can have a branch-cut. In specific models these can be understood, just like boundary conditions, as continuum versions of corresponding structures in lattice models of statistical mechanics. (For instance, in the lattice version of the Ising model a topological defect is produced by changing the coupling along all bonds that cross a specified line from ferromagnetic to antiferromagnetic.)

Sigma models have indeed been a significant source of examples for quantum field theories, at least on a heuristic level. Conversely, having a sigma model interpretation for a given quantum field theory allows for a geometric interpretation of quantum field theoretic quantities.

A distinguished subclass of theories in which this relationship between quantum field theory and geometry can be studied are two-dimensional conformal field theories, or CFTs, for short, and among these in particular the rational conformal field theories for which there exists a rigorous representation theoretic approach. The structures appearing in that approach in the three situations mentioned above suggest new geometric notions for conformal sigma models. Below we will investigate these notions with the help of standard geometric principles. Before doing so we formulate, in representation theoretic terms, the relevant aspects of the quantum field theories in question.

4.2. Rational conformal field theory. The conformal symmetry, together with further, so-called chiral, symmetries of a CFT can be encoded in the structure of a conformal vertex algebra $\mathcal{V}$. For any conformal vertex algebra one can construct (see e.g. [FrB]) a chiral CFT; in mathematical terms, a chiral CFT is a system of *conformal blocks*, i.e., sheaves over the moduli spaces of curves with marked points. These sheaves of conformal blocks are endowed with a projectively flat connection, the Knizhnik–Zamolodchikov connection, which in turn furnishes representations of the fundamental groups of the moduli spaces, i.e., of the mapping class groups.

Despite the physical origin of its name, a chiral conformal field theory is mathematically rigorous. On the other hand, from the two-dimensional point of view it is, despite its name, not a conventional quantum field theory, as one deals with (sections of) bundles instead of local correlation functions. In particular, it must not be confused with a full local conformal field theory, which is the relevant structure to enter our discussion of holonomies.

Chiral conformal field theories are particularly tractable when the vertex algebra $\mathcal{V}$ is rational in the sense of [Hu, Theorem 2.1]. Then the representation category $\mathcal{C}$ of $\mathcal{V}$ is a modular tensor category, and the associated chiral CFT is a *rational chiral CFT*, or chiral RCFT. In this situation, we can use the tools of three-dimensional topological quantum field theory (TFT). A TFT is, in short, a monoidal functor $\mathtt{tft}_{\mathcal{C}}$ [Tu, Chapter IV.7] that associates a finite-dimensional vector space $\mathtt{tft}_{\mathcal{C}}(\mathrm{E})$ to any (extended) surface E, and a linear map from $\mathtt{tft}_{\mathcal{C}}(\mathrm{E})$ to $\mathtt{tft}_{\mathcal{C}}(\mathrm{E}')$ to any (extended) cobordism $\mathcal{M}\colon \mathrm{E} \longrightarrow \mathrm{E}'$.

More precisely, a three-dimensional TFT is a projective monoidal functor from a category $\mathcal{C}ob_{\mathcal{C}}$ of decorated cobordisms to the category of finite-dimensional complex vector spaces. The modular tensor category $\mathcal{C}$ provides the decoration data for $\mathcal{C}ob_{\mathcal{C}}$. Specifically, the objects E of $\mathcal{C}ob_{\mathcal{C}}$ are extended surfaces, i.e.,[1] compact closed oriented two-manifolds with a finite set of embedded arcs, and each of these arcs is marked by an object of $\mathcal{C}$. A morphism $\mathrm{E} \longrightarrow \mathrm{E}'$ is an *extended cobordism*, i.e., a compact oriented three-manifold $\mathcal{M}$ with $\partial\mathcal{M} = (-\mathrm{E}) \sqcup \mathrm{E}'$, together with an oriented ribbon graph $\Gamma_{\mathcal{M}}$ in $\mathcal{M}$ such that at each marked arc of $(-\mathrm{E}) \sqcup \mathrm{E}'$ a ribbon of $\Gamma_{\mathcal{M}}$ is ending. Each ribbon of $\Gamma_{\mathcal{M}}$ is labeled by an object of $\mathcal{C}$, while each coupon of $\Gamma_{\mathcal{M}}$ is labeled by an element of the morphism space of $\mathcal{C}$ that corresponds to the objects of the ribbons which enter and leave the coupon. Composition in $\mathcal{C}ob_{\mathcal{C}}$ is defined by gluing, the identity morphism id_{E} is the cylinder over E, and the tensor product is given by disjoint union of objects and cobordisms.

A topological field theory furnishes, for any extended surface, a representation of the mapping class group. Our approach relies on the fundamental conjecture (which is largely established for a broad class of models) that, for $\mathcal{C}$ the representation category of a rational vertex algebra $\mathcal{V}$, the mapping class group representation given by $\mathtt{tft}_{\mathcal{C}}$ is equivalent to the one provided by the Knizhnik–Zamolodchikov connection on the conformal blocks for the vertex algebra $\mathcal{V}$.

4.3. The TFT construction of full RCFT. Let us now turn to the discussion of full local conformal field theories, which are the structures to be compared to holonomies. A *full* CFT is, by definition, a consistent system of *local correlation functions* that satisfy all sewing constraints (see e.g. [FjFRS2, Definition 3.14]). According to the principle of holomorphic factorization, every full RCFT can be understood with the help of a corresponding chiral CFT. The relevant chiral CFT is, however, not defined on world sheets Σ (which may be unoriented or have a non-empty boundary), but rather on their *complex doubles* $\widehat{\Sigma}$, which can be given the structure of extended surfaces; this affords a geometric separation of left- and right-movers. The double $\widehat{\Sigma}$ of Σ is, by definition, the orientation bundle over Σ modulo identification of the two points in the fibre over each boundary point of Σ. The world sheet Σ can be obtained from $\widehat{\Sigma}$ as the quotient by an orientation-reversing involution τ. To give some examples, when Σ is closed and orientable, then $\widehat{\Sigma}$ is just the disconnected sum $\widehat{\Sigma} = \Sigma \sqcup -\Sigma$ of two copies of Σ with opposite orientation, and the involution τ just exchanges these two copies; the double of both the disk and the real projective plane is the two-sphere (with τ being given, in standard complex coordinates, by $z \longmapsto \overline{z}^{-1}$ and by $z \longmapsto -\overline{z}^{-1}$, respectively); and the double of both the annulus and the Möbius strip is a two-torus. Further, when Σ comes with field insertions, that is, embedded

[1] Here various details are suppressed. Detailed information, e.g. the precise definition of a ribbon graph or the reason why $\mathtt{tft}_{\mathcal{C}}$ is only projective, can be found in many places, such as [Tu], [BK], [KRT] or [FFFS, Section 2.5–2.7].

arcs labeled by objects of either $\mathcal{C}$ (for arcs on $\partial\Sigma$) or pairs of objects of $\mathcal{C}$ (for arcs in the interior of Σ), then corresponding arcs labeled by objects of $\mathcal{C}$ are present on $\hat{\Sigma}$.

Given this connection between the surfaces relevant to chiral and full CFT, the relationship between the chiral and the full CFT can be stated as follows: A correlation function $C(\Sigma)$ of the full CFT on Σ is a specific element in the appropriate space of conformal blocks of the chiral CFT on the double $\hat{\Sigma}$. A construction of such elements has been accomplished in [FRS1], [FRS2], [FRS3], [FjFRS1]. The first observation is that they can be computed with the help of the corresponding TFT, namely as

$$C(\Sigma) = \mathtt{tft}_{\mathcal{C}}(\mathcal{M}_\Sigma)\, 1 \ \in \mathtt{tft}_{\mathcal{C}}(\hat{\Sigma}). \tag{4.1}$$

Here $\mathcal{M}_\Sigma \equiv \emptyset \xrightarrow{\mathcal{M}_\Sigma} \hat{\Sigma}$, the *connecting manifold* for the world sheet Σ, is an extended cobordism that is constructed from the data of Σ. Besides the category $\mathcal{C}$, the specification of the vector $C(\Sigma)$ needs a second ingredient: a (Morita class of a) symmetric special Frobenius algebra A in $\mathcal{C}$.

Let us give some details [2] of the construction of $C(\Sigma)$.

- As a three-manifold, $\mathcal{M}_\Sigma$ is the interval bundle over Σ modulo a $\mathbb{Z}_2$-identification of the intervals over $\partial\Sigma$. Explicitly,

$$\mathcal{M}_\Sigma = \big(\hat{\Sigma} \times [-1,1]\big)/\!\sim \quad \text{with } ([x,\mathrm{or}_2],t) \sim ([x,-\mathrm{or}_2],-t). \tag{4.2}$$

  It follows in particular that $\partial\mathcal{M}_\Sigma = \hat{\Sigma}$ and that Σ is naturally embedded in $\mathcal{M}_\Sigma$ as $\imath\colon \Sigma \xrightarrow{\simeq} \Sigma \times \{t{=}0\} \hookrightarrow \mathcal{M}_\Sigma$. Indeed, $\imath(\Sigma)$ is a deformation retract of $\mathcal{M}_\Sigma$, so that the topology of $\mathcal{M}_\Sigma$ is completely determined by the one of Σ.

- A crucial ingredient of the construction of the ribbon graph $\Gamma_{\mathcal{M}_\Sigma}$ in $\mathcal{M}_\Sigma$ is a (dual) oriented triangulation Γ of the submanifold $\imath(\Sigma)$ of $\mathcal{M}_\Sigma$. This triangulation is labeled by objects and morphisms of $\mathcal{C}$. It is here that the Frobenius algebra A enters: Each edge of $\Gamma \setminus \imath(\partial\Sigma)$ is covered with a ribbon labeled by the object A of $\mathcal{C}$, while each (three-valent) vertex is covered with a coupon labeled by the multiplication morphism $m \in \mathrm{Hom}_{\mathcal{C}}(A \otimes A, A)$. In addition, whenever these assignments in themselves would be in conflict with the orientations of the edges, a coupon with morphism in either $\mathrm{Hom}_{\mathcal{C}}(A \otimes A, \mathbf{1})$ or $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A \otimes A)$ is inserted. Such morphisms are part of the data for a Frobenius structure on A. Assuming, for now, that the world sheet Σ is oriented, independence of $C(\Sigma)$ from the choice of triangulation Γ amounts precisely to the statement that the object A carries the structure of a symmetric special Frobenius algebra.

- If Σ has *non-empty boundary*, the prescription for Γ is amended as follows. Each edge e of $\Gamma \cap \imath(\partial\Sigma)$ is covered with a ribbon labeled by a (left, say) A-module

[2] For another brief summary, with different emphasis, see Section 7 of [FRS4]. An in-depth exposition, including for instance the relevance of various orientations, can e.g. be found in Appendix B of [FjFRS1].

$N = N(e)$, while each vertex lying on $\imath(\partial\Sigma)$ is covered with a coupon that has incoming N- and A-ribbons as well as an outgoing N-ribbon and that is labeled by the representation morphism $\rho_N \in \mathrm{Hom}_{\mathcal{C}}(A \otimes N, N)$. The physical interpretation of the A-module N is as the *boundary condition* that is associated to a component of $\partial\Sigma$. That the object N of $\mathcal{C}$ labeling a boundary condition carries the structure of an A-module and that the morphism ρ_N is the corresponding representation morphism is precisely what is required (in addition to A being a symmetric special Frobenius algebra) in order to get independence of $C(\Sigma)$ from the choice of triangulation Γ.

- If Σ is *unoriented*, then as an additional feature one must ensure independence of $C(\Sigma)$ from the choice of local orientations of Σ. As shown in [FRS2], this requires an additional structure on the algebra A, namely the existence of a morphism $\sigma \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ that is an algebra isomorphism from the opposite algebra A^{opp} to A and squares to the twist of A, i.e., satisfies

$$\sigma \circ \eta = \eta, \quad \sigma \circ m = m \circ c_{A,A} \circ (\sigma \otimes \sigma), \quad \sigma \circ \sigma = \theta_A, \tag{4.3}$$

  where $\eta \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A)$, $\theta_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ and $c_{A,A} \in \mathrm{Hom}_{\mathcal{C}}(A \otimes A, A \otimes A)$ denote the unit morphism, the twist, and the self-braiding of A, respectively. This way A becomes a braided version of an algebra with involution. A symmetric special Frobenius algebra endowed with a morphism σ satisfying (4.3) is called a *Jandl algebra*.

- In the presence of *topological defect lines* on Σ a further amendment of the prescription is in order. The defect lines partition Σ into disjoint regions, and to the regions to the left and to the right of a defect line one may associate different (symmetric special Frobenius) algebras A_l and A_r, such that the part of the triangulation Γ in one region is labeled by the algebra A_l, while the part of Γ in the other region is labeled by A_r. The defect lines are to be regarded as forming a subset Γ^D of Γ themselves; each edge of Γ^D is covered with a ribbon labeled by some object B of $\mathcal{C}$, while each vertex of Γ lying on Γ^D is covered with a coupon labeled by a morphism $\rho \in \mathrm{Hom}_{\mathcal{C}}(A_l \otimes B, B)$, respectively $\varrho \in \mathrm{Hom}_{\mathcal{C}}(B \otimes A_r, B)$. Consistency requires that these morphisms endow the object B of $\mathcal{C}$ that labels a defect line with the structure of an A_l-A_r-bimodule. (Below we will concentrate on the case $A_l = A_r =: A$, so that we deal with A-bimodules.)

- There are also rules for the morphisms of $\mathcal{C}$ that label bulk, boundary and defect fields, respectively.

The prescription summarized above allows one to construct the correlator (4.1) for any arbitrary world sheet Σ. The so obtained correlators can be proven [FjFRS1] to satisfy all consistency conditions that the correlators of a CFT must obey. Thus, specifying the algebra A is sufficient to obtain a consistent system of correlators. The assignment of a (suitably normalized) correlator $C(\Sigma)$ to Σ actually depends only on the Morita class of the symmetric special Frobenius algebra A. Conversely, any consistent set of correlators can be obtained this way [FjFRS2].

Topological defects admit a number of interesting operations. In particular, they can be fused – on the algebraic side this corresponds to the tensor product $B \otimes_A B'$ of bimodules. The bimodule morphisms $\mathrm{Hom}_{A|A}(B \otimes_A B', B'')$ appear as labels of vertices of defect lines. Defect lines can also be fused to boundaries; depending on the relative situation of the defect line and the boundary, this is given on the algebraic side by the tensor product $B \otimes_A N$ of a bimodule with a left module, or by the tensor product $N \otimes_A B$ with a right module, respectively.

In the table below we collect some pertinent aspects of the construction and exhibit the geometric structures on the sigma model target space M that correspond to them.

geometric situation	algebraic structure in the category $\mathcal{C}$	geometric structure on M
Σ closed oriented	symmetric special Frobenius algebra A	bundle gerbe $\mathcal{G}$ with connection
Σ unoriented	Jandl structure $\sigma\colon A^{\mathrm{opp}} \longrightarrow A$	Jandl gerbe
boundary condition	A-module	$\mathcal{G}$-D-brane
topological defect line	A-bimodule	$\mathcal{G}$-bi-brane

Jandl gerbes, D-branes and bi-branes will be presented in Sections 5, 6 and 7, respectively.

5. Jandl gerbes: Holonomy for unoriented surfaces

We have defined trivial bundle gerbes with connection as 2-forms because 2-forms can be integrated over oriented surfaces. Closing the 2-category of trivial bundle gerbes under descent has lead us to bundle gerbes. *Jandl gerbes* are bundle gerbes with additional structure, whose holonomy is defined for closed surfaces without orientation, even for unorientable surfaces [SSW]. In particular, Jandl gerbes provide Wess–Zumino terms for unoriented surfaces. Comparing the geometric data with the representation theoretic ones from Section 4, bundle gerbes with connection correspond to Frobenius algebras, while Jandl gerbes correspond to Jandl algebras.

The appropriate quantity that has to replace 2-forms in order to make integrals over an unoriented surface well-defined is a *2-density*. Every surface Σ has an oriented double covering $\mathrm{pr}\colon \hat{\Sigma} \longrightarrow \Sigma$ that comes with an orientation-reversing involution $\sigma\colon \hat{\Sigma} \longrightarrow \hat{\Sigma}$ which exchanges the two sheets and preserves the fibres. A 2-density on Σ is a 2-form $\omega \in \Omega^2(\hat{\Sigma})$ such that

$$\sigma^*\omega = -\omega. \tag{5.1}$$

A 2-density on Σ can indeed be integrated without requiring Σ to be oriented. One chooses a dual triangulation Γ of Σ and, for each face f of Γ, one of its two preimages under $\mathrm{pr}\colon \hat{\Sigma} \longrightarrow \Sigma$, denoted f_{or}. Then one sets

$$\int_\Sigma \omega := \sum_f \int_{f_{\mathrm{or}}} \omega. \tag{5.2}$$

Owing to the equality (5.1) the so defined integral does not depend on the choice of the preimages f_{or} nor on the choice of triangulation Γ. If Σ can be endowed with an orientation, the preimages f_{or} can be chosen in such a way that $\mathrm{pr}|_{f_{\mathrm{or}}}\colon f_{\mathrm{or}} \longrightarrow f$ is orientation-preserving. Then the integral of a 2-density ω_ρ coincides with the ordinary integral of the 2-form ρ.

Next we want to set up a 2-category whose objects are related to 2-densities. To this end we use the 2-category of trivial bundle gerbes introduced in Section 3.2. Thus, one datum specifying an object is a 2-form $\omega \in \Omega^2(\hat{\Sigma})$. In the context of 2-categories, demanding strict equality as in (5.1) is unnatural. Instead, we replace equality by a 1-morphism

$$\eta\colon \sigma^*\omega \longrightarrow -\omega, \tag{5.3}$$

i.e., a 1-form $\eta \in \Omega^1(\hat{\Sigma})$ such that $\sigma^*\omega = -\omega + \mathrm{d}\eta$. As we shall see in a moment, we must impose equivariance of the 1-morphism up to some 2-morphism, i.e., we need in addition a 2-isomorphism

$$\phi\colon \sigma^*\eta \Longrightarrow \eta, \tag{5.4}$$

in other words a smooth function $\phi\colon M \longrightarrow \mathrm{U}(1)$ such that $\eta = \sigma^*\eta - \mathrm{i}\,\mathrm{dlog}\,\phi$. This 2-isomorphism, in turn, must satisfy the equivariance relation

$$\sigma^*\phi = \phi^{-1}. \tag{5.5}$$

Thus the objects of the 2-category are triples (ω, η, ϕ). Let us verify that they still lead to a well-defined notion of holonomy. We choose again a dual triangulation Γ of Σ as well as a preimage f_{or} for each of its faces. The expression (5.2) is now no longer independent of these choices, because every change creates a boundary term in the integrals of the 1-form η. To resolve this problem, we involve *orientation-reserving edges*: these are edges in Γ whose adjacent faces have been lifted to opposite sheets. Since Γ is a dual triangulation, its orientation-reversing edges form a disjoint union of piecewise smooth circles $c \subset \Sigma$. For each of these circles, we choose again a preimage c_{or}. It may not be possible to choose c_{or} to be closed, in which case there exists a point $p^c \in \Sigma$ which has two preimages in c_{or}. We choose again one of these preimages, denoted p^c_{or}. Then

$$\mathrm{Hol}_{\omega,\eta,\phi} := \exp\Big(2\pi\mathrm{i}\Big(\sum_f \int_{f_{\mathrm{or}}} \omega + \sum_c \int_{c_{\mathrm{or}}} \eta\Big)\Big) \prod_c \phi(p^c_{\mathrm{or}}) \tag{5.6}$$

is independent of the choice of the lifts f_{or}, c_{or} and p_{or}, and is independent of the choice of the triangulation.

More generally, let $\mathcal{M}an_+$ be the category of smooth manifolds with involution, whose morphisms are equivariant smooth maps. (The involution is not required to act freely.) In a first step, we want to define a presheaf

$$\mathcal{J}antriv^{\nabla} \colon \mathcal{M}an_+^{\mathrm{opp}} \longrightarrow \mathcal{C}at \tag{5.7}$$

of trivial Jandl gerbes. For (M,k) a smooth manifold with involution $k\colon M \longrightarrow M$, a trivial Jandl gerbe involves as a first datum a trivial bundle gerbe $\mathcal{I}_\omega$, but as explained in Section 1 we replace the 1-morphism η from (5.3) by a line bundle L over M with connection of curvature

$$\mathrm{curv}(L) = -\omega - k^*\omega, \tag{5.8}$$

and we replace the 2-isomorphism ϕ from (5.4) by an isomorphism $\phi\colon k^*L \longrightarrow L$ of line bundles with connection, still subject to the condition (5.5). Notice that the pair (L,ϕ) is nothing but a k-equivariant line bundle with connection over M. After this step, we still have the holonomy (5.6), which now looks like

$$\mathrm{Hol}_{\mathcal{I}_\omega,L,\phi} = \exp\Big(2\pi\mathrm{i}\sum_f \int_{f_{\mathrm{or}}} \omega\Big) \prod_c \mathrm{Hol}_{\bar{L}}(c), \tag{5.9}$$

where we have used the fact that, since the action of $\langle k\rangle$ on c_{or} is free, the k-equivariant line bundle (L,ϕ) descends to a line bundle $\bar{L}$ with connection over the quotient $c = c_{\mathrm{or}}/\langle k\rangle$. This formula is now manifestly independent of the choices of c_{or} and p^c_{or}. Its independence under different choices of faces f_{or} is due to (5.8).

Now we close the presheaf $\mathcal{J}antriv^{\nabla}(M)$ under descent to allow for non-trivial bundle gerbes. To do so, we need to introduce duals of bundle gerbes, 1-morphisms and 2-isomorphisms see [Wa1]; for the sake of brevity we omit these definitions here.

Definition 4. Let M be a smooth manifold with involution $k\colon M \longrightarrow M$. A *Jandl gerbe* is a bundle gerbe $\mathcal{G}$ over M together with a 1-isomorphism $\mathcal{A}\colon k^*\mathcal{G} \longrightarrow \mathcal{G}^*$ to the dual gerbe and a 2-isomorphism $\varphi\colon k^*\mathcal{A} \Longrightarrow \mathcal{A}^*$ that satisfies $k^*\varphi = \varphi^{*-1}$.

Jandl gerbes form a sheaf

$$\mathcal{J}an^{\nabla} \colon \mathcal{M}an_+^{\mathrm{opp}} \longrightarrow \mathcal{C}at. \tag{5.10}$$

The gluing axiom for this sheaf has been proved in [GSW2]. We remark that the 1-isomorphism $\mathcal{A}$ may be regarded as the counterpart of a Jandl structure σ on the Frobenius algebra A that corresponds to the bundle gerbe $\mathcal{G}$, if one accepts that the dual gerbe plays the role of the opposed algebra.

Suppose we are given a Jandl gerbe $\mathcal{J}$ over a smooth manifold M with involution k. If Σ is a closed surface, possibly unoriented and possibly unorientable, and

$$\Phi\colon (\hat{\Sigma}, \sigma) \longrightarrow (M, k) \tag{5.11}$$

is a morphism in $\mathcal{M}an_+$, we can pull back the Jandl gerbe $\mathcal{J}$ from M to $\hat{\Sigma}$. As in the case of ordinary surface holonomy, it then becomes trivial as a gerbe for dimensional reasons, and we can choose an isomorphism

$$\mathcal{T}: \Phi^*\mathcal{J} \xrightarrow{\sim} (\mathcal{I}_\omega, L, \phi). \tag{5.12}$$

Then we define

$$\mathrm{Hol}_{\mathcal{J}}(\Phi) := \mathrm{Hol}_{\mathcal{I}_\omega, L, \phi}. \tag{5.13}$$

This is independent of the choice of $\mathcal{T}$, because any other choice $\mathcal{T}'$ gives rise to an isomorphism $\mathcal{T}' \circ \mathcal{T}^{-1}$ in $\mathcal{J}antriv^\nabla(\hat{\Sigma}, \sigma)$ under which the holonomy stays unchanged.

We have now seen that every Jandl gerbe $\mathcal{J}$ over a smooth manifold M with involution k has holonomies for unoriented closed surfaces and equivariant smooth maps $\Phi\colon \hat{\Sigma} \longrightarrow M$. We thus infer that sigma models on M whose fields are such maps, are defined by Jandl gerbes $\mathcal{J}$ over M rather than by ordinary bundle gerbes $\mathcal{G}$. This makes it an interesting problem to classify Jandl gerbes.

Concerning the existence of a Jandl gerbe $\mathcal{J}$ with underlying bundle gerbe $\mathcal{G}$, the 1-isomorphism $\mathcal{A}\colon k^*\mathcal{G} \longrightarrow \mathcal{G}^*$ requires the curvature H of $\mathcal{G}$ to satisfy

$$k^*H = -H. \tag{5.14}$$

Apart from this necessary condition, there is a sequence of obstruction classes [GSW2]. Reduced to the case that M is 2-connected, there is one obstruction class $o(\mathcal{G}) \in H^3(\mathbb{Z}_2, \mathrm{U}(1))$, the group cohomology of $\mathbb{Z}_2$ with coefficients in $\mathrm{U}(1)$, on which $\mathbb{Z}_2$ acts by inversion. If $o(\mathcal{G})$ vanishes, then inequivalent Jandl gerbes with the same underlying bundle gerbe $\mathcal{G}$ are parameterized by $H^2(\mathbb{Z}_2, \mathrm{U}(1))$.

These results can be made very explicit in the case of WZW models, for which the object in $\mathcal{M}an_+$ is a connected compact simple Lie group G equipped with an involution $k\colon G \longrightarrow G$ acting as

$$k\colon g \longmapsto (zg)^{-1} \tag{5.15}$$

for a fixed 'twist element' $z \in Z(G)$. It is easy to see that the 3-form $H_k \in \Omega^3(G)$, which is the curvature of the level-k bundle gerbes $\mathcal{G}$ over G, satisfies the necessary condition (5.14). All obstruction classes $o(\mathcal{G})$ and all parameterizing groups have been computed in dependence of the twist element z and the level k [GSW1]. The numbers of inequivalent Jandl gerbes range from two (for simply connected G, per level and involution) to sixteen (for PSO($4n$), for every even level).

Most prominently, there are two involutions on SU(2), namely $g \longmapsto g^{-1}$ and $g \longmapsto -g^{-1}$, and for each of them two inequivalent Jandl gerbes per level. On SO(3) there is only a single involution, but the results of [SSW], [GSW1] exhibit four inequivalent Jandl gerbes per even level. This explains very nicely why SU(2) and SO(3) have the same number of orientifolds, despite a different number of involutions. These results reproduce those of the algebraic approach (see e.g. [FRS2]); for the precise comparison, Jandl structures related by the action of the trivial line bundle with either of its two equivariant structures have to be identified.

6. D-branes: Holonomy for surfaces with boundary

We now introduce the geometric structure needed to define surface holonomies and Wess–Zumino terms for surfaces with boundary. When one wants to define holonomy along a curve that is not closed, one way to make the parallel transport group-valued is to choose trivializations at the end points. To incorporate these trivializations into the background, one can choose a submanifold $\dot{\mathcal{D}} \subset M$ together with a trivialization $E|_{\dot{\mathcal{D}}} \longrightarrow \mathbf{1}_A$. Admissible paths $\gamma\colon [0,1] \longrightarrow M$ are then required to start and end on this submanifold, $\gamma(0), \gamma(1) \in \dot{\mathcal{D}}$. The same strategy has proven to be successful for *surfaces* with boundary.

Definition 5. Let $\mathcal{G}$ be a bundle gerbe with connection over M. A *$\mathcal{G}$-D-brane* is a submanifold $\dot{\mathcal{D}} \subset M$ together with a 1-morphism

$$\mathcal{D}\colon \mathcal{G}|_{\dot{\mathcal{D}}} \longrightarrow \mathcal{I}_\omega \tag{6.1}$$

to a trivial bundle gerbe $\mathcal{I}_\omega$ given by a two-form ω on $\dot{\mathcal{D}}$.

The morphism $\mathcal{D}$ is called a $\mathcal{G}$-module, or twisted vector bundle. Notice that if H is the curvature of $\mathcal{G}$, the 1-morphism $\mathcal{D}$ enforces the identity

$$H|_{\dot{\mathcal{D}}} = \mathrm{d}\omega. \tag{6.2}$$

This equality restricts the possible choices of the world volume $\dot{\mathcal{D}}$ of the $\mathcal{G}$-D-brane.

Suppose that Σ is an oriented surface, possibly with boundary, and $\Phi\colon \Sigma \longrightarrow M$ is a smooth map. We require that $\Phi(\partial\Sigma) \subset \dot{\mathcal{D}}$. As described in Section 3.3, we choose a trivialization $\mathcal{T}\colon \Phi^*\mathcal{G} \longrightarrow \mathcal{I}_\rho$. Its restriction to $\partial\Sigma$ and the $\mathcal{G}$-module $\mathcal{D}$ define a 1-morphism

$$\mathcal{I}_\rho\big|_{\partial\Sigma} \xrightarrow{\mathcal{T}^{-1}|_{\partial\Sigma}} \Phi^*\mathcal{G}\big|_{\partial\Sigma} = \Phi^*(\mathcal{G}\big|_{\dot{\mathcal{D}}}) \xrightarrow{\Phi^*(\mathcal{D})} \Phi^*(\mathcal{I}_\omega). \tag{6.3}$$

According to the definition (3.13), this 1-morphism is nothing but a hermitian vector bundle E with connection over $\partial\Sigma$ and its curvature is $\mathrm{curv}(E) = \omega - \rho$. Then we consider

$$\mathrm{Hol}_{\mathcal{G},\mathcal{D}}(\Phi) := \exp\left(2\pi\mathrm{i}\int_\Sigma \rho\right)\mathrm{tr}(\mathrm{Hol}_E(\partial\Sigma)), \tag{6.4}$$

where the trace makes the holonomy of E independent of the choice of a parameterization of $\partial\Sigma$. This expression is independent of the choice of the trivialization $\mathcal{T}$: if $\mathcal{T}'$: $\mathcal{G} \longrightarrow \mathcal{I}_{\rho'}$ is another one and E' is the corresponding vector bundle, we have the transition isomorphism L from (3.27) with curvature $\rho' - \rho$, and an isomorphism $E' \otimes L \cong E$. It follows that

$$\exp\left(2\pi\mathrm{i}\int_\Sigma \rho\right)\mathrm{tr}(\mathrm{Hol}_E(\partial\Sigma)) = \exp\left(2\pi\mathrm{i}\left(\int_\Sigma \rho' - \mathrm{curv}(L)\right)\right)\mathrm{tr}(\mathrm{Hol}_{E'\otimes L}(\partial\Sigma)), \tag{6.5}$$

and on the right hand side the unprimed quantities cancel by Stokes' theorem.

Important results on D-branes concern in particular two large classes of models, namely free field theories and again WZW theories. The simplest example of a free field theory is the one of a compactified free boson, in which M is a circle $S^1_R \cong \mathbb{R} \bmod 2\pi R\,\mathbb{Z}$ of radius R. As is well known, there are then in particular two distinct types of D-branes: D0-branes $\mathcal{D}^{(0)}_x$, whose support is localized at a position $x \in S^1_R$, and D1-branes $\mathcal{D}^{(1)}_\alpha$, whose world volume is all of S^1_R and which are characterized by a Wilson line $\alpha \in \mathbb{R} \bmod \frac{1}{2\pi R}\,\mathbb{Z}$, corresponding to a flat connection on S^1_R.

For WZW theories, which are governed by a bundle gerbe $\mathcal{G}$ over a connected compact simple Lie group G, preserving the non-abelian current symmetries puts additional constraints on the admissible D-branes: their support $\dot{\mathcal{D}}$ must be a conjugacy class $\mathcal{C}_h$ of a group element $h \in G$. This can e.g. be seen by studying the scattering of bulk fields in the presence of the D-brane. On such conjugacy classes one finds a canonical 2-form $\omega_h \in \Omega^2(\mathcal{C}_h)$. Additionally, the 1-morphism $\mathcal{D}$: $\mathcal{G}|_{\mathcal{C}_h} \longrightarrow \mathcal{I}_{\omega_h}$ of a symmetric D-brane must satisfy a certain equivariance condition [Ga2]. Interestingly, only on those conjugacy classes $\mathcal{C}_h$ for which

$$h = \exp(2\pi\mathrm{i}\,\tfrac{\alpha+\rho}{k+g^\vee}), \tag{6.6}$$

with α an *integrable* highest weight, admit such 1-morphisms. Here ρ denotes the Weyl vector and $g^\vee$ the dual Coxeter number of the Lie algebra $\mathfrak{g}$ of G. Thus in particular the possible world volumes of symmetric D-branes form only a discrete subset of conjugacy classes.

We finally remark that the concepts of D-branes and Jandl gerbes can be merged [GSW2]. The resulting structures provide holonomies for unoriented surfaces with boundary, and can be used to define D-branes in WZW orientifold theories.

7. Bi-branes: Holonomy for surfaces with defect lines

7.1. Gerbe bimodules and bi-branes. In the representation theoretic approach to rational conformal field theory, boundary conditions and defect lines are described as modules and bimodules, respectively. The fact that the appropriate target space structure for describing boundary conditions, D-branes, is related to gerbe modules, raises the question of what the appropriate target space structure for defect lines should be. The following definition turns out to be appropriate.

Definition 6. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be bundle gerbes with connection over M_1 and M_2, respectively. A $\mathcal{G}_1$-$\mathcal{G}_2$-*bi-brane* is a submanifold $\dot{\mathcal{B}} \subset M_1 \times M_2$ together with a $(p_1^*\mathcal{G}_1)|_{\dot{\mathcal{B}}}$-$(p_2^*\mathcal{G}_2)|_{\dot{\mathcal{B}}}$-bimodule, i.e., with a 1-morphism

$$\mathcal{B}\colon (p_1^*\mathcal{G}_1)|_{\dot{\mathcal{B}}} \longrightarrow (p_2^*\mathcal{G}_2)|_{\dot{\mathcal{B}}} \otimes \mathcal{I}_\varpi \tag{7.1}$$

with $\mathcal{I}_\varpi$ a trivial bundle gerbe given by a two-form ϖ on $\dot{\mathcal{B}}$.

Similarly as in (6.2) it follows that the two-form ϖ on $\dot{\mathcal{B}}$ obeys

$$p_1^* H|_{\dot{\mathcal{B}}} = p_2^* H|_{\dot{\mathcal{B}}} + \mathrm{d}\varpi. \tag{7.2}$$

We call $\dot{\mathcal{B}}$ the world volume and ϖ the curvature of the bimodule. With the appropriate notion of duality for bundle gerbes (see Section 1.4 of [Wa1]), a $\mathcal{G}_1$-$\mathcal{G}_2$-bimodule is the same as a $(\mathcal{G}_1 \otimes \mathcal{G}_2^*)$-module. For a formulation in terms of local data, see (B.8) of [FSW].

As an illustration, consider again the free boson and WZW theories, restricting attention to the case $M_1 = M_2$. For the free boson compactified on a circle S^1_R of radius R, one finds that the world volume of a bi-brane is a submanifold $\dot{\mathcal{B}}_x \subset S^1_R \times S^1_R$ of the form

$$\dot{\mathcal{B}}_{x,\alpha} := \{(y, y-x) \mid y \in S^1_R\} \tag{7.3}$$

with $x \in S^1_R$. The submanifold $\dot{\mathcal{B}}_{x,\alpha}$ has the topology of a circle and comes with a flat connection, i.e., with a Wilson line α. Thus the bi-branes of a compactified free boson are naturally parameterized by a pair (x, α) taking values in two dual circles that describe a point on S^1_R and a Wilson line.

In the WZW case, for which the target space is a compact connected simple Lie group G, a scattering calculation [FSW] similar to the one performed for D-branes indicates that the world volume of a (maximally symmetric) bi-brane is a *biconjugacy class*

$$\dot{\mathcal{B}}_{h,h'} := \big\{(g, g') \in G \times G \mid \exists\, x_1, x_2 \in G\colon\, g = x_1 h x_2^{-1},\, g' = x_1 h' x_2^{-1}\big\} \subset G \times G \tag{7.4}$$

of a pair (h,h') of group elements satisfying $h\,(h')^{-1} \in \mathcal{C}_{h_\alpha}$ with h_α as given in (6.6). The biconjugacy classes carry two commuting G-actions, corresponding to the presence of two independent conserved currents in the field theory. Further, a biconjugacy class can be described as the preimage

$$\dot{\mathcal{B}}_{h,h'} = \tilde{\mu}^{-1}(\mathcal{C}_{hh'^{-1}}) = \big\{(g,g') \in G \times G \mid gg'^{-1} \in \mathcal{C}_{hh'^{-1}}\big\} \tag{7.5}$$

of the conjugacy class $\mathcal{C}_{hh'^{-1}}$ under the map

$$\tilde{\mu}\colon G \times G \ni (g_1,g_2) \longmapsto g_1 g_2^{-1} \in G. \tag{7.6}$$

Finally, the relevant two-form on $\dot{\mathcal{B}}_{h,h'}$ is

$$\varpi_{h,h'} := \tilde{\mu}^*\omega_{hh'^{-1}} - \tfrac{k}{2}\,\langle p_1^*\theta \wedge p_2^*\theta\rangle. \tag{7.7}$$

Here k is the level, θ is the left-invariant Maurer–Cartan form, p_i are the projections to the factors of $G \times G$, and ω_h is the canonical 2-form (see Section 6) on the conjugacy class $\mathcal{C}_h$. One checks that $\varpi_{h,h'}$ is bi-invariant and satisfies (7.2).

Examples of symmetric bi-branes can be constructed from symmetric D-branes using a multiplicative structure on the bundle gerbe $\mathcal{G}$ [Wa2]. Another important class of examples are Poincaré line bundles. These describe T-dualities; an elementary relation between T-duality and Poincaré line bundles is provided [SaS] by the equation of motion [RS] in the presence of defects.

7.2. Holonomy and Wess–Zumino term for defects. The notion of bi-brane allows one in particular to define holonomy also for surfaces with defect lines.

The simplest world sheet geometry involving a defect line consists of a closed oriented world sheet Σ together with an embedded oriented circle $S \subset \Sigma$ that separates the world sheet into two components, $\Sigma = \Sigma_1 \cup_S \Sigma_2$. Assume that the defect S separates regions that support conformally invariant sigma models with target spaces M_1 and M_2, respectively, and consider maps $\phi_i\colon \Sigma_i \longrightarrow M_i$ for $i \in \{1,2\}$ such that the image of

$$\begin{aligned} \phi_S\colon S &\longrightarrow M_1 \times M_2 \\ s &\longmapsto (\phi_1(s),\phi_2(s)) \end{aligned} \tag{7.8}$$

is contained in the submanifold $\dot{\mathcal{B}}$ of $M_1 \times M_2$. The orientation of Σ_i is the one inherited from the orientation of Σ, and without loss of generality we take $\partial\Sigma_1 = S$ and $\partial\Sigma_2 = -S$.

We wish to find the Wess–Zumino part of the sigma model action, or rather the corresponding holonomy $\mathrm{Hol}_{\mathcal{G}_1,\mathcal{G}_2,\mathcal{B}}$, that corresponds to having bundle gerbes $\mathcal{G}_1$ and $\mathcal{G}_2$ over M_1 and M_2 and a $\mathcal{G}_1$-$\mathcal{G}_2$-bi-brane $\mathcal{B}$. The pullback of the bimodule (7.1) along the map $\phi_S\colon S \longrightarrow \dot{\mathcal{B}}$ gives a $(\phi_1^*\mathcal{G}_1)|_S$-$(\phi_2^*\mathcal{G}_2)|_S$-bimodule

$$\phi_S^*\mathcal{B}\colon (\phi_1^*\mathcal{G}_1)|_S \longrightarrow (\phi_2^*\mathcal{G}_2)|_S \otimes \mathcal{I}_{\phi_S^*\varpi}. \tag{7.9}$$

The pullback bundle gerbes $\phi_i^*\mathcal{G}_i$ over Σ_i are trivializable for dimensional reasons, and a choice $\mathcal{T}_i\colon \phi_i^*\mathcal{G}_i \longrightarrow \mathcal{I}_\rho$ of trivializations for two-forms ρ_i on Σ_i produces a vector bundle E over S. We then define

$$\mathrm{Hol}_{\mathcal{G}_1,\mathcal{G}_2,\mathcal{B}}(\Sigma,S) := \exp\left(2\pi\mathrm{i}\int_{\Sigma_1}\rho_1\right)\,\exp\left(2\pi\mathrm{i}\int_{\Sigma_2}\rho_2\right)\,\mathrm{tr}(\mathrm{Hol}_E(S)) \in \mathbb{C} \tag{7.10}$$

to be the holonomy in the presence of the bi-brane $\mathcal{B}$. As shown in Appendix B.3 of [FSW], for similar reasons as in the case of D-branes the number $\mathrm{Hol}_{\mathcal{G}_1,\mathcal{G}_2,\mathcal{B}}(\Sigma,S)$ is independent of the choice of the trivializations $\mathcal{T}_1$ and $\mathcal{T}_2$.

7.3. Fusion of defects. In the field theory context of section 4 there are natural notions of the fusion of a defect (an A-bimodule) with a boundary condition (a left A-module), yielding another boundary condition, and of the fusion of two defects, yielding another defect. Both of these are provided by the tensor product over the relevant Frobenius algebra A. These representation theoretic notions of fusion have a counterpart on the geometric side as well.

Consider first the fusion of a defect with a boundary condition. We allow for the general situation of a defect described by an M_1-M_2-bi-brane with different target spaces M_1 and M_2. Thus take an M_1-M_2-bi-brane with world volume $\dot{\mathcal{B}} \subseteq M_1\times M_2$ and an M_2-D-brane with world volume $\dot{\mathcal{D}} \subseteq M_2$. The action of correspondences on sheaves suggests the following ansatz for the world volume of the fusion product:

$$(\mathcal{B}\star\mathcal{D})^{\boldsymbol{\cdot}} := p_1\big(\dot{\mathcal{B}} \cap p_2^{-1}(\dot{\mathcal{D}})\big) \tag{7.11}$$

with p_i the projection $M_1\times M_2 \longrightarrow M_i$. The corresponding ansatz for the fusion of an M_1-M_2-bi-brane $\mathcal{B}$ of world volume $\dot{\mathcal{B}}$ with an M_2-M_3-bi-brane $\mathcal{B}'$ of world volume $\dot{\mathcal{B}}'$ uses projections p_{ij} from $M_1\times M_2\times M_3$ to $M_i\times M_j$:

$$(\mathcal{B}\star\mathcal{B}')^{\boldsymbol{\cdot}} := p_{13}\big(p_{12}^{-1}(\dot{\mathcal{B}}) \cap p_{23}^{-1}(\dot{\mathcal{B}}')\big). \tag{7.12}$$

In general one obtains this way only subsets, rather than submanifolds, of M_1 and $M_1\times M_3$, respectively. On a heuristic level one would, however, expect that owing to quantization of the branes a finite superposition of branes is selected, which should then reproduce the results obtained in the field theory setting.

We illustrate this again with the two classes of models already considered, i.e., free bosons and WZW theories, again restricting attention to the case $M_1 = M_2$. First, for the theory of a compactified free boson, the D-brane is of one of the types $\mathcal{D}_x^{(0)}$ or $\mathcal{D}_\alpha^{(1)}$ (see Section 6) and the bi-brane world volume is of the form $\dot{\mathcal{B}}_{x,\alpha}$ given in (7.3). For D-branes of type $\mathcal{D}_x^{(0)}$ the prescription (7.11) thus yields

$$\mathcal{B}_{(x,\alpha)} \star \mathcal{D}_y^{(0)} = \mathcal{D}_{x+y}^{(0)}. \tag{7.13}$$

For the fusion of a bi-brane $\mathcal{B}_{(x,\alpha)}$ and a D1-brane $\mathcal{D}_\beta^{(1)}$, one must take the flat line bundle on the bi-brane into account. We first pull back the line bundle on $\dot{\mathcal{D}}_\beta^{(1)}$ along p_2 to a line bundle on $S_R^1 \times S_R^1$, then restrict it to $\dot{\mathcal{B}}_{(x,\alpha)}$, and finally tensor this restriction with the line bundle on $\dot{\mathcal{B}}_{(x,\alpha)}$ described by the Wilson line α. This results in a line bundle with Wilson line $\alpha+\beta$ on the bi-brane world volume, which in turn can be pushed down along p_1 to a line bundle on S_R^1, so that

$$\mathcal{B}_{(x,\alpha)} \star \mathcal{D}_\beta^{(1)} = \mathcal{D}_{\alpha+\beta}^{(1)}. \tag{7.14}$$

In short, the fusion with a defect $\mathcal{B}_{(x,\alpha)}$ acts on D0-branes as a translation by x in position space, and on D1-branes as a translation by α in the space of Wilson lines. Similarly, the prescription (7.12) leads to

$$\mathcal{B}_{(x,\alpha)} \star \mathcal{B}_{(x',\alpha')} = \mathcal{B}_{(x+x',\alpha+\alpha')} \tag{7.15}$$

for the fusion of two bi-branes $\mathcal{B}_{(x,\alpha)}$ and $\mathcal{B}_{(x',\alpha')}$, i.e., both the position and the Wilson line variable of the bi-branes add up.

For WZW theories, besides the quantization of the positions of the branes another new phenomenon is that multiplicities other than zero or one appear in the field theory approach. In that context they arise from the decomposition $B_\alpha \otimes_A B_\beta = \bigoplus_\gamma \mathcal{N}_{\alpha\beta}^{\ \ \gamma} B_\gamma$ of a tensor product of simple A-bimodules into a finite direct sum of simple A-bimodules, and analogously for the case of mixed fusion (in rational CFT, both the category of A-modules and the category of A-bimodules are semisimple). Moreover, for simply connected groups, the multiplicities appearing in both types of fusion are in fact the same as the chiral fusion multiplicities which are given by the Verlinde formula.

By analogy with the field theory situation we expect fusion rules

$$\mathcal{B}_\alpha \star \mathcal{B}_\beta = \sum_\gamma \mathcal{N}_{\alpha\beta}^{\ \ \gamma} \mathcal{B}_\gamma \tag{7.16}$$

of bi-branes, and analogously for mixed fusion of bi-branes and D-branes. In the particular case of WZW theories on simply connected Lie groups one can in addition invoke the duality $\alpha \longmapsto \alpha^\vee$ which in that case exists on the sets of branes as well as defects that preserve all current symmetries, so as to work instead with fusion coefficients of type $\mathcal{N}_{\alpha\beta\gamma} = \mathcal{N}_{\alpha\beta}^{\ \ \gamma^\vee}$. Then for the case of two D-branes $\mathcal{D}_\alpha$ and $\mathcal{D}_\gamma$ with world volumes given by conjugacy classes $\mathcal{C}_{h_\alpha}$ and $\mathcal{C}_{h_\gamma}$ of G, as well as a bi-brane $\mathcal{B}_\beta$ whose world volume is the biconjugacy class $\tilde{\mu}^{-1}(\mathcal{C}_{h_\beta})$, one is lead to consider the subset

$$\begin{aligned} \Pi_{\alpha\beta\gamma} &:= p_1^{-1}(\mathcal{C}_\alpha) \cap \tilde{\mu}^{-1}(\mathcal{C}_\beta) \cap p_2^{-1}(\mathcal{C}_\gamma) \\ &= \{(g,g') \in G \times G \mid g \in \mathcal{C}_\alpha,\ g' \in \mathcal{C}_\gamma,\ gg'^{-1} \in \mathcal{C}_\beta\} \end{aligned} \tag{7.17}$$

of $G \times G$. Combining the adjoint action on g and on g' gives a natural G-action on $\Pi_{\alpha\beta\gamma}$. And since both D-branes and the bi-brane are equipped with two-forms ω_α, ω_γ and ϖ_β, $\Pi_{\alpha\beta\gamma}$ comes with a natural two-form as well, namely with

$$\omega_{\alpha\beta\gamma} := p_1^*\omega_\alpha|_{\Pi_{\alpha\beta\gamma}} + p_2^*\omega_\gamma|_{\Pi_{\alpha\beta\gamma}} + \varpi_\beta|_{\Pi_{\alpha\beta\gamma}}. \tag{7.18}$$

By comparison with the field theory approach, this result should be linked to the fusion rules of the chiral WZW theory and thereby provide a physically motivated realization of the Verlinde algebra. To see how such a relation can exist, notice that fusion rules are dimensions of spaces of conformal blocks and as such can be obtained by geometric quantization from suitable moduli spaces of flat connections which arise in the quantization of Chern–Simons theories (see e.g. [ADW]). The moduli space relevant to us is the one for the three-punctured sphere $S^2_{(3)}$, for which the monodromy of the flat connection around the punctures takes values in conjugacy classes $\mathcal{C}_\alpha$, $\mathcal{C}_\beta$ and $\mathcal{C}_\gamma$, respectively. The relations in the fundamental group of $S^2_{(3)}$ imply the condition $g_\alpha g_\beta g_\gamma = 1$ on the monodromies $g_\alpha \in \mathcal{C}_\alpha$, $g_\beta \in \mathcal{C}_\beta$ and $g_\gamma \in \mathcal{C}_\gamma$. Since monodromies are defined only up to simultaneous conjugation, the moduli space that matters in classical Chern–Simons theory is isomorphic to the quotient $\Pi_{\alpha\beta\gamma}/G$.

It turns out that the range of bi-branes appearing in the fusion product is correctly bounded already before geometric quantization. Indeed, the relevant product of conjugacy classes is

$$\mathcal{C}_h * \mathcal{C}_{h'} := \{gg' \mid g \in \mathcal{C}_h,\ g' \in \mathcal{C}_{h'}\}, \tag{7.19}$$

and for the case of $G = \mathrm{SU}(2)$ it is easy to see that this yields the correct upper and lower bounds for the SU(2) fusion rules [JW], [FSW]. A full understanding of fusion can, however, only be expected after applying geometric quantization to the so obtained moduli space: this space must be endowed with a two-form, which is interpreted as the curvature of a line bundle, and the holomorphic sections of this bundle are what results from geometric quantization. In view of this need for quantization it is a highly non-trivial observation that the two-form (7.18) furnished by the two branes and the bi-brane is exactly the same as the one that arises from classical Chern–Simons theory.

In terms of defect lines, the decomposition (7.16) of the fusion product of bi-branes corresponds to the presence of a *defect junction*, which constitutes a particular type of *defect field*. A sigma model description for world sheets with such embedded defect junctions has been proposed in [RS].

We have demonstrated how structural analogies between the geometry of bundle gerbes and the representation theoretic approach to rational conformal field theory lead to interesting geometric structure, including a physically motivated realization of the Verlinde algebra. The precise form of the latter and its relation with the realization of the Verlinde algebra in the context of supersymmetric conformal field theory [FHT] remain to be understood. But in any case the parallelism between classical actions and full quantum theory exhibited above remains intriguing and raises the hope that some of the structural aspects discussed in this contribution are generic features of quantum field theories.

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5ECM Amsterdam 2008

Topological field theories in 2 dimensions

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Abstract. We discuss a formality result for 2-dimensional topological field theories which are based on a semi-simple Frobenius algebra: namely, when sufficiently constrained by structural axioms, the complete theory is determined by the Frobenius algebra and the grading information. The structural constraints apply to Gromov–Witten theory of a variety whose quantum cohomology is semi-simple. Some open questions about semi-simple field theories are mentioned in the final section.

Mathematics Subject Classification (2000). Primary 57R56; Secondary 14N35, 81T45.

Keywords. Topological field theory, Gromov–Witten theory, Deligne–Mumford space.

1. Introduction

The notion of a *Topological Field Theory* (TFT) was formalised by Atiyah and Witten [W] and modelled on Graeme Segal's definition of a *Conformal Field Theory*. It was intended to supply the structural framework for the new topological invariants of the 1980s, in particular Donaldson theory and 3-dimensional Chern–Simons theory.

The distinguishing feature of the new invariants was their *multiplicativity* under unions, rather than the additivity common to classical algebraic topology invariants, such as characteristic classes. The source of additivity is the *Mayer–Vietoris sequence* for homology.

Quantum field theory explains this behaviour heuristically: the invariants of a manifold X are integrals, not over X, but over a space of *fields* on X, these fields being maps to another, fixed space. The space of fields is "multiplicative in pieces of X." Formulated naïvely, this can only be carried out for target spaces X with strong finiteness properties (such as finite sets, or finite groupoids, leading to gauge theory for finite groups), but good use of geometric or analytic information can lead to a notion of integration over more interesting spaces of fields. This happens for instance in Gromov–Witten theory, where a fundamental class for integration in the space of continuous maps from a closed surface to a Kähler manifold M is defined by the finite-dimensional cycle of holomorphic maps.

While in recent years, the notion of TFT has experienced substantial refinements, passing through the *open-closed* TFT's of Moore and Segal [MS], [C], to an all-dimensional notion relying on higher category theory [Lu], the original notion is the best starting point:

1.1 Definition. A TFT is a symmetric monoidal functor from the oriented n-dimensional bordism category to the category of complex vector spaces. The monoidal structures are given by disjoint union, respectively tensor product.

Thus, to each closed oriented $(n-1)$-manifold (object in the n-bordism category) there is assigned a vector space, to disjoint unions we assign tensor products, to an n-dimensional bordism we assign linear maps between the boundary spaces, and the gluing of bordisms leads to the composition of linear maps.

It can be said that the interest and activity surrounding the notion of TFT has not been met by commensurate applications along the lines originally intended. In dimension 3, a classification of TFT's with special properties was given by Reshetikhin–Turaev in terms of *modular tensor categories*; the most famous example is Chern–Simons theory. By contrast, in higher dimension, there seem to be no interesting theories: all examples are built from characteristic classes. The sophisticated 4-dimensional TFT's constructed from gauge theories and their Floer homologies, or from Khovanov's categorified knot invariants, do *not* satisfy Definition 1.1, but only a variant of it.

One striking exception is the case of surfaces ($n = 2$), where a natural extension of the notion of TFT, that of a *cohomological field theory*, has framed the solution of a classical enumerative question of algebraic geometry, the problem of counting curves of fixed degree and genus in a projective manifold, with specified incidence properties.

1.1. Two dimensions. The classification of (compact, connected, oriented) topological surfaces has long been known: the invariants are the number of boundary components and the Euler characteristic. TFT's in dimension 2 were initially studied as a toy model and their structure was understood early on. Recall that an (associative) algebra A is called *Frobenius* if it comes equipped with a *trace* $\theta : A \to \mathbb{C}$ for which $a, b \mapsto \theta(a \cdot b)$ gives a perfect, symmetric paring. In particular, $\dim A < \infty$.

1.2 Theorem (folklore, see [A]). *A 2-dimensional oriented TFT with vector space A assigned to the circle is equivalent to the datum of a commutative Frobenius algebra A over $\mathbb{C}$.*

The multiplication on A is defined by the pair of pants with two boundary circles incoming and one outgoing, and the trace by the disk with incoming boundary.

Divertingly enough, in spite of this simple classification, it is in 2D that the original notion of TFT has seen powerful applications.

2. Gromov–Witten theory

My application is to *Gromov–Witten theory*, which generalises a classical and delicate question in enumerative geometry: counting algebraic curves in a projective manifold, with prescribed degree and intersection conditions.

For example, there is a unique linear map $\mathbb{P}^1 \to \mathbb{P}^n$ sending the points $0, 1, \infty$ to three linear subspaces placed in general position and with total dimension $n-1$. GW theory encodes this information by *deforming* the ordinary cohomology algebra

$$H^*(\mathbb{P}^n) = \mathbb{C}[\omega]/\langle \omega^{n+1} \rangle$$

into the (small) *quantum cohomology algebra*, parametrised by $q \in \mathbb{C}^*$,

$$QH^*(\mathbb{P}^n) = \mathbb{C}[\omega]/\langle \omega^{n+1} - q \rangle.$$

The coefficient 1 in front of q indicates the uniqueness of the map, while its exponent 1 is the degree. Seen most naturally, q lives in $\mathbb{C}^* = H^2(\mathbb{P}^n; \mathbb{C}^*)$, which can be seen as the exponentiated image of $H^2(\mathbb{P}^n; \mathbb{C})$. The reader is probably familiar with the definition of small quantum cohomology for any projective manifold X: using the Poincaré duality bilinear form on $H := H^*(X; \mathbb{C})$, the datum of a multiplication $H \otimes H \to H$ is equivalent to that of a vector in $H^{\otimes 3}$. On three homology classes α, β, γ, this vector takes the value

$$\sum\nolimits_{\delta \in H_2(X)} N_{\alpha\beta\gamma}(\delta) \cdot q^{\delta}, \tag{2.1}$$

where $N_{\alpha\beta\gamma}(\delta)$ is the number of maps of degree δ from $\mathbb{P}^1$ to X sending 0, 1, ∞ to three cycles representing α, β, γ placed in general position. When this number is not finite, it is set to zero. The variable q lives in the torus $H^2(X; \mathbb{C}^*)$ and can be raised to powers $\delta \in H_2(X; \mathbb{Z})$.

The precise definition of the numbers $N_{\alpha\beta\gamma}(\delta)$ for general manifolds requires more sophisticated tools – construction of the moduli space of stable maps and its virtual fundamental class – which we shall overlook here. The proof that the multiplication is associative, while formally a consequence of the existence of a "four-point function", also relies on those constructions. Convergence of the series in (2.1) seems to be an open question for general target manifolds.

2.1. Frobenius structure on quantum cohomology. Just like ordinary cohomology, the quantum cohomology of a projective manifold X is a Frobenius algebra, with the (classical) Poincaré duality pairing on H. In fact, we get a family of 2D TFT's parametrised by $q \in H^2(X;\mathbb{C}^*)$. There is a general method to extend the space of parameters to the rest of $H^{\mathrm{ev}}(X;\mathbb{C})$, see for instance [T, §6]. This extended, "big" quantum cohomology incorporates the count of $\mathbb{P}^1$'s with arbitrarily many marked points and incidence conditions. The properties of the resulting structure on the space H were abstracted into the notion of a *Frobenius manifold* [D1], [D2]; see also [G1], [M1].

One important ingredient incorporates the *grading* on cohomology. Literally, this is broken by quantum multiplication – see the case of $QH^*(\mathbb{P}^n)$ above – but the grading is restored in the entire *family* of multiplications by grading the functions on the parameter space $H^{\mathrm{ev}}(X;\mathbb{C})$: thus, one declares $\deg q^\delta = \langle c_1(X)|\delta\rangle$, using the first Chern class of X, and grades the rest of cohomological parameters by the shifted normalised degree $\deg/2 - 1$. The quantum multiplication, when viewed as a tensor field on $H^{\mathrm{ev}}(X;\mathbb{C})$, is homogeneous of degree 1 in this grading. Thus, $\deg q = n+1$ in the $\mathbb{P}^n$ example, rendering the identity defining QH^* homogeneous of degree n, once we remember that q stands for $q \cdot 1 \in QH^0$, where 1 has degree (-1). (To explain this slightly counter-intuitive shift, see Remark 2.2 below.) Odd cohomology can be incorporated using the language of supermanifolds.

The more general Gromov–Witten invariants $\bar{Z}_g^n$ mentioned in §3 below are also homogeneous, of prescribed weights. When spelt out, the homogeneity property encodes the fact that the virtual fundamental cycles of the moduli space of stable maps have topologically determined dimensions, depending on the degrees of the maps and on $c_1(X)$.

2.2 Remark. Geometrically, the grading is implemented by the *Euler vector field* on the Frobenius manifold H. Specifically, given $u \in H^{ev}(X)$, let $u' \in H^*(X)$ be obtained from u by re-scaling its degree d part by $d/2$; the value of the *Euler vector* at u is $v := c_1(X) + u'/2 - u$. Dubrovin's general theory of Frobenius manifolds shows that, near any point with semi-simple quantum multiplication, the eigenvalues of quantum v-multiplication give a local coordinate system on the parameter space, [D2, Theorem 3.1].

The Frobenius manifold of quantum cohomology contains answers to most enumerative questions about rational curves in X, concerning incidence and tangency conditions with generally positioned cycles in X at any specified collection of marked points. (Actually, without some positivity property of the target manifold X, the curves must be counted in a virtual sense, taking account of obstructions.) A small exception concerns the case of one and two marked points, where extra information in the form of a *calibration* of the Frobenius manifold is needed [G1] (see also Remark 3.3 below). The precise formulation and explanation for this fact lies in the theorem that *genus zero cohomological field theories in two dimensions are equiva-*

lent to germs of Frobenius manifolds. Manin's book [M1] is largely devoted to the proof of this.

2.2. Givental's reconstruction conjecture. Quantum cohomology generalises to incorporate curves of any genus, with any number of marked points and incidence invariants, defined by the same method used for rational curves. The fundamental result (Ruan, Tian, Li, McDuff, Salomon) ensures that the numerically defined GW invariants are governed by the structure of an *all-genus cohomological field theory* (CohFT). We will briefly review that notion below, but for now let us focus on one important consequence of this structure. Recall that a commutative algebra over $\mathbb{C}$ is called *semi-simple* if it is isomorphic to a direct sum of copies of $\mathbb{C}$. Semi-simple Frobenius algebras are easy to classify: the trace must be diagonal in the basis of projectors, so all we need to know are the projector traces, a collection of non-zero complex numbers. Classical cohomology of a manifold is never semi-simple, but the *quantum* multiplication of some important projective manifolds, such as projective space above, is semi-simple for generic values of the parameter (see §2.3 below for more examples).

It is for this class of manifolds that Givental formulated (and proved, in the toric Fano case) the following remarkable reconstruction conjecture:

2.3 Conjecture (Givental, [G2]). *For a compact symplectic manifold X whose quantum cohomology algebra is semi-simple at generic values of the parameter $u \in H^{ev}(X)$, all Gromov–Witten invariants are explicitly determined from genus zero information.*

Loosely speaking: counting rational curves determines the answer to enumerative questions for curves of all genera. Givental predicted a formula for the generating function of all GW invariants in his framework of quantised quadratic Hamiltonians.

2.4 Theorem ([T]). *Givental's conjecture holds. More precisely, the GW* (*ancestor*) *invariants are determined by a recursive relation from the quantum multiplication law at a* single *semi-simple value of the parameter u, and from the first Chern class $c_1(X)$.*

We will review the recursion in §3 below. I refer to Givental [G2] for the discussion of 'ancestor' versus 'descendent' Gromov–Witten invariants; suffices to say here that the ancestor invariants are missing precisely the calibration of the Frobenius manifold we mentioned earlier.

2.5 Remark. Givental [G1] spells out the reconstruction using the Frobenius manifold. But his discussion shows that the knowledge of a *single* semi-simple quantum multiplication law suffices.

Theorem 2.4 follows from a structural result which classifies abstract cohomological field theories based on semi-simple Frobenius algebras (§3 below). The essential and difficult ingredient of the proof is the *Mumford conjecture*, proved by Madsen and Weiss [MW].

2.3. When does the theorem apply? Semi-simplicity of quantum cohomology is a very strong restriction. Here are the main examples.

(i) Projective spaces, Grassmannians, generalised flag varieties of GL.

(ii) Projective toric varieties always have semi-simple *big* quantum cohomology.

(iii) A method used by Givental in the toric case applies to varieties which have a circle action with isolated fixed-points: the circle-equivariant cohomology then gives a semi-simple deformation of the classical (and hence also the quantum) cohomology ring.

(iv) Bayer [B] showed that semi-simplicity is preserved by blowing up points; in particular, there exist non-Fano examples.

(v) 36 of the 59 families of 3D Fanos with no odd cohomology have been checked [AM], [Ci].

(vi) The abstract version of Theorem 2.4 applies to CohFT's constructed from *Landau–Ginzburg B-models* for potentials with isolated critical points. This has not been sufficiently exploited; see open questions, below.

(vii) On the negative side, if the *even* part of quantum cohomology is semi-simple, then the manifold has no odd cohomology, and, in the algebraic case, all cohomology is of type (p, p) [HMT]. This contradicts claims in the literature about some complete intersection Fanos.

2.4. Dubrovin's conjecture. A remarkable conjecture has gathered strong experimental support. Recall that an ordered collection $\{\mathcal{E}_i\}$ of objects in a triangulated $\mathbb{C}$-linear category is *exceptional* if $\mathrm{Ext}^*(\mathcal{E}_i, \mathcal{E}_i) = \mathbb{C}$, concentrated in degree 0, while $\mathrm{Ext}^k(\mathcal{E}_j, \mathcal{E}_i) = 0$ for all k, when $j > i$. The collection is *complete* if it generates the (triangulated) category.

2.6 Conjecture (Dubrovin). *A projective manifold has generically semi-simple quantum cohomology iff its derived category of coherent sheaves contains a complete exceptional collection.*

2.7 Remark. Dubrovin also relates the Euler characteristics of the Ext-groups to quantum cohomology data.

For instance, the conjecture is known for rational surfaces [D1], [B]. Ciolli [Ci] also checks this for 36 families of 3-dimensional Fanos. The conjecture is also confirmed for projective toric manifolds, by Kawamata's theorem [K] on the existence of exceptional objects and knowledge of their quantum cohomology.

The conjecture would follow from sufficiently optimistic formulations of mirror symmetry: the mirror partner for Gromov–Witten theory for a manifold X as in the conjecture would be a *Landau–Ginzburg B-model* whose potential had isolated Morse singularities (at generic values of the deformation space). But then, its Fukaya–Seidel category should contain a complete exceptional collection, which by the opposite mirror symmetry would give the desired exceptional objects in the derived category of X. This method has been essentially confirmed for projective toric manifolds by Abouzaid [Ab], who proves the required mirror symmetry (B-side on X is equivalent to A-side on the Landau–Ginzburg model) *without a priori* reference to exceptional objects (and, in particular, recovers Kawamata's theorem).

2.5. Related results of Kontsevich. In the 1990s, Kontsevich initiated a programme, *Homological Mirror Symmetry*, which should give a far-reaching adaptation of Givental's reconstruction conjecture without the semi-simplicity assumption. (The programme preceded Givental's cited work, but converged with it later.) For a recent update, see [KKP].

A key step is to replace the notion of cohomological field theory with that of *chain-level, open-closed field* theory. Costello's work [C] gives an implementation of these ideas. This more sophisticated approach seems necessitated by the fact that cohomological field theories seem unclassifiable with our limited understanding of Deligne–Mumford spaces. The semi-simple classification was a pleasant surprise.

Applying this programme to Gromov–Witten theory requires the construction a good *Fukaya category* for compact symplectic manifolds. This has not yet been accomplished in general. On the other hand, Kontsevich's programme does appear to settle the partner side, *B-model* of mirror symmetry, where the analogue of the Fukaya category, the derived category of coherent sheaves, is much better understood.

3. Cohomological field theory

A cohomological field theory generalises the notion of a Frobenius algebra to the situation where surfaces vary in families, and the values of the theory are (matrices in) the cohomology classes in the base space of the family, instead of numbers. CohFT's were introduced by Kontsevich and Manin [KM] precisely with Gromov–Witten theory in mind, although related notions (such as Segal's topological conformal field theory, taken up by Costello and Kontsevich) had an independent development.

I refer to [KM] for precise definitions and to the account in [T] of several possible variations, but here are the salient points. We start with a Frobenius algebra A, and require that:

- A family of closed surfaces over a base B gives a class in $H^*(B;\mathbb{C})$.
- With m input and n output boundaries, we get a class in $H^*\big(B;\mathrm{Hom}(A^{\otimes m};A^{\otimes n})\big)$.
- "Gluing boundaries = composition" applies in families.
- Nodal degenerations of surfaces (Lefschetz fibrations) are permitted.
- Everything is functorial in the base B.
- To make contact with the notion of [KM], all surfaces are assumed to be stable.
- The *flat vacuum* condition asks that "inserting a vacuum state should do nothing"; see (3.1) below.
- Finally, there is a homogeneity condition with respect to a specified *Euler vector field* on the Frobenius manifold of the CohFT (see §2.1); I will not spell out here.

Functoriality ensures that it suffices to specify the classes for the universal Lefschetz fibrations of stable surfaces, namely the universal curves $\bar{C}_g^n$ over the *Deligne–Mumford spaces* $\bar{M}_g^n$ of *stable curves* of genus g with n marked points. The collection of these classes is then subject to a set of constraints, which can be concisely formulated by stating that A is an algebra over the homology operad of the collection of Deligne–Mumford spaces.[1]

More precisely, to each pair (n,g) describing the genus and number of marked points on a stable pointed curve, we must assign a class $\bar{Z}_g^n \in H^*(\bar{M}_g^n; A^{\otimes n})$. Restriction to boundary divisors is subject to *factorisation rule*. Thus, on a boundary stratum $\bar{M}_{g'}^{n'} \times \bar{M}_{g''}^{n''}$, corresponding to a splitting of the curve into two components of genera g', g'' joined at a node, the restricted $\bar{Z}_g^n$ is described in terms of $\bar{Z}_{g'}^{n'} \wedge \bar{Z}_{g''}^{n''}$: namely, the latter product must be contracted with the Frobenius bilinear form applied to the two arguments corresponding to the nodes (one in each factor). A similar constraint arises at boundary strata corresponding to irreducible nodal curves. The flat vacuum condition demands that

$$1 \dashv \bar{Z}_g^{n+1} = f_1^* \bar{Z}_g^n, \tag{3.1}$$

where $f_1 \colon \bar{M}_g^{n+1} \to \bar{M}_g^n$ is the morphism forgetting the first marked point, and $1 \dashv$ stands for contraction with the "vacuum", the identity $1 \in A$. (This is the only sensible implementation of the "vacuum does nothing" command, as we need an extra point to insert the vacuum vector!) The homogeneity constraint pertains to the classes $\bar{Z}_g^n$, when viewed as tensors on the Frobenius manifold of the CohFT [T, §7].

[1] We must regard this as a modular operad.

3.2 Theorem ([T]). *A CohFT based on a semi-simple Frobenius algebra A is determined by a power series* $R(z) \in \mathrm{End}(A)[\![z]\!]$, $R = \mathrm{Id} + O(z)$, *which is subject to* Givental's symplectic constraint $R(z)R^*(-z) \equiv \mathrm{Id}$.

The additional homogeneity constraint enforce a restriction on R (its kth Taylor coefficient R_k has degree $(-k)$, as a tensor on the Frobenius manifold). This constraints turns out to determine R uniquely. For example, in Gromov–Witten theory, A is the quantum cohomology $QH^*(X)$ at some semi-simple point u of the Frobenius manifold. The Taylor coefficients of R are then recursively determined from $R_0 = \mathrm{Id}$ by the equations[2]

$$\left[R_{k+1}, (v\,\cdot)\right] = (\mu + k) \circ R_k,$$

where $(v\,\cdot)$ is quantum multiplication by the Euler vector v of Remark 2.2, and μ is the linear operator $(\deg/2 - \dim_{\mathbb{C}} X \cdot \mathrm{Id})$ on $A = H^*(X)$.

3.3 Remark. The Taylor series R can be interpreted in relation to *Dubrovin's monodromy data* description of the Frobenius manifold. Namely, the $\mathrm{End}(A)$-valued expression

$$F(z) := R(1/z) \circ \exp\left(z(v\,\cdot)\right) \tag{3.4}$$

describes the asymptotics around $z = \infty$ of solutions to the *quantum differential equation*

$$\frac{\partial F}{\partial z} = \left((v\,\cdot) + \frac{\mu}{z}\right) F. \tag{3.5}$$

In other words, in suitable sectors centred at $z = \infty$ and for suitable solutions F, $R(1/z)$ gives the asymptotic expansion of the function $F(z) \circ \exp(-z(v\,\cdot))$. In the case of quantum cohomology, a distinguished solution to this equation can be described from the 1*-point* $(J\text{-})$ *function*, [D2, Example 2.3]. (This is essentially the calibration of the Frobenius manifold, mentioned in §2.1.) The genuine solutions to (3.5) are multi-valued, with monodromy $z^{c_1(X)\cup}$ (classical cup-product); so the expression for F in (3.4), which is formally single-valued, cannot be a genuine function, and so R is not convergent.

3.1. Moral interpretation of R.

The Frobenius algebra A is associated to the circle in a 2D field theory. Heuristically, it should be viewed as the cohomology of a space Y with circle action. In known applications, A is the Hochschild cohomology of a category (the category of boundary states in the open-closed theory), and so its chain-level model has an algebraic circle action. The series R is intended to give a *splitting* of the S^1-equivariant cohomology:

$$H^*_{S^1}(Y) \cong H^*(Y) \otimes \mathbb{C}[\![z]\!], \tag{3.6}$$

[2] When $(v\,\cdot)$ has repeated eigenvalues, solvability imposes constraints on μ.

where z is the generator of $H^*_{S^1}(\text{point})$. Equivalently, this singles out the space of primary fields, a copy of A, inside the circle-equivariant version of A: the latter vector space is where the states to be inserted at points on the surface genuinely live.

The *existence* of such a splitting is *necessary* if one is to extend a cohomological field theory from the open stratum M_g^n of Deligne–Mumford moduli space over its boundary. This is not difficult to see, as follows. Consider for simplicity the setting of a space Y, as above. Boundaries of Deligne–Mumford space arise by degenerating handles on the curve to nodes. The cylinder with one incoming end and one outgoing end describes the identity map on $H^*_{S^1}(Y)$. The Deligne–Mumford degeneration of this cylinder into two crossing disks must extend this to a map

$$H^*_{S^1\times S^1}(Y) \to H^*_{S^1\times S^1}(Y),$$

but with the peculiar feature that, on the left, $S^1 \times S^1$ acts on Y via the first factor only, while the right, it acts only via the second factor. This is because the incoming and outgoing copies of Y are attached to the two different disks! The reader can now check that the existence of such an extension forces the existence of a splitting as in (3.6). A similar argument applies in a chain-level theory, enforcing a homotopy trivialisation of the circle action.

The core of the classification is the statement that a choice of splitting *determines* this boundary extension uniquely at the level of cohomology classes, in a semi-simple theory.

3.2. Construction of the theory. The theory based on a given R can be constructed using the Morita–Mumford–Miller (tautological) classes and the so-called boundary classes on $\overline{M}_g^n$; here, I just give a flavour of the universal formula and refer to [T, §5] for more details. Recall first that the κ-classes are integrals, along fibres of the universal surface bundle $\overline{C}_g^n \to \overline{M}_g^n$, of powers of the relative Euler class, and therefore behave *additively* under gluing of surfaces.

Start with the multiplicative class $\exp(\sum_j a_j\kappa_j) \in H^*(\overline{M}_g^n; A)$. Co-multiply this expression out to $A^{\otimes n}$, where one factor is attached to each of the n marked points. (It turns out that the coefficients $a_j \in A$ are pinned down from R and the flat vacuum condition.) Twist now each output factor by $R(\psi_i)$, with the ψ-class at the respective marked point.

As it stands, the class obtained turns out to violate the factorisation rules on Deligne–Mumford boundaries, and this must be corrected by the addition of boundary classes as follows. On each boundary divisor of $\overline{M}_g^n$, note the two Chern classes $\psi_\pm$ of the cotangent lines at the node of the curve. Also note the class $\overline{Z}_{g'}^{n'} \wedge \overline{Z}_{g''}^{n''}$ coming from the lower-genus moduli spaces;[3] the two factors are assumed to have been recursively corrected. We now contract the linear operator $(\mathrm{Id} - R(\psi_+)^* R(\psi_-))/(\psi_+ + \psi_-)$

[3] Or its counterpart $\overline{Z}_{g-1}^{n+2}$ on a boundary produced by a non-splitting handle of the curve.

with $\overline{Z}^{n'}_{g'} \wedge \overline{Z}^{n''}_{g''}$, at the two factors of A attached to the node. We push this forward to $\bar{M}^n_g$ using Thom class of the boundary divisor, to produce the boundary correction term for $\overline{Z}^n_g$.

This construction can be captured more concisely by defining an action of the matrices $R(z)$ on the cohomology of Deligne–Mumford spaces. This action lifts Givental's loop group action on Gromov–Witten potentials, defined in his framework of quantised quadratic Hamiltonians [G2].

3.3. What makes the classification work? The *Euler class* of a Frobenius algebra A is the product of the co-product of 1: $1 \mapsto A \otimes A \mapsto A$. Pictorially, this is represented by a torus with one outgoing boundary. For the cohomology ring of a manifold, this is the usual Euler class, and it always squares to zero. However, the quantum Euler class *can* be *invertible*: this happens precisely when the quantum multiplication is semi-simple.

Hence, in the semi-simple case, one can increase the genus of surfaces without loss of information in the CohFT. Now, the *Mumford conjecture* (Madsen–Weiss) describes the complex cohomology of the open moduli space M^n_g of smooth curves in the $g \to \infty$ limit as a free $\mathbb{C}$-algebra in the *tautological* classes κ_j, ψ_i. From here, we can classify the restriction to M^n_g of semi-simple CohFT's.

Finally, it follows from a result of Looijenga's [L] that, in large g the boundary divisors of $\bar{M}^n_g$ have Euler classes which are not zero-divisors. This controls the problem of extending cohomology classes to the boundary.

4. Some open questions

- *Reconstruction from classical data.* The reconstruction theorem requires some quantum cohomology information. A similar-looking result in classical singularity theory starts from more "classical" data. Namely, Saito's work shows how to produce Frobenius manifold structures on the unfolding space of a function f with an isolated critical point. The associated CohFT is the *Landau–Ginzburg B-model* of mirror symmetry. In this model, the analogue of $H^*(X)$ is the Jacobian ring of f, and in some ways the first Chern class is the residue class of f therein. The quantum cohomologies correspond to the Jacobian rings of deformations of f. Now, a theorem of Scherk [Sc] asserts that, up to a change of coordinates in the ambient space, the singularity is determined from the Jacobian ring and the residue class of f. This does not quite determine the Frobenius manifold, but carries quite a bit of information (the *F-manifold*, missing the metric); the missing information has been completely described by Saito, as the choice of a *primitive volume form*.

Question: Is the quantum cohomology F-manifold determined by the classical $H^*(X)$ and $c_1(X)$, at least for varieties with generically semi-simple quantum cohomology? What minimal extra data is needed to recover the Frobenius manifold?

- *Degeneration.* Semi-simple theories come in families with non-semi-simple degenerations: classical cohomology for GW theory, or the Jacobian ring for the Landau–Ginzburg B-model potential with an isolated critical point. The Givental data for semi-simple theories degenerates at such a classical point. Nonetheless, some theories are continuous.

  Problem: Understand this phenomenon; in particular, provide a formula for the higher-genus part of the Landau–Ginzburg B-model.

- *Formality.* Gromov–Witten theory can be defined at *chain level*, because the virtual fundamental classes are define from actual spaces with obstruction bundles. A cohomological classification leaves open the possibility of higher operations, in the style of Massey products in cohomology.

  Question: Can we have higher operations in semi-simple field theories?

  The expected answer is no. Formality of Deligne–Mumford spaces erases the difference between the chain operad and its homology. The proof of the classification theorem uses Euler classes to produce homological splittings, which appear to work at chain level. However, it is a bit more difficult than it seems to define the chain-level theory with all the trappings of a CohFT – mind, for instance, the fact that the quantum cup-product breaks the grading!

- *Twisted Frobenius manifolds.* Gromov–Witten K-theory and the twisted Gromov–Witten invariants (introduced by Coates and Givental) and other generalised cohomology examples do not fit the standard definition of Frobenius manifolds. Variations of the notion have been studied by Dubrovin and Manin [M2].

  Problem: Describe Givental's higher genus reconstruction in generalised cohomology.

  This is not an idle generalisation, as K-theory seems particularly suited to equivariant calculations.

Acknowledgements. It is my pleasure to thank the numerous people who, in patient conversations, have improved my understanding of the matter. Among those are Mohammed Abouzaid, Arend Bayer, Tom Coates, Kevin Costello, Alexander Givental, Hiroshi Iritani, Yurii Manin, Graeme Segal and Dennis Sullivan.

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Lecture on Invitation by the KWG

5ECM Amsterdam 2008

The revolution of 1907 – Brouwer's dissertation

Dirk van Dalen

Abstract. Brouwer's dissertation contains both the germs of his topological and his foundational work. We concentrate here on the latter. The extraordinary rich thesis contains comments and critique on his contemporaries, and a novel approach to many foundational issues. Between the lines one finds the genesis of the continuum and the natural numbers via the ur-intuition, the constructive interpretation of logic, choice sequences, and a precise discussion of the language, logic, and mathematics levels. The present paper provides a survey of the material and comments on Brouwer's innovations.

Mathematics Subject Classification (2000). 01, 03.

Keywords. Continuum, ur-intuition, Korteweg, intuitionism, proof interpretation, choice sequences.

1. Introduction

A century ago a young, unknown Dutchman submitted to the faculty of mathematics and physics a dissertation that contained the germs of a revolution in mathematics, which soon became known under the name of *Intuitionism*. In fact the student was neither young, according to our standards, nor totally unknown. He was, although undeniably a brilliant student, almost 26 when he got his Ph.D. There was a reason for his protracted study (he enrolled at 16, and got his *doctorandus degree*, say 'master's degree', at 23): he suffered from nervous breakdowns and accompanying physical disorders, as a result of spending his vacations in the compulsory military service.[1]

Nonetheless he had already published on decompositions of rotations in four-dimensional space, on higher dimensional vectordistributions (including Stokes' theorem), and on potential theory in non-Euclidean spaces. Moreover he had given in 1905 a series of lectures with a strong mystical flavour, published as *Life, Art, and Mysticism*, [Brouwer 1905].

The dissertation was supervised by the leading Dutch mathematician at the time, D. J. Korteweg. Korteweg was a prominent applied mathematician, and he had hoped that Brouwer would continue his research in the direction of his earlier publications.

[1] Students could accomplish their service during vacations.

Brouwer, however had the lofty ideal of putting mathematics on a better basis, like some of the great men of the period – Borel, Cantor, Frege, Hilbert, Poincaré, Russell, Zermelo, …. The result was a dissertation that basically consisted of two parts: geometry and foundations. The geometrical part contained of a solution of a special case of Hilbert's fifth problem of eliminating the differentiability conditions from the definition of Lie groups; he solved Hilbert 5 for continuous groups acting on a one-dimensional manifold. His investigations of Lie group theory led him to topology, and to his stunning innovations in the years following 1909. To do justice to the evolution of Brouwer's topological ideas, see [van Dalen 1999], Chapters 3, 4, and 5, is beyond the present occasion; it has to be left to 2010. Let us point out that Lie groups did not come in by accident. In order to create the mathematical 'measurable' continuum out of the intuitive continuum, continuous group theory was exactly the tool needed.

Figure 1. Student Brouwer.

Before we move on to discuss the dissertation let us make a few observations. Some parts of it give the impression of being written in haste. This is indeed the case, as confirmed by the correspondence between student and adviser. After the mystical intermezzo Brouwer returned to his thesis; in September 1906 he informed Korteweg that he was about to start to order his ideas and notes for the thesis, and in October he sent him the list of the six chapters, and a synopsis of the first chapter. The chapters were

(1) The building of mathematics.

(2) Its emergence in connection with experience.

(3) Its philosophical meaning.
(4) Its founding it on axioms
(5) Its value for society.
(6) Its value for the individual

This somewhat grand scheme was in the end reduced to three chapters: The building of mathematics, Mathematics and experience, Mathematics and logic.

From then on Brouwer forced himself to deliver regularly drafts of the text, and if we take into account that he defended his dissertation on February 19, 1907, it is no exaggeration to say that he was working against the clock. Indeed, on January 23 he was still discussing the part on logic with Korteweg. Apparently publishers worked very efficient in those days – the book was ready in time for the public defence.

The mystic intermezzo had not been a contemplative pastime; some of the philosophical tenets of the thesis had recognisable connections with the 1905 lectures. We will briefly mention some topics of *Life, Art, and Mysticism*, and comment on them. The central message was: one should live content for oneself, and not interfere with, or dominate, nature or fellow human beings. The ultimate aim of the subject is introspection – get rid of the separation between object and subject. What stands in the way of the ultimate return into oneself is morally to be condemned.

Like a true mystic Brouwer was far from impressed by the world and its inhabitants: "The life of mankind as a whole, is an arrogant eating away of its nests all over the perfect earth, a meddling with her mothering vegetation, gnawing, spoiling, sterilising her rich creative powers, until it has gnawed away all life, and the human cancer withers away over the barren earth."[2] A profound difference of view, where language and communication were concerned, separated Brouwer from his modern contemporaries. In his words. "No two persons will experience exactly the same feeling, and even in the most restricted sciences, logic and mathematics, which can properly speaking not be separated, no two [persons] will think the same thing in the case of the basic notions from which logic and mathematics are built." In a less colorful formulation: if a person carries out a mental mathematical construction, formulates it in language and communicates it to a second person, then there is no reason to believe that the second person will be able to carry out the original construction. In short, there is no such thing as perfect communication. This, of course, was a powerful motivation to be sceptical with respect to the universally preached virtues of formalization, and the role of language.

One of the central points of Brouwer's philosophy was the role of the individual, called the *subject*. The subject creates in acts of introspection the objects of mathematics (and far more, but we will leave that out here).

In introspection, the distinction between subject and object gradually disappears into a unordered flow of impressions:

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[2]Almost verbatim repeated in [Brouwer 1949].

> And in that merging sea of colours, without separation, without permanence and yet without movement, that chaos without disorder, you know a Direction, which you follow spontaneously, and which you could just as well not follow. You recognise your "Free Will", in so far as it was free to withdraw itself from the world, in which there was causality, and then remains free, and yet only then has a really determined Direction, which it reversibly follows in freedom. [...] The phenomena follow each other in time, bound by causality, because you yourself want, shrouded in clouds, the phenomena in that regularity ([Brouwer 1905], p. 14).

The free will is mentioned here not as a metaphor, but as an actual asset of the subject. Free will plays a role in the dissertation, and it later appears in full force at the introduction of choice sequences.

2. The ur-intuition

For Brouwer, and for each of us, the construction or creation of the basic objects of mathematics is of the greatest importance. At this point he had a long way to go before he could, or would, express himself with the desired clarity. The dissertation opens with a brief recapitulation of the by then accepted means of creating the number systems out of the natural numbers. This is followed by the bold step to introduce the basic material of which mathematics is made through the act of intuition.

> In the following chapters we shall go further into the ur-intuition of mathematics (and of every activity of the intellect) as the substratum, divested of all quality, of any perception of change, a unity of the continuous and the discrete, a possibility of thinking together several entities, connected by a 'between', which is never exhausted by the insertion of new entities. Since continuity and discreteness occur in this ur-intuition as inseparable complements, both having equal rights and being equally clear, it is impossible to stay clear of one of them as a primitive entity, and then to construct it from the other one, the latter being considered by itself; in fact it is already impossible to consider it by itself. Having recognised the intuition of continuity, the "flowing" as primitive, as well as the conceiving of several things as one, the latter being at the basis of every mathematical structure, we are able to state properties of the continuum as a "matrix of points joined together in thought."

This terse introduction of the continuum has the characteristics of a recapitulation of a notion explained to the reader at an earlier occasion. As it stands, it is mystifying rather than clarifying. There actually is a simple explanation: the dissertation originally contained a philosophical introduction (with a definite mystical flavour), which mentioned the intuition of time, the notion of (causal) sequence[3] and the jump from end to means. In this introduction, which actually came at the opening of Chapter 2, the ur-intuition is given its place as *intuition of time*.

> But man is endowed with a faculty, that accompanies all his interactions with nature, that is the ability of objectifying the world, seeing in the world recurrences of sequences, seeing in the world causal systems in time.
>
> The ur-phenomenon is the intuition of time by itself, in which iteration, as "thing in time, and one more thing", is possible, but in which (and this is a phenomenon, which is outside mathematics) also a sensation can fall apart into composing qualities, so that a single moment of life can be lived as a sequence of qualitative distinct things.

The fact that Brouwer, when ordered by Korteweg, dropped this elucidation, made the foundational part of the dissertation harder to grasp. Korteweg's motivation was simply that a text with such strong flavour of mysticism would not contribute to the general appreciation of the faculty members, and even harm his student's future in mathematics. As Brouwer put it in a letter to Korteweg, the topics of Chapter 2 suddenly appeared in the lime light to take the place of their former leader, "and it was not possible to dress all of them so that on their own they could together save the show."

This rather condensed explanation of the underlying motivation of the ur-intuition was elaborated more than twenty years later in print. We will commit an anachronism and discuss the later formulation here.

The subject experiences a flow of sensations, the basic phenomenon here is the "falling apart of a moment of life" or "the move of time", i.e. a present sensation gives way to another present sensation in such a way that the first one is retained in memory. Thus a temporary pair (*twoity* in Brouwer's terminology) comes into existence, and is separated from the ego (of the subject). Evidently this move of time can be repeated, and thus a triple emerges, etc. In this wealth of pairs, triples, … a measure of order is enforced through a mental act of the subject, later called the *causal act*, which identifies certain pairs, triples, etc. In the ultimate identification (abstraction) all pairs lose their individual character and become the "common substratum of all twoities, that forms the ur-intuition of mathematics, the self-unfolding of which introduces among other things the infinite as a mental reality, and in fact first of all the totality of the natural numbers, not to be commented on here; next that of the real numbers, and finally the whole of pure mathematics" ([Brouwer 1929], p. 154).

The natural numbers are in fact only one half of the ur-intuition; the other half, the continuum, is simultaneously produced in the act of the move of time. A year after

[3] Brouwer uses the term "volgreeks" in his early writings. For lack of a better word, we use "sequence" as a translation. In his later writings, for example [Brouwer 1929A,1933A], Brouwer introduces the term 'causal sequence', which denotes the 'equivalence class' of sequences under identification by the subject.

the dissertation, Brouwer gave a lecture at the international mathematics congress in Rome, where he discussed the set theory that belonged to his constructive universe, [Brouwer 1908].

> When one investigates how mathematical systems come about, one sees that they are constructed out of the ur-intuition of two-ity. The intuitions of the continuous and the discrete join here, as [simply] a second [thing] is thought not by itself, but under preservation of the recollection of the first. The first and the second are thus *kept together* and the intuition of the continuous (continere = keeping together) consists of this keeping together. This mathematical ur-intuition is nothing but the contentless abstraction of the sensation of time. I.e. the sensation of fixed and floating together. Or of remaining and changing together.

Here one sees that the continuous, let us say, the continuum, comes about in the passage from one sensation to the next – it *is* the passage. Hence Brouwer's claim that the two notions are inseparable: you cannot have the one without the other.

There is a problem that we have to face: did Brouwer in 1907 see the necessity of embracing infinity? To begin with, the infinity of the natural number sequence? The formulation of the ur-intuition in the dissertation does not mention the issue; it clearly allows the subject to construct the individual numbers, but does it allow for the natural numbers as a totality?

For explicit statements one has to turn to the publications in the late twenties; in the early days of Brouwer's intuitionism one has often to read between the lines to see what the extent of the natural numbers is.

Whatever may be in doubt, it is clear that one will not get the set of natural numbers with the same status as, say, the set of numbers less than 20. There is no such thing for the subject as a completed totality of natural numbers; the second best for the subject would be the recognition of the totality of the natural numbers as a potential infinite entity. There is a letter from Brouwer to the Utrecht mathematics professor J. de Vries (undated draft in the Brouwer archive), in which Brouwer gave a quick survey of the main points of the dissertation. Here he states "I put the 'mathematical construction act of complete induction' in the place of 'the axiom of complete induction', and show how this is nothing new after the intuition of time." Thus the act that yields the totality of natural numbers is implicit in the intuition of time. The above formulation of Brouwer foreshadows the modern practice that reduces complete induction to "complete recursion" or "iteration", where the "mathematical act of complete induction" takes the place of the recursor.

In the dissertation and various other places, Brouwer uses the term "the mathematical intuition *and so on*" (discussion of Russell), and at another place he comments on Dedekind: "Dedekind's system has no meaning; a logical meaning would require a consistency proof, which Dedekind does not give either, then he would have to appeal to the intuition of 'and so on'." In his later publications Brouwer specifies the generation of the number sequence as caused by the "self-unfolding of the act of the intellect".

We note that quite early on Brouwer was aware of the role of the natural number sequence, viewed as a mathematical object. Starting from the "and-so-on" intuition, he passed on to the more explicit 'self-unfolding' notion. This self-unfolding, so to speak, is part and parcel of the ur-intuition; it supplements the first acts of mathematical attention that yield the finite sequences. The discussion of Brouwer's views on the natural number sequence should help to alleviate the worries that Brouwer would in fact only be able to create the individual numbers, but not the whole of all natural numbers. The latter would have been serious, Brouwer would not have been able to quantify over the natural numbers.

The axiom of complete induction is explicitly dealt with in the list of theses appended to the dissertation.[4] The second one says, "It is not only impossible to prove the admissibility of the axiom of complete induction, but it ought not to have a place as a separate axiom or as an intuitive truth. Complete induction is an act of mathematical constructing, that has its justification already in the ur-intuition of mathematics."

In his *Intuitionism and Formalism* Brouwer returns to the issue of the objects of mathematics: "This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely; this gives rise to the smallest infinite ordinal ω." In fact this is the first step towards his version of the second number class.

The names "*intuitionism*" and "*formalism*", by the way, were not Brouwer's invention. They were coined by Felix Klein in his Evanstone lecture, [Klein 1894]; he had a fairly informal classification in mind, the one that fitted the general practice of his day. Brouwer adopted the terms in his review of Mannoury's book on the foundations of mathematics, *Methodologisches und Philosophisches zur Elementarmathematik*, [Brouwer 1910]. There he listed Dedekind, Peano, Russell, Hilbert, and Zermelo as formalists, and Poincaré, and Borel as intuitionists. Brouwer criticised the intuitionists for "rejecting every infinite number, including the denumerable" and for identifying mathematical existence with non-contradictority (Poincaré's slip). In his inaugural lecture of 1912, he considers formalism as 'largely German' and intuitionism as 'largely French', and his own brand of intuitionism is called there "*neo-intuitionism*". In his mature foundational program the 'neo' was dropped, and the French school was called "pre-intuitionist" ("semi-intuitionist" by some). Formalism was by then exclusively used for the Hilbert school.

Now let us look at the other twin, the continuum. Already before the ur-intuition was introduced, Brouwer discussed the continuum in his notebooks where he wrote

[4] A dissertation was always supplemented by a list of 'theses'. The topics of these were not necessarily connected to that of the body of the dissertation.

down his ideas, and tested technical details for the dissertation. In the notebooks he accepted the continuum as intuitively given, but without any details. Here are two quotes:

"The intuitive continuum as the opposite of the point; the unknown, about which lack of understanding is impossible."

"About the continuum I see intuitively that there are yet unknown assignments on it, as such it is the matrix of yet unborn points. It exists thus independent from the points to be built on it, is thus different from the set of those points, for otherwise its creation would follow that of those points."

He obviously saw that one needed more than just the definable (or lawlike) points. After all, Cantor had made it clear that any attempt to stick to a denumerable continuum was bound to fail; and although Brouwer found a satisfactory solution to the cardinality questions concerning the continuum only after the introduction of choice sequences and spreads, he was aware of the fact that the extra, non-lawlike points had to be incorporated. He called these extra points "unknown points". For example, "One also defines unknown irrationals as limits of unknown series. On assigns the familiar ordering relation, and only afterwards one has to introduce the continuity postulate, in order to carry out the operations on these irrationals."

The unknown points of the continuum are used in the dissertation to refute the well-ordering theorem.

> "Now we know that besides the denumerable sets, for which the theorem *certainly holds*, there exists only the continuum, for which the theorem *certainly does not hold*, firstly because the greater part of the elements of the continuum must be considered as unknown, and consequently can never be individually ordered, secondly, because every well-ordered set is denumerable. Thus this question also turns out to be illusory." ([Brouwer 1907], p. 153)

A similar use is made in the discussion of the Cantor–Bernstein theorem.

3. Choice sequences

It is an anachronism to place the choice sequences in the dissertation; yet, there are certain associations that are hard to ignore. The first question to ask is, was Brouwer aware of the earlier discussion of the phenomenon? At the time of Brouwer's dissertation the topic of 'choice' was widely discussed in connection with Zermelo's axiom of choice. Emile Borel, in particular, took up the idea of sequences of numbers determined by choice. In the discussion following Zermelo's paper, he explicitly rejected arbitrarily large choice functions, i.e. the possibility of non-denumerably many choices; by default one might conclude that he endorsed ω-sequences of choices. The crucial point here is the location of these choice objects, are they to be found in an idealised mathematical world, or inside the human being? Borel took the human side of the issue serious, in his 1908 Rome lecture, *Sur les Principes de la Théorie des Ensembles*, he admitted that denumerably many choices could not be effected in their totality; nonetheless he was willing to allow certain thought experiments on them. But, as the infinite totality of choices (evidently considered as choice sequences *avant la lettre*) is non-denumerable, his conclusion was that it could not be admitted as a mathematical entity for use in mathematical arguments.

Apart from the discussion around the axiom of choice, there was at the time of Brouwer's dissertation only Du Bois-Reymond's book on Function theory, which explicitly discussed "lawless sequences". Neither the dissertation, nor the notebooks mention Du Bois-Reymond; hence we may safely assume that Brouwer was not familiar with Du Bois-Reymond's lawless sequences; as Du Bois-Reymond introduced the notion in the context of analysing the continuum, Brouwer would certainly have seen the relevance, had he been aware of the text. In Brouwer's notebooks there is mention of *kansrij* (chance sequence), and Brouwer uses the term "prendre au hasard". Apparently he did not see how to make use of the notion of choice. There is no explicit mention in the dissertation, but one meets choices in the part where perfect sets are discussed. Brouwer's procedure there is familiar in the proofs of Cantor's fundamental theorem. On page 65 there is a figure that, slightly anachronistically, may be called a "fan". And, indeed, the step from one node to a next one is made by *choice*. Hence choice sequences *avant la lettre* do appear in the dissertation, albeit anonymously. The context of Cantor's fundamental theorem makes this unavoidable; sticking to the legitimate sets of Brouwer's classification would deny all sense to the theorem. Indeed, the 'fan' is used to deal with certain subsets of the continuum, and it is just a tool to describe these subsets. Fans and spreads, as objects in their own right, are only introduced in the *Begründungs*-paper in 1918.

Within a year after the dissertation, Brouwer returned to subsets of the continuum, [Brouwer 1908]. By an analysis as sketched in the dissertation, he showed that these are denumerable or of the cardinality of the continuum. The argument, however, is not constructively acceptable, as Brouwer soon realised. In his own reprint of "Intuitionism and Formalism", [Brouwer 1913], he wrote in the margin that the process of weeding out isolated branches was not legitimate, as one cannot in general decide whether a branch will eventually become isolated. The point is elaborated in the subsequent "Addenda and Corrigenda", [Brouwer 1917]. By that time Brouwer had come to his mature intuitionistic program with choice sequences and spreads. The notion of "deconstructible spread", i.e. the spread in which the weeding process can be carried out, returned in the second paper of the *Begründungs* series; after that the notion was not mentioned again, except briefly in Brouwer's correspondence with Fraenkel[5].

[5] Brouwer to Fraenkel 12.I.1927, see [van Dalen 2000].

It is quite reasonable, in the light of the existing publications and notes of Brouwer to conclude that the notion of choice sequence was already in his mind at the time of the dissertation. It had to wait, however until he found the key principle for handling functionals – the continuity principle.

4. Brouwer on logic

There are, even today, persistent rumours that Brouwer disapproved of logic in general. It certainly is the case that he was critical in more than one sense. He denied, for example, that mathematics was based on logic. Logic, he said, is based on mathematics. Mathematics, according to him, is a constructional activity that creates the objects of mathematics, and operates on them to obtain mathematical structures of a large variety. The properties of these structures (*buildings* in his terminology) are expressed in the language of mathematics and established, or verified by means of a certain kind of construction that usually is called a *proof*. For simple statements the required construction is easy enough, think of a construction that establishes $5 + 4 = 9$. For more complicated properties a more systematic approach becomes desirable. This approach is embodied in the *proof interpretation*, which was formulated by Heyting, and which is based on ideas of Brouwer that go back to the dissertation. The chapter "Mathematics and Logic" opens with a discussion of logic, in particular the hypothetical judgement. The reading of it is far from obvious, and there are different ways to look at Brouwer's formulations, see [Atten, van 2008] and [van Dalen 2004]. The basic ingredient in establishing $A \to B$ is to transform a proof(construction) a for the structure A into a proof(construction) b for B. And this should be done in a general manner, in the sense that we have a construction f that converts a into $f(a) = b$. The problem here is 'should one have the construction a before one can proceed to the construction b?' This, evidently, would put normal practice in danger. Heyting gave an instructive example of a questionable case: let $A =$ "there is a sequence of decimals 0123456789876543210 in π",[6] and $B =$ "there is a sequence of decimals 012345678987654321 in π". We have an obvious construction that converts a proof of A into a proof of B. But we have no proof of A; this clearly would be asking too much.

One has to see, so to speak, Brouwer's logic in action. There are quite a few places where Brouwer applies the hypothetical argument, so we may safely assume that he supported Heyting's formulation of the interpretation of the logical operators. The beauty of the Brouwer–Heyting–Kolmogorov view of logic is that logic is internalised in mathematics; it becomes a calculus of constructions.

So what is Brouwer criticism of logic? Mathematical logic busies itself with the mathematical language; it draws consequences from given statements about mathe-

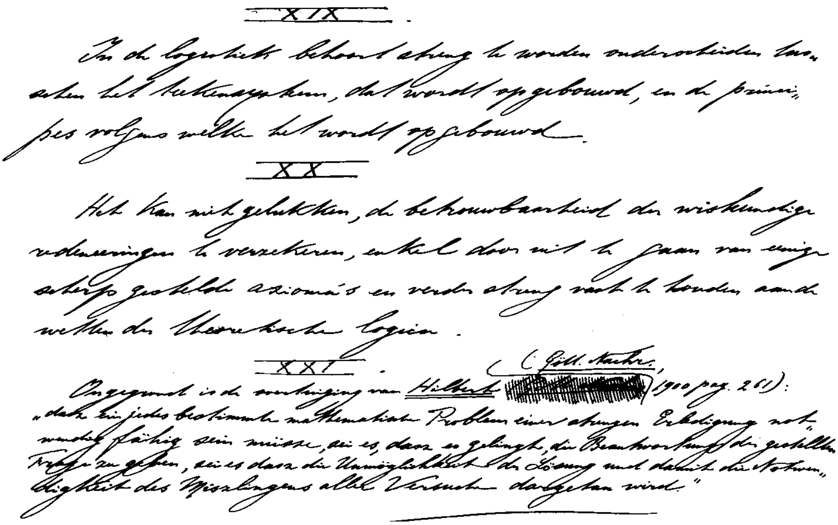
[6]We have changed the example slightly. Heyting's original sequence 0123456789 *does* occur in π.

matical objects and relations. Given Brouwer's dim views of the language, in science as well as in daily life, it is no surprise that he denies it the exactness that his constructional activity requires. Some logical principles are justified by the proof interpretation, some are not. In the dissertation Brouwer did not measure up to his own standards; the principle that he rejected whole heartedly a year later, to wit the *principle of the excluded middle*, PEM, was accepted. The matter is curious, whereas he rejected with good reasons *Hilbert's Dogma*, i.e. the solvability of all mathematical problems, he claimed that PEM was harmless and uninformative, because, he said, $A \vee \neg A$ is equivalent to $\neg A \rightarrow \neg A$. This puzzling identification was, I conjecture, borrowed from the lectures on logic of the Amsterdam philosopher Bellaar-Spruit. In his logic course Bellaar-Spruit had stated PEM in the form "If Alexander is not a great man, then Alexander is not a great man". Whatever the reasons may have been for this formulation (and none of them could be valid), Brouwer apparently forgot to test the equivalence in his own logical interpretation. However, a year later Brouwer published the paper "The unreliability of logic", in which he dropped PEM and identified it with Hilbert's dogma. The identification is, from an intuitionistic point of view easy to see. Let us abbreviate a is a proof of A as $a : A$. If $a : A \vee \neg A$, then a had to consist of two parts, a number $a_0 < 2$ and a construction a_1, such that $a_1 : A$ if $a_0 = 0$ and $a_1 : \neg A$ if $a_0 = 1$. Now $p : \neg A$ is defined as $p : A \rightarrow 0 = 1$ (where instead of 0=1 any contradiction may be taken). Since there is no construction that identifies 0 and 1, A can have no proof q. Hence $p : \neg A$ is equivalent to "A has no proof", a quite reasonable constructive viewpoint. Summing up: $A \vee \neg A$ has a proof $\Leftrightarrow$ A has a proof or there is no proof of A. And indeed, Hilbert's dogma of the solvability of all problems A says "we can show (prove) A, or we can show that A has no proof."

Brouwer's dissertation contains extensive critical discussions of existing foundational approaches. On all points he was quite correct from his constructive view point, but he must have shocked his readers. We mention a few points.

The *axiomatic method* was criticized because the setting up of an axiom system, sticking conscientiously to it and showing it to be consistent, gave no guarantee whatsoever, that there would be a mathematical structure satisfying the axioms. Given a structure, such as e.g. Euclidean space, it would be quite alright to formulate axioms describing the structure. The theorems derived from these axioms (in Brouwer's case by intuitionistic means) would then automatically be correct in the structure. But the connection between structure and axiom system remained a one-way matter. On page 141 Brouwer explicitly mentions that "it has nowhere been proved that if a finite number has to satisfy a system of conditions, of which can be shown that they are consistent, that then such a number indeed exists." With the hindsight of Gödel's theorems this sounds plausible and familiar.

Cantor's *set theory* shares the same fate. The countable sets, ordinals of the second number class, and the continuum are accepted, but higher ordinals and cardinals are



XIX.

In de logistiek behoort streng te worden onderscheiden tusschen het teekensysteem, dat wordt opgebouwd, en de principes volgens welke het wordt opgebouwd.

XX.

Het kan niet gelukken, de betrouwbaarheid der wiskundige redeneeringen te verzekeren, enkel door uit te gaan van eenige scherp gestelde axioma's en verder streng vast te houden aan de wetten der theoretische logica.

XXI.

Ongegrond is de overtuiging van Hilbert [illegible] (C. Gött. Nachr. 1900 pag. 261): „dass ein jedes bestimmte mathematische Problem einer strengen Erledigung notwendig fähig sein müsse, sei es, dass es gelingt, die Beantwortung der gestellten Frage zu geben, sei es dass die Unmöglichkeit der Lösung und damit die Notwendigkeit des Misslingens aller Versuche dargetan wird."

Figure 2. Three theses belonging to the dissertation.

ruled out as being beyond mental construction. In particular is the power-set out of bounds. Brouwer goes here further than the French intuitionists; he accepts the full continuum, as opposed to Borel who stuck to a 'definable' continuum.

Logicism has no interest for intuitionists as it studies a symbolic language and hence "remains irrevocably separated from mathematics" It restricts itself to "*the language of mathematics*, which *itself is no mathematics*, but just an imperfect tool for people to communicate mathematics to each other, and to support their memory for mathematics." The discussion of Hilbert's consistency methods is interesting as it points out in a precise manner the shortcomings of the Heidelberg paper.[7] Hilbert wished to develop logic and mathematics simultaneously *ab ovo*, in particular without making use of any mathematics. However, in his treatment he had to use the principle of induction in dealing with the usual syntactic arguments on the meta level. The step forward of Hilbert, compared to axiomatics and logicism, is that he explicitly moves from mathematics to a second order mathematics that deals with the formal system of the mathematics in the first order. The basic problem remains even for Hilbert, that a consistency proof of the formal system does not guarantee the existence of a model, or as Brouwer puts it, "the consistency of the language system, shown on the basis of the mathematical intuition, does not prove (justify) the mathematical intuition that it accompanies".

[7] Hilbert's lecture at the Heidelberg congress, 1904, [Hilbert 1905]. It was the first "proof theoretic" paper, and it dealt with a fragment of arithmetic.

The discussion of Hilbert's paper ends with a detailed enumeration and discussion of the levels of mathematics, language, logic.

It may safely be assumed that Brouwer's foundational ideas did not travel far. The dissertation was in Dutch, and although in the twenties a few mathematicians outside of the Netherlands even went so far as to learn enough Dutch to be able to read the dissertation, we may safely assume that Brouwer's foundational work only started to draw attention after 1918, when the first papers in German started to appear. Hilbert and Weyl were informed at an earlier stage. In the summer of 1909 Hilbert was on vacation at the North sea coast in Scheveningen, where Brouwer visited him and told him his views on the various levels and the (proof) theoretic consequences.

Conceptually Brouwer was probably closest to Poincaré, who had expressed similar objections to a wide variety of foundational ideas; there was however one essential point on which Brouwer could not follow him, that is Poincaré's view that "existence = consistency". He stuck to his "*existence* in mathematics means *intuitively constructed*", and deplored that the above identification showed "how little Poincaré thinks of taking the *intuitive* construction of mathematics as the point of departure for his criticism", [Brouwer 1907], p. 177.

5. Life and career

The young Brouwer was a precocious child; he was mainly taught at home by his mother, who had been a school teacher. His father was a headmaster with a solid reputation for pedagogy. The boy grew up in an atmosphere of study and responsibility. The late nineteenth century schoolmasters were as a whole a dedicated group, believing in the values and benefits of education; they were the true champions of enlightenment. In this environment Brouwer developed into a mature boy with a penchant for learning and for sports (the rougher, the better). At the high school age he mixed in Haarlem with artists and scholars. There were virtually no topics that he despised; before entering the university he was already well read in philosophy, and in his study and research he made good use of the fact. Given the circumstance that Dutch mathematics and philosophy could not compete with that of the international centres, where new ideas and subjects flourished, he had to teach himself a great deal. The dissertation showed that he had managed very well for himself. And so the student Brouwer, who presented his thesis, could face the faculty with self-assurance.

The public defence of his Ph.D. thesis presented no problems, even if the examiners had been able to punch a few holes in his line of argument, the wealth of ideas and the perfect command of the necessary technical skills would have been enough to convince the committee. As it was, he was awarded the doctorate *cum laude*. He had not been an easy student for Korteweg. The correspondence shows the temper of the young man that was wisely and with tact overruled by the older master. The foundational material, and in particular the underlying mystical, subjective motivation of the central points were perhaps too much for him. Brouwer, later in life, told that Korteweg could no longer than a quarter of an hour on end read these parts, they made him feel dizzy. In one case Korteweg used his authority and ordered Brouwer to delete part of his manuscript. He wrote, "But really Brouwer, this wont do. A kind of pessimistic and mystic philosophy of life has been woven into it, that is no longer mathematics, and has also nothing to do with mathematics. …" Brouwer gave in, much to the detriment of future readers – some of the basic ideas and notions now came out of the blue. Nonetheless student and teacher remained close for years to come. Korteweg was well aware that a student of this sort comes along only once in a Blue Moon, so he tried to further the interests of the young man. The first step was to make him a *privaat docent*[8]. Brouwer accepted the post with a public lecture *On the nature of geometry* (1909). After that prolonged and difficult negotiations with the board of the university followed. Eventually Korteweg decided

Figure 3. Privaat docent Brouwer (1909).

to make a detour via the Royal Academy of the Netherlands, the idea being that once Brouwer was a member of the Academy, the university could hardly refuse to appoint him. With recommendations of Hilbert, Poincaré, Klein, and Borel, the membership of the Academy was secured in 1912. In the same year the appointment to extraordinary professor at Amsterdam followed. At that occasion Brouwer gave

[8]After the German *Privatdozent*. A kind of free lance university teacher with a small fee.

his inaugural lecture *Intuitionism and Formalism*. As the lecture was translated into English, this was the first occasion for an international readership to get acquainted with Brouwer's program. A year later Korteweg made the generous offer to give up his full professorship for Brouwer and to exchange it for the extraordinary chair. From then on Brouwer was more or less in command of the mathematics section.

It has often been suggested that Brouwer's topological career was a deliberate strategy to acquire enough status to make his foundational program respectable. All evidence points the other way; he became involved in topology partly through his work on Hilbert 5. When he decided to submit his results to the Mathematische Annalen he found out that he had been relying on Schoenflies' somewhat deficient 1908 *Bericht* on point sets. Thus he had to clean up the basics of topology, and once he got involved in the subject, he fell in love with its fascinating geometric features. In an incredible tempo he mastered everything a topologist at that time should know. Being a stubborn man with a strong feeling of justice, not tempered by any considerations of misplaced respect, he was often incited by attempts to belittle him, or to cheat him. This was a strong incentive to show his mastership in the face of adversity. Whatever his motivations were, love or justice, he managed to reach the top of his profession already before World War one. It brought him ample recognition, e.g. he was the third on the list for the succession of Felix Klein. Leyden offered him a chair in 1915, and in 1919 he got offers from Göttingen and Berlin.

Brouwer's topological activity was cut short at the beginning of the World War; there is a number of possible reasons: (1) after the successful wrapping up of the dimension problems, he may have lost interest in further research, (2) he may have wanted to return to the foundations of mathematics, (3) the isolation as a consequence of the war. Probably each of those contributed towards his return to intuitionism. It started with his lectures on point set theory – after all, he was the professor in set theory, function theory, and axiomatics (the last topic seems a curious duty for an intuitionist) – which were at first middle of the road constructive, but suddenly in 1916 turned into full blooded intuitionism. That was the beginning of his mature intuitionistic program.

After the war he simultaneously worked on topology and intuitionism, but gradually the topology part receded.

Characterising Brouwer's brand of constructive mathematics, one has to acknowledge that it is the constructivism of a topologist. Whereas most constructivists stuck to the discrete part of mathematics, Brouwer ventured out into infinitary realms, such as the continuum, Baire space, Cantor space and the various function spaces. Her hardly ever practised number theory or combinatorics, and algebra appeared mostly in a geometric or topological context. His introduction of spreads and fans betrays the hand and the taste of a topologist.

Already before acquiring a position that was generally considered a prerequisite for marriage, Brouwer spread his wings. In 1904, after passing his final exam, he bought some land in a picturesque part of the country, and had a friend of his design a cottage for him. During that summer he married an older divorcee with a daughter. And so he settled in the countryside where he could devote himself to his research. His wife was the daughter of a family doctor, who had died early. His widow had carried on the pharmacy that belonged to the practice. The idea was that the Brouwer's wife would eventually be in charge of the pharmacy; for this purpose she studied pharmacy at the Amsterdam University. In a truly enterprising spirit Brouwer bought in 1905 the pharmacy from his mother in law for his wife. The self made mystic was thus solidly tied to the materialistic world, even before he had secured himself a place in the scientific world. In practice he remained a free man, who travelled as before. For a time Göttingen became his second home, where he made friends with the mathematicians. In those happy days he had friends all over Germany, but in particular in Göttingen and Berlin. Like a true Göttingen man he got himself a place in the Harz,[9] and to even things out, also a house in Berlin. He usually lived in the cottage in Blaricum, while his wife stayed during the week in Amsterdam, minding

Figure 4. The young couple in Blaricum.

the pharmacy. In due time he acquired more property in Blaricum. Colleagues and friends were always welcome; quite a number of top mathematicians stayed with him, among others Hilbert, Carathéodory, Weyl; and Einstein too was an occasional guest. Slowly Brouwer adapted to the bohemian life of *het Gooi*, as the area was called, mixing with writers, painters, journalists, politicians. Within some ten years the introvert student had become a man of the world, loved and admired for his wit

[9] That is to say, the house in Harzburg was the property of his protegé/secretary/girl friend, Cor Jongejan.

and universal knowledge, feared for his sharp tongue. The man who entered the second phase in his career had come a long way, without leaving his personal and scientific loyalties behind.

It is obvious that a brief account of even the beginning of Brouwer's activities can not do justice to the colorful career and life of such a complicated personality. The reader who wants to know more cannot do better than consult the two-volume biography [van Dalen 1999], [van Dalen 2005].

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Plenary Lectures

5ECM Amsterdam 2008

New developments in combinatorial number theory and applications

Jean Bourgain

Abstract. This is a survey of a line of research in arithmetic combinatorics. It is centered around so-called sum-product phenomena in various settings and its applications to problems in number theory, computer science, spectral and ergodic theory. More specifically, the sum-product results in finite fields and residue rings lead to new bounds on exponential sums of various types and in fact provide the first non-trivial estimates. A typical result in this spirit are bounds on Gauss sums for small multiplicative subgroups. Product theorems in matrix spaces derived from the scalar theory enable one to prove various conjectures on the expansion of Cayley graphs and the existence of spectral gaps for Hecke operators, most notably in $\mathrm{SL}_2(q)$ and $\mathrm{SU}(2)$. Those in turn lead to an extension of Selberg's theorem for congruence subgroups and new results on prime sieving in non-elementary subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Finally, Furstenberg's "stiffness problem" for toral actions of subgroups of $\mathrm{SL}_2(\mathbb{Z})$ as well as a quantitative equidistibution property of the orbits are described.

Mathematics Subject Classification (2000). 11B30, 20G40, 11L07.

Keywords. Sum-product, exponential sum, expander graph.

Introduction

This exposé is a partial overview of a recent line of research in combinatorial number theory centered around the so-called "sum-product phenomena" in various settings.

The basic philosophy of the sum-product theorem is that either the sum set $A+A = \{x+y \mid x \in A, y \in A\}$ or the product set $A \cdot A = \{x \cdot y \mid x, y \in A\}$ will be substantially "larger" than A, putting aside obvious obstructions of algebraic (or metrical) nature.

If $A \subset \mathbb{Z}$ is a finite set of integers, Erdös and Szemeredi conjectured in [35] that for all $\varepsilon > 0$
$$|A+A| + |A \cdot A| > c_\varepsilon |A|^{2-\varepsilon}.$$
This problem is still far from resolved and the best general result to date is due to J. Solymosi [59], with
$$|A+A| + |A \cdot A| > |A|^{4/3} (\log |A|)^{-1}.$$

In [28], a nontrivial sum-product result is obtained for subsets $A \subset \mathbb{F}_p$ of a prime field (also the appropriate version for arbitrary finite fields).

We will not discuss here the background for the renewed interest in that type of question, which has to do with the 3-dimensional Kakeya problem (see [5], [48]).

It turns out however that the real interest of the results from [28] are their further developments related to a variety of subjects, including

(i) the theory of exponential sums and their applications;

(ii) the theory of expander graphs and spectral gaps of Hecke operators;

(iii) problems of derandomization in computer science;

(iv) prime sieving in certain thin varieties;

(v) equidistribution properties of orbits of linear groups.

We will survey some of this work. This report, as well as the reference list, is by no means exhaustive however. Also the lack of space will not make it possible to get into proof.

1. Sum-product theorem in finite fields

The following result was proven in [28] and in a more precise form in [27]. See also [61] for a different approach.

Theorem 1.1 ([28] and [27]). *For all $\varepsilon > 0$, there is $\delta > 0$ such that if $A \subset \mathbb{F}_p$ and $|A| < p^{1-\varepsilon}$, then*

$$|A + A| + |A \cdot A| > c|A|^{1+\delta},$$

where $c > 0$ is an absolute constant.

To be pointed out that formulations with explicit exponents have been obtained, but we will not mention them here.

If we try to generalize Theorem 1.1 to arbitrary finite fields, there is the obvious obstruction of a nontrivial subfield. As is clear from the next result, this is the only one.

Theorem 1.2 ([28]). *Assume $S \subset \mathbb{F}_q$ and $|S| > q^{\delta}$, where $\delta > 0$ is arbitrary and*

$$|S + S| + |S \cdot S| < K|S|.$$

Then there is a subfield G of $\mathbb{F}_q$ and $\xi \in \mathbb{F}_q^*$ such that

$$|G| < K^C|S| \text{ and } |S \backslash \xi G| < K^C,$$

where $C = C(\delta)$.

Further generalizations (with an appropriate formulation) to Cartesian products $\mathbb{F}_p \times \mathbb{F}_p$, residue rings $\mathbb{Z}/q\mathbb{Z}$ and, more generally, O/I with I an ideal in the integers O of a number field, will also play a role in the discussion below.

2. Exponential sums over multiplicative subgroups

A first significant application of the result of section 1 is to the theory of exponential sums over finite fields, leading to nontrivial results in situations, where classical methods do not seem to apply. The first progress obtained along these lines appear in [29] and [27].

Theorem 2.1 ([29] and [27]). *For all $\varepsilon > 0$, there is $\delta > 0$ such that if H is a multiplicative subgroup of $\mathbb{F}_p^*$ ($H < \mathbb{F}_p^*$ for short) and $|H| > p^\varepsilon$, then*

$$\max_{(a,p)=1} \Big| \sum_{x \in H} e_p(ax) \Big| < cp^{-\delta} |H|.$$

Earlier results cover the range up to $\varepsilon > \frac{1}{4}$; see [45], [49]. The technique used in those papers are variants of Stepanov's method.

Remark. Nontrivial bounds of the form

$$\max_{(a,p)=1} \Big| \sum_{x \in H} e_p(ax) \Big| = o(|H|)$$

may be obtained provided $\log |H| > C \frac{\log p}{\log \log p}$, for some constant C. This seems to be the limitation of our method. It is a challenging problem to obtain results below this threshold.

Theorem 2.1 is of course equivalent to the following formulation for Gauss sums.

Corollary 2.2. *For all $\delta > 0$, there is $\delta' > 0$ such that if $(k, p-1) < p^{1-\delta}$, then*

$$\max_{(a,p)=1} \Big| \sum_{x=1}^{p} e_p(ax^k) \Big| < cp^{1-\delta'}.$$

Remark. Gauss classical bound by $(k, p-1)\sqrt{p}$ is trivial if $(k, p-1) \geq \sqrt{p}$.

More generally, one has Weil's inequality for $f(x) \in \mathbb{F}_p[X]$ of degree d, namely

$$\Big| \sum_{1 \leq x \leq p} e_p\big(f(x)\big) \Big| \leq d\sqrt{p}. \tag{2.1}$$

This inequality is again trivial for $d \geq \sqrt{p}$. Obtaining nontrivial exponential sum bounds for general polynomials when $d \geq \sqrt{p}$ is a major open problem.

3. Extensions to "almost groups"

It turns out that our methods apply equally well to certain incomplete sums. In particular one obtains estimates on short exponential sums involving exponential functions.

Theorem 3.1 ([6]). *For all $\delta > 0$, there is $\delta' > 0$ such that if $\theta \in \mathbb{Z}_+$ satisfies*

$$(\theta, p) = 1 \text{ and } \mathcal{O}_p(\theta) \geq t > p^\delta,$$

where we denote $\mathcal{O}_p(\theta)$ the multiplicative order of $\theta \bmod p$, then

$$\max_{(a,p)=1} \Big| \sum_{s=1}^{t} e_p(a\theta^s) \Big| < tp^{-\delta'}.$$

A similar result, with the appropriate necessary assumptions, may be obtained for arbitrary finite fields $\mathbb{F}_q$.

Let $q = p^m$ and denote for $x \in \mathbb{F}_q$ the trace

$$\mathrm{Tr}(x) = x + x^p + \cdots + p^{m-1}.$$

Let $\psi(x) = e_p(\mathrm{Tr}(x))$ be the additive character.

Theorem 3.2 ([16]). *Let $\theta \in \mathbb{F}_q^*$ be of order t and let $t \geq t_1 > q^\varepsilon$. Assume*

$$\max_{\substack{1 \leq \nu < m \\ \nu | m}} (p^\nu - 1, t) < q^{-\varepsilon} t,$$

where $\varepsilon > 0$ is arbitrary and fixed. Then

$$\max_{a \in \mathbb{F}_q^*} \Big| \sum_{j \leq t_1} \psi(ag^j) \Big| < Cq^{-\delta} t_1,$$

where $\delta = \delta(\varepsilon) > 0$.

Both Theorems 3.1 and 3.2 have numerous applications, as described for instance in the book [50]. Because shorter sums can be handled, several results in [50] can now be stated with less restrictive assumptions and in some the conclusion may be strengthened. Before listing a few of those issues, we state some further developments of our techniques.

4. More on exponential sums

We first consider the analogue of Theorem 2.1 for subgroups H of the unit group $\mathbb{Z}_q^*$ of the ring $\mathbb{Z}/q\mathbb{Z}$ of residues modulo q. A number of restricted results have been obtained here; see also [15]. We only state the general one.

Theorem 4.1 ([8]). *Let q be an arbitrary modulus. For all $\varepsilon > 0$, there is $\delta = \delta(\varepsilon)$ such that if $H < \mathbb{Z}_q^*$ satisfies*

$$|H| > q^{\varepsilon},$$

then

$$\max_{\xi \in \mathbb{Z}_q^*} \Big| \sum_{x \in H} e_q(\xi x) \Big| < q^{-\delta} |H|.$$

Remarks. (1) Note that the statement in Theorem 4.1 is uniform in the modulus q. The case of specific moduli q involving a bounded number of prime factors had been treated previously in [15]. A classical example of such sums are Heilbronn sums

$$S(a) = \sum_{x=0}^{p-1} e_{p^m}(a x^{p^{m-1}}),$$

where $m \geq 2$ is fixed, and their generalizations; see [57], [44], [15], [10].

(2) Note also that in Theorem 4.1, we only make the assumption $|H| > q^{\varepsilon}$. However, extension to incomplete sums, in the spirit of Theorem 3.1, is more restrictive if the modulus is composite; see [15], [11]. This is clear, letting, say $q = p^2, \theta = 1 + p$ and $H = \{\theta^j \mid 0 \leq j \leq \frac{p}{10}\}$.

An analogue of Theorem 3.1 for general modulus is the following:

Theorem 4.2. *Let $\theta \in \mathbb{Z}_q^*$ satisfy $\mathcal{O}_{q_1}(\theta) \geq q_1^{\delta}$ for $q_1 | q$ ($\delta > 0$ an arbitrary given exponent). Then*

$$\max_{(a,q)=1} \Big| \sum_{s=1}^{t} e_q(a\theta^s) \Big| < t p^{-\delta'}$$

assuming $t > q^{\delta}$ and where $\delta' = \delta'(\delta) > 0$.

Going beyond monomials, extensions of the combinatorial method based on sum-product results permit us to treat also certain "sparse" polynomials in the spirit of Mordell [56].

The following statement is essentially optimal, as a qualitative result, if only assumptions on the exponents are made.

Theorem 4.3 ([7]). *Let*

$$f(x) = \sum_{i=1}^{r} a_i x^{k_i} \in \mathbb{Z}[X],$$

where $(a_i, p) = 1$ *and* r *is fixed. Assume*

$$(k_i, p-1) < p^{1-\delta} \quad (1 \le i \le r) \tag{4.1}$$

and

$$(k_i - k_j, p-1) < p^{1-\delta} \quad (1 \le i \ne j < r), \tag{4.2}$$

with $\delta > 0$ *arbitrary and fixed. Then*

$$\Big| \sum_{x=1}^{p} e_p\big(f(x)\big) \Big| < C p^{1-\delta'},$$

where $\delta' = \delta'(r, \delta) > 0$.

More generally, there is the following variant for incomplete sums involving exponential functions.

Theorem 4.4. *Assume* $\theta_1, \dots, \theta_r \in \mathbb{F}_p^*$ *satisfying*

$$\mathcal{O}_p(\theta_i) > p^{\delta} \quad (1 \le i \le r)$$

and

$$\mathcal{O}_p(\theta_i \theta_j^{-1}) > p^{\delta} \quad (1 \le i < j \le r).$$

Let $t > p^{\delta}$. *Then*

$$\max_{(a_i, p)=1} \Big| \sum_{s=1}^{t} e_p(a_1 \theta_1^s + \cdots + a_r \theta_r^s) \Big| < t p^{-\delta'}$$

with $\delta' = \delta'_r(\delta) > 0$.

Remarks. (1) A condition on the difference of the exponents such as (4.2) needs to be imposed and assuming (4.1) only is not sufficient. An example is the binomial sum

$$S = \sum_{x=0}^{p-1} e_p(x^{(p+1)/2} + x)$$

for which $|S| \sim p$; see [31], [32].

(2) The sum-product theorem underlying Theorem 4.2 is a certain extension of Theorem 1.1 to subsets of Cartesian squares $\mathbb{F}_p \times \mathbb{F}_p$; the statement here is necessarily more complicated.

Replacing condition (4.2) by a much weaker assumption, it turns out to be still possible to state a result on the solvability of certain congruences (short of getting an exponential sum bound).

Theorem 4.5. *Given $r \in \mathbb{Z}_+$, $r \geq 2$ and $\delta > 0$, there is a constant B and $\varepsilon > 0$ such that if $k_1, \dots, k_r$ satisfy* (4.1) *and moreover*

$$(k_i - k_j, p - 1) > B \quad \textit{for } i \neq j, \tag{4.3}$$

the following holds. Let $(a_i, p) = 1$ and $\ell_i \in \mathbb{Z} (1 \leq i \leq r)$. Then the system of congruences

$$a_i x^{k_i} \equiv \ell_i + y_i \pmod p \quad (1 \leq i \leq r)$$

has a solution in $1 \leq x < p$ and $y_i \in [0, p^{1-\varepsilon}] \cap \mathbb{Z}$.

See [14]. The proof combines Theorem 4.3 and the geometry of Fermat varieties.

Finally, we state a more abstract result on uniform distribution in finite commutative rings. Its original motivation was the study of ideal quotients in algebraic number fields.

Let R be a finite commutative ring with unit and assume $|R| = q$, where q has no small prime divisors (hence Theorem 4.6 below does not cover Theorem 4.1). Denote R^* the group of invertible elements of R. The following trichotomy holds, [10].

Theorem 4.6. *Let $H < R^*$ and $|H| > q^\delta$, where δ is arbitrarily fixed. For all $\varepsilon > 0$, there is $\varepsilon' = \varepsilon'(\varepsilon) \to 0$, as $\varepsilon \to 0$, such that one of the following alternatives holds:*

(i) *We have*

$$\max_{\chi \neq \chi_0} \Big| \sum_{x \in H} \chi(x) \Big| < |H|^{1-\varepsilon},$$

where χ refers to the additive characters of R.

(ii) *There is a nontrivial ideal I in R with*

$$|H \cap (1 + I)| > |H|^{1-\varepsilon'}.$$

(iii) *There is a nontrivial subring R_1 of R such that*

$$|H \cap R_1| > |H|^{1-\varepsilon'}.$$

5. Some applications

In this section, we list a few topics where the previous estimates turn out to be relevant. We do not go into any detail or state the results. The reader may also wish to consult the bibliography of the references cited here for more background. Many of the items discussed below may be found in [50].

5.1. Cryptography and pseudo-randomness. Our first application concerns the distribution of Diffie–Hellman triples; see [6]. Let p be a prime and $\theta \in \mathbb{Z}_p^*$. We consider the distribution in $[0, 1]^3$ of the triples

$$\left(\left\{\frac{\theta^x}{p}\right\}, \left\{\frac{\theta^y}{p}\right\}, \left\{\frac{\theta^{xy}}{p}\right\}\right)_{1\leq x,y\leq t},$$

where $(p, \theta) = 1$ and $\mathcal{O}_p(\theta) = t > p^\delta$. The results obtained in [6] complement those of [30, 1] and permit to establish some of these unconditionally.

A second application concerns the uniform distribution of power generators; see [37], [38], [7]. We consider a modulus of the form $q = p\ell$, where $p \neq \ell$, $p \sim \ell$ prime (called a Blum integer). Take $e \in \mathbb{Z}_q^*$, with $\big(e, (p-1)(\ell-1)\big) = 1$. We then define an RSA generator $(u_0, u_1, \ldots)$ by $u_0 = \theta \in \mathbb{Z}_q^*$ and inductively by $u_{n+1} = u_n^e$. Such sequences were proposed as deterministic models for random sequences. Theorem 4.4 turns out to be what is precisely needed to prove their uniform and joint uniform distribution properties; see also the references cited in [7].

Based on Theorem 4.4, one may establish nontrivial bounds on certain other exponential sums arising in this context, for instance sums of the form

$$\sum_{0\leq x<p} e_p(ag^x + bg^{x^2}).$$

The original motivation for Theorem 4.5 was the Goresky–Klapper conjecture [42] in coding theory (see also [18]).

5.2. Number fields. An issue discussed in [50], [17], [10] concerns the minimum norm representatives in residue classes and the Euclidean division algorithm in algebraic number fields (Egami's problem).

5.3. Coding theory. The Odlyzko–Stanley enumeration problem (see [50], [6]).

5.4. Hyperelliptic curves. Kodama's problem on supersingularity of the modulo p reduction, see [50], [6].

5.5. Quantum unique ergodicity (QUE). The ergodicity of the Hanney–Berry quantum cat map and refinements of the work of Kulberg and Rudnick on the QUE problem, see [43], [51], [12].

6. Exponential sums over general sets and the theory of extractors in computer science

Assume p a prime number. The following sharp result, obtained in [13], makes the link between Theorems 1.1 and 2.1.

Theorem 6.1. *Given $\varepsilon > 0$, there is $\delta = \delta(\varepsilon)$ such that for arbitrary sets $A_1, \dots, A_k \subset \mathbb{F}_p$ satisfying $|A_j| > p^\varepsilon$ for $1 \le j \le k$ and $|A_1| \cdots |A_k| > p^{1+\varepsilon}$,*

$$\max_{a \in \mathbb{F}_p^*} \Big| \sum_{x_1 \in A_1} \cdots \sum_{x_k \in A_k} e_p(a x_1 \dots x_k) \Big| < p^{-\delta} |A_1| \dots |A_k|. \tag{6.1}$$

Observe that when $k = 2$ the statement is elementary and well known. Indeed one has

$$\max_{a \in \mathbb{F}_p^*} \Big| \sum_{x \in A} \sum_{y \in B} e_p(axy) \Big| \le (p|A|\,|B|)^{1/2}. \tag{6.2}$$

It turns out that the inequality (6.1) is of interest to certain issues in theoretical computer science such as explicit construction of extractors (see [2], [3]). In this language, inequality (6.1) provides a k-source extractor at entropy ratio $\varepsilon > \frac{1}{k}$; (6.2) yields a 2-source extractor at ratio $\varepsilon > 1/2$.

The problem of providing explicit 2-source extractors below $1/2$-entropy ratio assignments has been open for some time. Some progress was made in [9] where the $1/2$-barrier was broken. An example is given by the following.

Theorem 6.2 ([9]). *There is $\gamma > 0$ such that if $A, B \subset \mathbb{F}_p$ and $|A| \sim |B| \sim p^{1/2}$, then*

$$\max_{a,b \in \mathbb{F}_p^*} \Big| \sum_{x \in A} \sum_{y \in B} e_p(axy + bx^2y^2) \Big| < p^{1-\gamma}. \tag{6.3}$$

The 2-source question for arbitrary entropy ratio remains open. It is reasonable to believe that in the following statement the assumption $|A| \sim |B| \sim p^{1/2}$ could be weakened to $|A|, |B| > p^\varepsilon$.

Theorem 6.3 ([9]). *There is $\gamma > 0$ such that if $A, B \subset \mathbb{F}_p$ and $|A| \sim |B| \sim p^{1/2}$, then, assuming $g \in \mathbb{F}_p^*$ a generator, we have*

$$\max_{a,b \in \mathbb{F}_p^*} \Big| \sum_{x \in A} \sum_{y \in B} e_p(axy + bg^{x+y}) \Big| < p^{1-\gamma}. \tag{6.4}$$

7. $\mathrm{SL}_2(p)$ results and generalizations

It turned out that sum-product theorems in finite fields lead to product theorems in semi-simple Lie groups. A first breakthrough result in this direction was obtained by H. A. Helfgott [46].

Theorem 7.1 ([46]). *Let $A \subset \mathrm{SL}_2(\mathbb{F}_p)$, $|A| < p^{3-\delta}$ and assume A is not contained in any proper subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$. Then*

$$|A \cdot A \cdot A| > c|A|^{1+\varepsilon},$$

with $c, \varepsilon > 0$ only depending on δ.

Remark. Denote $A_{(s)} = (A \cup A^{-1}) \cdots (A \cup A^{-1})$ the s-fold product set. It can be shown that if

$$|A \cdot A \cdot A| < |A|^{1+\varepsilon},$$

and

$$|A_{(s)}| < |A|^{1+c(s)\varepsilon}$$

for all given s.

Theorem 7.1 is a set-theoretical statement. It permits us to deduce new convolution inequalities, following a similar procedure as in the deduction of Theorem 2.1 from Theorem 1.1. A key tool is provided by the Balog–Szemeredi–Gowers lemma and its non Abelian extensions, see [61].

The following results on expander graphs in $\mathrm{SL}_2(p)$ are due to A. Gamburd and the author; see [20].

Theorem 7.2 ([20]). *Let S be a symmetric generating subset of $\mathrm{SL}_2(\mathbb{F}_p)$ satisfying the girth condition*

$$\mathit{girth}\big(\mathcal{G}(\mathrm{SL}_2(p), S)\big) > \rho \log p,$$

where $\rho > 0$ is an arbitrary fixed constant. Then the expansion coefficient $c(\mathcal{G})$ satisfies

$$c(\mathcal{G}) > c(\rho) > 0.$$

Example. Fix $S \subset \mathrm{SL}_2(\mathbb{Z})$ generating a free group.

Problem (A. Lubotzky). Is there for given $k \in \mathbb{Z}_+$ a constant $\rho > 0$ such that

$$c\big(\mathcal{G}(\mathrm{SL}_2(p), S)\big) > \rho$$

whenever $|S| = k$ and S generates $\mathrm{SL}_2(p)$?

The next two statements are easy consequences of Theorem 7.2. The first hints towards a positive answer to previous problem.

Corollary 7.3. *For any $k \geq 2$ random Cayley graphs of $\mathrm{SL}_2(p)$ on k generators are expanders.*

Corollary 7.4. *Let S be a subset of $\mathrm{SL}_2(\mathbb{Z})$. Then $\mathcal{G}(\mathrm{SL}_2(\mathbb{F}_p), S_p)$ form a family of expanders if and only if $\langle S \rangle$ is non-elementary, i.e. the limit set of $\langle S \rangle$ consists of more than two points, or equivalently, $\langle S \rangle$ does not contain a solvable subgroup of finite index.*

Remark. For subgroups of $\mathrm{SL}_2(\mathbb{Z})$ of finite index, the result is due to A. Selberg [58]. An extension of Selberg's theorem for "big" subgroups of $\mathrm{SL}_2(\mathbb{Z})$, more precisely assuming the dimension of the limit set larger than $5/6$ has been obtained by A. Gamburd [40].

Theorem 7.2 was generalized in [26] to arbitrary square free modulus, with uniform estimates; see also [25]. This extension uses essentially the sum-product theory in $\mathbb{Z}/q\mathbb{Z}$.

Theorem 7.5 ([26]). *Assume $S \subset \mathrm{SL}_2(\mathbb{Z})$ generates a free subgroup. There is $q_0 = q_0(S) \in \mathbb{Z}$ and $c = c(S) > 0$ such that*

$$c\big(\mathcal{G}(\mathrm{SL}_2(q), S)\big) > c$$

for q squarefree and $(q, q_0) = 1$.

It turns out that Theorem 7.5 is exactly the tool needed to carry out prime and pseudo-prime sieving in certain "thin varieties" defined from subgroups of $\mathrm{SL}_2(\mathbb{Z})$ which may have arbitrary small dimension.

A few samples of results obtained along these lines (the reader should consult [25] and [26] for the full account).

Theorem 7.6 ([26]). *Assume Λ is a non-elementary subgroup of $\mathrm{SL}_2(\mathbb{Z}) \subset \mathbb{Z}^4$. Further, let $f \in \mathbb{Q}[x_1, x_2, x_3, x_4]$ taking integer values on Λ and not a multiple of $g(x_1, x_2, x_3, x_4) = x_1x_4 - x_2x_3 - 1$. There is $r = r(\Lambda) \in \mathbb{Z}_+$ such that*

$$\{x \in \Lambda \mid f(x) \textit{ has at most } r \textit{ prime factors}\}$$

is Zariski dense in SL_2.

Define $r(z) =$ number of prime factors of $z \in \mathbb{Z}\backslash\{0\}$.

Theorem 7.7 ([24]). *Under the assumption of Theorem 7.6, there is $C = C(\Lambda) \in \mathbb{Z}_+$ such that for $N \to \infty$*

$$|\{x \in \Lambda \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 \le N^2 \text{ and } r(x_1x_2x_3x_4) < C\}| \gtrsim \frac{N^{2\delta}}{(\log N)^4},$$

where $\delta = \delta\big(L(\Lambda)\big)$ is the dimension of the limit set of Λ.

Note that there is no other assumption on $\delta\big(L(\Lambda)\big)$ than positivity in previous statements.

Selberg's combinatorial sieve may be applied with balls defined by the word-metric (as in Theorem 7.6)

$$N_\Lambda(z) = \{x \in \Lambda \mid \ell(x) \le z\}$$

or with the Archimedean metric (Theorem 7.7)

$$N'_\Lambda(z) = \Big\{x \in \Lambda \mid \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} \le z\Big\}.$$

In the Archimedean setting, a distinction is made between the cases $\delta > \frac{1}{2}$ and $\delta \le \frac{1}{2}$. If $\delta > \frac{1}{2}$, there is a L^2-spectral theory for $\mathbb{H}/\Lambda$ and we rely on the Lax-Phillips asymptotic for $N'_\Lambda(z)$, [53], together with an extension of Selberg's spectral gap theorem based on Theorem 7.5. For $\delta \le \frac{1}{2}$, we follow Lalley's symbolic dynamics approach [54] based on renewal theorems and the meromorphic extension of Ruelle's transfer operator.

In [26], Theorem 7.6 and related statements are casted in a broader context of "non-Abelian" Dirichlet theorems. At this point they remain conjectural and only their "pseudo-prime version" *cf.* Theorem 7.6, has been established.

Returning to Corollary 7.4, Lubotzky and Weiss made the conjecture that if S is a finite subset of $\mathrm{SL}_d(\mathbb{Z})$ generating a Zariski dense subgroup G of SL_d, then the family of Cayley graphs

$$\mathcal{G}\big(\mathrm{SL}_d(\mathbb{Z}/q\mathbb{Z}), \pi_q(S)\big)$$

forms an expander family, provided we take q to satisfy $(q, q_0) = 1$, where $q_0 \in \mathbb{Z}$ depends on Λ. Recall indeed that according to the strong approximation property, there is $q_0 = q_0(\Lambda) \in \mathbb{Z}$ such that

$$\pi_q(G) = \mathrm{SL}_d(\mathbb{Z}/q\mathbb{Z}) \text{ if } (q, q_0) = 1$$

(connectedness of the graph $\mathcal{G}$).

Thus Theorem 7.5 establishes this conjecture for $\mathrm{SL}_2(q)$, restricting q to square-free moduli. At the other end, the family $\mathrm{SL}_2(p^n)$ (p prime, $n \in \mathbb{Z}_+$) was treated in [22], using a different approach combining sum-product theory with the Solovay–Kitaev algorithm. This approach generalizes to $\mathrm{SL}_d(p^n)$, $d \ge 2$ (p fixed and

$n \to \infty$), see [23]. Helfgott [47] recently extended his Theorem 7.1 to $\mathrm{SL}_3(p)$. Combined with results from [23], this enables us to add the family $\mathrm{SL}_3(p)$ (p-prime) to the list satisfying the Lubotzky–Weiss conjecture.

8. SU(2)

It turns out that the true counterpart of Theorem 1.1 for subsets of $\mathbb{R}$ is the "discretized ring conjecture" made in [48] and proven in [4], rather than statements of [35] type.

Theorem 8.1 ([4], [21]). *For all $0 < \sigma < 1$ and $\kappa > 0$, there is $\varepsilon = \varepsilon(\sigma, \kappa) > 0$ such that if $A \subset [0, 1]$ is a union of δ-intervals, where $\delta > 0$ is small, satisfying*

$$|A| = \delta^{1-\sigma},$$

and for all $\delta < \rho < \delta^{\varepsilon}$,

$$\max_t |A \cap B(t, \rho)| < \rho^{\kappa} |A|, \tag{8.1}$$

then

$$|A + A| + |A \cdot A| > \delta^{1-\sigma-\varepsilon}.$$

Remark. A "non-concentration" assumption such as (8.1) is easily seen to be necessary for such a statement to hold.

While the initial motivation of Theorem 8.1 was to progress on the dimension conjecture for Kakeya sets in $\mathbb{R}^3$ and other problems discussed in [48], it appeared that Theorem 8.1 as a counterpart of Theorems 1.1 and 1.2, permits us to carry out [46], [20] in the Archimedian setting of SU(2).

First we need to recall the non-Abelian diophantine condition from [41] that will play the role of the "large girth" assumption.

Definition 8.2. For $k \geq 2$, we say that the set of elements $g_1, \dots, g_k \in \mathrm{SU}(2)$ are *diophantine* if there is $D > 0$ such that for any $m \geq 1$ and any word R_m in $g_1, \dots, g_k$ of length m, and such that $R_m \neq \pm e$, we have

$$\|R_m \pm e\| \geq D^{-m}.$$

Example ([41]). Take $g_1, \dots, g_k \in \mathrm{SU}(2) \cap M_2(\bar{\mathbb{Q}})$ generating a free group.

We may now state the main result from [21].

Theorem 8.3 ([21]). *Let* $\{g_1, \dots, g_k\}$ *be a set of elements in* $\mathrm{SU}(2)$ *generating a free group and satisfying a diophantine property. Then*

$$z_{g_1,\dots,g_k} = g_1 + g_1^{-1} + \cdots + g_k + g_k^{-1}$$

has a spectral gap, with lower bound depending on k *and* D *only.*

Corollary 8.4. *If* $g_1, \dots, g_k \in \mathrm{SU}(2) \cap M_2(\bar{\mathbb{Q}})$ *generate a free group, then* $z_{g_1,\dots,g_k}$ *has a spectral gap.*

There are several applications, including to geometry and quantum computation (see [21]). Here are a few:

(1) A purely analytical solution to the Banach–Ruziewiez problem on finitely additive invariant measures on S_2 (see [21], [60]).

(2) Proof of the exponential mixing rate in the Conway–Radin quaquaversal tiling of $\mathbb{R}^3$.

(3) ε-approximation by words of length $\sim \log \frac{1}{\varepsilon}$ in fault tolerant gates in quantum computation (improving on the Solovay–Kitaev algorithm).

(4) Construction of explicit "dimension expanders".

9. Actions of linear groups on tori [19]

Combining Theorem 8.1 and classical theory of random matrix products enables us to answer affirmatively certain questions raised by Furstenberg and Guivarch on stationary measures and equidistribution for toral actions of linear groups. Let $S = \{g_1, \dots, g_k\}$ be elements in $\mathrm{SL}_d(\mathbb{Z})$ generalizing a Zariski dense subgroup of SL_d and denote

$$\nu = \frac{1}{|S|} \sum_{g \in S} \delta_g.$$

We consider the action of $\mathrm{SL}_d(\mathbb{Z})$ on the d-dimensional torus $\prod^d$.

Theorem 9.1. *Let* $\xi \in \prod^d$ *be irrational. Then for* $\nu^{(\infty)}$*-almost every sequence* $(x_1, x_2, \dots)$*, the sequences*

$$x_r x_{r-1} \cdots x_1 \xi \text{ and } x_1 x_2 \dots x_r \xi \ (r \to \infty)$$

are equidistributed in $\prod^d$.

Application of next statement in the situation $\langle S \rangle = \mathrm{SL}_d(\mathbb{Z})$ solves affirmatively Furstenberg's "stiffness problem" [39].

Corollary 9.2. *Let μ be a probability measure on $\prod^d$ which is ν-stationary, i.e.*

$$\mu = \mu * \nu = \sum_g \nu(g) g_*[\mu].$$

Then μ is a combination of Haar measure and an atomic measure supported by rational points and μ is $\langle \nu \rangle$-invariant.

We also retrieve the description of $\langle \nu \rangle$-invariant compact subsets K of $\prod^d$, due to Starkov, Muchnik, Guivarch.

Corollary 9.3. *Let ν be as above and $K \subset \prod^d$ a $\langle \nu \rangle$-invariant compact set. Then K is finite or $K = \prod^d$.*

Guivarch's equidistribution question may in fact be addressed in the following quantitative form (which is the main result from [19]).

Theorem 9.4. *Given ν as above, there are constants $c > 0$ and $C < \infty$, such that if $\xi \in \prod^d \backslash\{0\}$ and $b \in \mathbb{Z}^d \backslash \{0\}$, $\|b\| < e^{cn}$, then*

$$(*) = \Big| \sum_g \nu^{(n)}(g) e^{2\pi i \langle b, g\xi \rangle} \Big| < e^{-cn}$$

unless $\|\xi - \frac{a}{q}\| < e^{-cn}$ with $q < e^{\frac{c}{4}n}$, in which case

$$(*) < \frac{|b|^C}{q^c}.$$

A key step in proving Theorem 9.4 is to exhibit certain well-distributed sets in $\mathbb{Z}^d$ of large Fourier coefficients. This is the place where Theorem 8.1 enters. The most elegant approach is to rely on a consequence of Theorem 8.1, which has the form of a Marstrand-type projection theorem. For simplicity, we formulate the result in terms of Hausdorff dimension. The application above requires however a box-dimension version, which is slightly more technical to state.

Theorem 9.5. *Given $d \in \mathbb{Z}, d \geq 2$ and $0 < \rho < d$, $\varepsilon > 0$ there is $\delta > 0$ such that the following holds. Let $A \subset [0,1]^d$ with H-dim $A > \rho$. Let η be a probability measure on S^{d-1} of dimension at least ε. Denote P_ξ the orthogonal projection on $\xi \in S^{d-1}$. Then the set $\{\xi \in S^{d-1} \mid \dim P_\xi A < \frac{\rho}{d} + \delta\}$ has zero η-measure.*

In our application, η will be the ν-stationary measure on $P_{d-1}(\mathbb{R})$ given by the theory of random matrix products.

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5ECM Amsterdam 2008

Large random planar maps and their scaling limits

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Abstract. We discuss scaling limits of random planar maps chosen uniformly at random in a certain class. This leads to a universal limiting space called the Brownian map, which is viewed as a random compact metric space. The Brownian map can be obtained as a quotient of the continuous random tree called the CRT, for an equivalence relation which is defined in terms of Brownian labels assigned to the vertices of the CRT. We discuss the known properties of the Brownian map. In particular, we give a complete description of the geodesics starting from the distinguished point called the root. We also discuss applications to various properties of large random planar maps.

Mathematics Subject Classification (2000). Primary 60C05, 05C80; Secondary 60F17, 05C12, 05C10.

Keywords. Random planar map, scaling limit, Brownian map, continuum random tree, geodesic, Gromov–Hausdorff convergence.

1. Introduction

The main purpose of the present article is to survey recent developments about scaling limits of large planar maps chosen uniformly at random in a suitable class. Recall that planar maps are just (finite) graphs embedded in the plane. A planar map is thus the kind of object one would draw on a sheet of a paper if asked to give an example of a graph.

To explain what a scaling limit is, consider a combinatorial object, such as a path, a tree or a graph, and suppose that it is chosen at random in the class of all objects of size n. Often the resulting random object can be rescaled as $n \to \infty$ in such a way that it becomes close to a continuous model. For instance, one may consider all discrete paths with length n starting from the origin on the integer lattice $\mathbb{Z}^d$. If one chooses uniformly at random a path in this collection, then modulo a suitable rescaling (essentially by the factor $1/\sqrt{n}$) it will become close to a continuous Brownian path. More precisely, for any set A in the path space, satisfying certain regularity assumptions, the probability that the rescaled discrete path of length n belongs to A will converge to the probability that the Brownian path belongs to A as $n \to \infty$.

Studying such scaling limits is all the more interesting as they are universal, meaning that the same continuous model corresponds to the limit of many different classes of discrete objects. A fundamental example of this universality property is Brownian motion, which is well known to be the scaling limit of many different classes of random paths. The study of scaling limits is motivated by at least two important reasons:

- Often the continuous model is of interest in its own. For instance, Brownian motion has numerous applications, independently of the fact that it is the scaling limit of random walks.
- Knowing the continuous model gives insight into the properties of the large discrete objects. Lots of interesting distributional asymptotics for long random paths can be derived from explicit calculations on Brownian motion.

In the present work, we discuss scaling limits first for random trees and then for random planar maps. The reason for considering random trees first comes from our specific approach, which involves bijections between planar maps and certain classes of decorated trees. The scaling limits of trees and maps both lead to remarkable probabilistic objects. In the case of trees, the scaling limit is the CRT (Continuum Random Tree), which has been introduced and studied by Aldous [A1], [A2] in the early nineties. The scaling limit of random planar maps, which we call the Brownian map, is then described as the quotient of the CRT for a certain (random) equivalence relation. The Brownian map may be thought of as the relevant probabilistic model for a random surface in the same sense as Brownian motion is the right model for a purely random continuous path. Indeed, one conjectures that the Brownian map appears as the continuous limit of many classes of planar maps, which are natural discretizations of surfaces.

Let us recall some basic definitions. A planar map is a proper embedding of a finite connected graph in the two-dimensional sphere $\mathbb{S}^2$. Loops and multiple edges are a priori allowed. The faces of the map are the connected components of the complement of the union of edges. A planar map is rooted if it has a distinguished oriented edge called the root edge, whose origin is called the root vertex. In what follows, we consider only rooted planar maps, even if this is not mentioned explicitly. Rooting maps avoids certain technical difficulties and is believed to have no influence on the problems we will be addressing.

Two rooted planar maps are said to be equivalent if the second one is the image of the first one under an orientation-preserving homeomorphism of the sphere, which also preserves the root edges. Two equivalent rooted planar maps will always be identified.

Given an integer $p \geq 3$, a p-angulation is a planar map where each face has degree p, that is p adjacent edges. One should count edge sides, so that if an edge lies entirely inside a face it is counted twice: For instance, the face in the upper right corner of Figure 1 below has degree 4, although it seems to be adjacent to only 3 edges. We denote by $\mathbb{M}_n^p$ the set of all rooted p-angulations with n faces. Thanks to the preceding identification, the set $\mathbb{M}_n^p$ is finite. A 3-angulation is called a triangulation, and a 4-angulation is called a quadrangulation. Figure 1 below shows a quadrangulation with 7 faces.

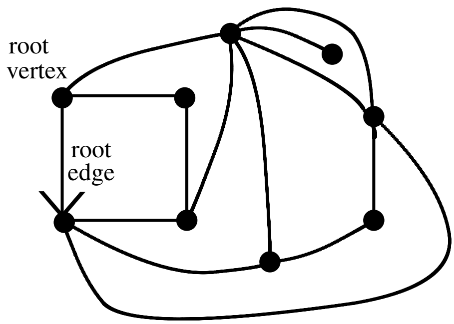

Figure 1

Consider a planar map M. Let $V(M)$ denote the vertex set of M. A path in M with length k is a finite sequence $(a_0, a_1, \ldots, a_k)$ in $V(M)$ such that a_i and a_{i-1} are connected by an edge of the map, for every $i \in \{1, \ldots, k\}$. The graph distance $d_{\mathrm{gr}}(a, a')$ between two vertices a and a' is the minimal k such that there exists a path $\gamma = (a_0, a_1, \ldots, a_k)$ with $a_0 = a$ and $a_k = a'$. A path $\gamma = (a_0, a_1, \ldots, a_k)$ is called a discrete geodesic (from a_0 to a_k) if $k = d_{\mathrm{gr}}(a_0, a_k)$. The set $V(M)$ equipped with the metric d_{gr} is a (finite) metric space. Clearly, the map M is not determined by the metric space $(V(M), d_{\mathrm{gr}})$. Nonetheless, much information is contained in this metric space, and in what follows we will concentrate on the study of metric properties of planar maps.

Fix an integer $p \geq 3$ and, for every integer $n \geq 2$, let M_n be a random planar map chosen uniformly at random in the space $\mathbb{M}_n^p$. Following our initial discussion of scaling limits, one would like to prove that for a suitable choice of the positive constant α, the rescaled random metric spaces

$$(V(M_n), n^{-\alpha} d_{\mathrm{gr}}) \tag{1}$$

converge in some appropriate sense towards a (non-degenerate) limiting random compact metric space. Moreover the limiting space is believed to be independent of p, up to trivial scaling factors. This corresponds to the universality property mentioned above.

The rescaling factor $n^{-\alpha}$ in (1) is needed if we want to get a “continuous” limit and to stay within the framework of compact metric spaces. It also makes sense to study the limit of the spaces $(V(M_n), d_{\mathrm{gr}})$ without rescaling, and this gives rise to infinite random graphs (see Angel [An] and Angel and Schramm [AS] for the case of infinite triangulations, and Chassaing and Durhuus [CS] and Krikun [Kr] for infinite quadrangulations of the plane).

As stated above, the problem of the scaling limit for planar maps requires an adequate notion of the convergence of a sequence of compact metric spaces. Such a notion is provided by the Gromov–Hausdorff distance (Gromov [Gr], Burago, Burago and Ivanov [BBI]). Let (E_1, d_1) and (E_2, d_2) be two compact metric spaces. The Gromov–Hausdorff distance between (E_1, d_1) and (E_2, d_2) is

$$d_{\mathrm{GH}}(E_1, E_2) = \inf \big(d_{\mathrm{Haus}}(\varphi_1(E_1), \varphi_2(E_2))\big),$$

where the infimum is over all isometric embeddings $\varphi_1 \colon E_1 \to E$ and $\varphi_2 \colon E_2 \to E$ of E_1 and E_2 into the same metric space (E, d), and d_{Haus} stands for the usual Hausdorff distance between compact subsets of E. If $\mathbb{K}$ denotes the space of all isometry classes of compact metric spaces, then d_{GH} is a distance on $\mathbb{K}$, and moreover the metric space $(\mathbb{K}, d_{\mathrm{GH}})$ is Polish, that is separable and complete (see Chapter 7 of Burago, Burago and Ivanov [BBI] for a thorough discussion of the Gromov–Hausdorff distance).

Thanks to the previous discussion, it makes sense to study the convergence in distribution of the random metric spaces (1) as random variables with values in the Polish space $(\mathbb{K}, d_{\mathrm{GH}})$. This problem was stated in this form for triangulations by Schramm [Sc]. The general idea of finding a continuous limit for large random planar maps had appeared earlier, especially in the pioneering paper of Chassaing and Schaeffer [CS]. The latter paper proves a limit theorem showing that the radius, or maximal distance from the root, of a quadrangulation with n faces chosen at random, rescaled by the factor $n^{-1/4}$, converges in distribution towards a nondegenerate limit (see Corollary 3.4 below). This gives evidence of the fact that the proper value of the constant α in (1) should be $\alpha = 1/4$.

For reasons that will be explained below, it turns out to be easier to handle bipartite planar maps: A planar map is bipartite if and only if all its faces have even degree. In the remaining part of this introduction, we thus restrict our attention to the case when p is even.

In order to explain our main result about scaling limits of planar maps, we need to introduce some notation. Aldous' Continuum Random Tree (the CRT), viewed as a random compact metric space, is denoted by $(\mathcal{T}_{\boldsymbol{e}}, d_{\boldsymbol{e}})$. Its root ρ is a distinguished point of $\mathcal{T}_{\boldsymbol{e}}$. The reason for the notation $\mathcal{T}_{\boldsymbol{e}}$ comes from the fact that the CRT can be coded by a normalized Brownian excursion $\boldsymbol{e}$, as will be explained in Section 3 below. This coding makes it possible to introduce a lexicographical order on the tree $\mathcal{T}_{\boldsymbol{e}}$: If $a, b \in \mathcal{T}_{\boldsymbol{e}}$, one may consider the "lexicographical" interval $[a, b]$ which is informally defined as the subset of $\mathcal{T}_{\boldsymbol{e}}$ consisting of all points that are visited when going from a to b around the tree in clockwise order (see Section 4 for more rigorous definitions). Next, conditionally given $(\mathcal{T}_{\boldsymbol{e}}, d_{\boldsymbol{e}})$, we consider a centered Gaussian process $(Z_a)_{a\in\mathcal{T}_{\boldsymbol{e}}}$ such that $Z_\rho = 0$ and

$$E[(Z_a - Z_b)^2] = d_{\boldsymbol{e}}(a,b)$$

for every $a, b \in \mathcal{T}_{\boldsymbol{e}}$ (again this definition is slightly informal, as we consider a random process indexed by a random set – see Section 4 for a more rigorous presentation). The process Z should be understood as Brownian motion indexed by the tree $\mathcal{T}_{\boldsymbol{e}}$: Z_a is a "label" assigned to vertex a, and this label evolves as linear Brownian motion when varying a along a line segment of the tree. Finally, we define a random equivalence relation $\approx$ on $\mathcal{T}_{\boldsymbol{e}}$ by setting

$$a \approx b \quad \text{iff} \quad Z_a = Z_b = \min_{c\in[a,b]} Z_c \ \text{ or } \ Z_a = Z_b = \min_{c\in[b,a]} Z_c.$$

Then Theorem 4.1 below, taken from [L2], states that, from any sequence of values of n converging to $+\infty$, we can extract a subsequence along which we have the convergence in distribution

$$(V(M_n), n^{-1/4} d_{\mathrm{gr}}) \longrightarrow (\mathcal{T}_{\boldsymbol{e}} / \approx, D), \tag{2}$$

where D is a metric on the quotient $\mathcal{T}_{\boldsymbol{e}} / \approx$, which induces the quotient topology on that space. The limiting space $(\mathcal{T}_{\boldsymbol{e}} / \approx, D)$ is called the Brownian map (to be more precise, we should say that we use the name Brownian map for any of the limiting random metric spaces that can arise in (2) when we vary p and the subsequence). This terminology comes from Marckert and Mokkadem [MMo], who discussed limits of rescaled random quadrangulations, however in a different sense than the Gromov–Hausdorff convergence. Our terminology slightly differs from that in [MMo], where the Brownian map is defined as the space $\mathcal{T}_{\boldsymbol{e}} / \approx$ with a specified metric which may or may not coincide with D.

The need for a subsequence in (2) comes from the fact that the limiting random metric D has not been fully characterized, and so there might be different metrics D corresponding to different subsequences. Still one believes that it should not be necessary to take a subsequence, and that the limiting metric space should be the same independently of p (even if p is odd), thus confirming the universality property of the Brownian map. The recent results of Marckert and Miermont [MMi], Miermont [Mi1] and Miermont and Weill [MW] strongly support this conjecture.

Even though the distribution of the Brownian map has not been fully characterized, many of its properties can be investigated in detail. In Section 5 below, we give two theorems showing on one hand that the Hausdorff dimension of the Brownian map is a.s. equal to 4, and on the other hand that the Brownian map is a.s. homeomorphic to the two-dimensional sphere. The last result is maybe not surprising since we started from graphs drawn on the sphere. Still it implies that typical large p-angulations will not have "small bottlenecks" (see Corollary 5.3 for a precise statement). In

Section 6 we present recent results taken from [L3] about the structure of geodesics in the Brownian map. Here again, we provide applications to properties of large discrete planar maps, in the spirit of the observations made at the beginning of this introduction.

One may ask why the scaling limit of random planar maps should be related to the CRT. This can be understood from the existence of bijections between the sets $\mathbb{M}_n^p$ and various classes of labeled trees. In the particular case of quadrangulations, such bijections were discovered by Cori and Vauquelin [CS] and then studied extensively by Schaeffer [S]. More recently, Bouttier, Di Francesco and Guitter [BDG] provided a nice simple extension of the Cori–Vauquelin–Schaeffer bijection to bipartite planar maps (see Section 2 below). This result partly explains why we restrict our attention to bipartite planar maps: The bijections in the general case seem more difficult to use for technical reasons (see however Miermont [Mi1]). The scaling limit of the discrete trees that arise in the bijections with planar maps turns out to be given by the CRT (see Section 3). Since in the discrete setting vertices of the map are in one-to one correspondence with vertices of the associated tree, it is not surprising that the Brownian map can be constructed from the CRT. However, the correct definition requires identifying certain pairs of points in the CRT, via the introduction of the equivalence relation $\approx$. This is so because, already in the discrete setting, certain pairs of vertices that are far away from each other in the tree can be very close in the associated map. The principal difficulty in the proof of (2) is in fact to determine precisely those pairs of points that need to be identified in the continuous limit.

To conclude this introduction, let us briefly comment on the motivations for studying planar maps and their scaling limits. Planar maps were first studied by Tutte [Tu] in connection with his work on the four color theorem, and since then they have been studied extensively in combinatorics. Planar maps also have algebraic and geometric applications: See the book of Lando and Zvonkin [LZ] for more on this matter. Because of their relations with Feynman diagrams, planar maps soon attracted the attention of specialists of theoretical physics. The pioneering papers by 't Hooft [tH] and Brézin, Itzykson, Parisi and Zuber [BIP] related enumeration problems for planar maps with asymptotics of matrix integrals. The interest for random planar maps in theoretical physics grew significantly when these combinatorial objects were interpreted as models of random surfaces, especially in the setting of the theory of quantum gravity (see in particular the book of Ambjørn, Durhuus and Jonsson [ADJ]). Bouttier's thesis [Bo] describes applications of planar maps to the statistical physics of random surfaces. The recent papers [BG1], [BG2], [BG3] by Bouttier and Guitter address questions closely related to those of the present work from the perspective of theoretical physics. From the probabilistic point of view, the Brownian map appears to be a fascinating model of a random fractal surface, even if its properties are still far from being completely understood.

2. Bijections between maps and trees

Throughout the remaining part of this work, we fix an integer $p \geq 2$ and we deal with the set $\mathbb{M}_n^{2p}$ of all rooted $2p$-angulations with n faces. We will present a bijection between $\mathbb{M}_n^{2p}$ and and a certain set of labeled trees.

By definition, a plane tree τ is a finite subset of the set

$$U = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

of all finite sequences of positive integers (including the empty sequence $\emptyset$), which satisfies three obvious conditions: First $\emptyset \in \tau$, then, for every $v = (u_1, \dots, u_k) \in \tau$ with $k \geq 1$, the sequence $(u_1, \dots, u_{k-1})$ (the "parent" of v) also belongs to τ, and finally for every $v = (u_1, \dots, u_k) \in \tau$ there exists an integer $k_v(\tau) \geq 0$ (the "number of children" of v) such that the vertex $vj := (u_1, \dots, u_k, j)$ belongs to τ if and only if $1 \leq j \leq k_v(\tau)$. The generation of $v = (u_1, \dots, u_k)$ is denoted by $|v| = k$.

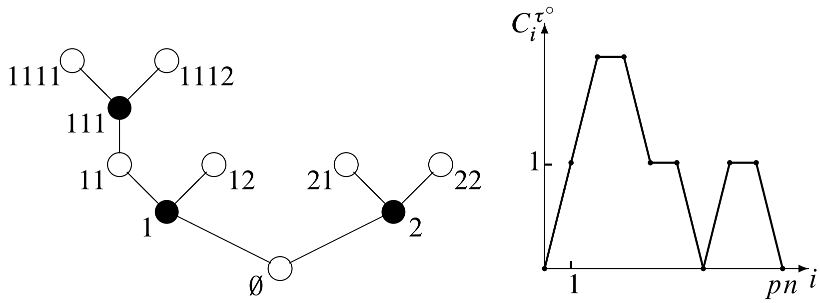

Figure 2. A 3-tree τ and the associated contour function C^{τ° of τ°.

A p-tree is a plane tree τ that satisfies the following additional property: For every $v \in \tau$ such that $|v|$ is odd, $k_v(\tau) = p - 1$.

If τ is a p-tree, vertices v of τ such that $|v|$ is even are called white vertices, and vertices v of τ such that $|v|$ is odd are called black vertices. We denote by τ° the set of all white vertices of τ and by $\tau^\bullet$ the set of all black vertices. See the left side of Figure 2 for an example of a 3-tree.

A labeled p-tree is a pair $\theta = (\tau, (\ell_v)_{v \in \tau^\circ})$ that consists of a p-tree τ and a collection of integer labels (taking values in $\mathbb{Z}$) assigned to the white vertices of τ, such that the following properties hold:

(a) $\ell_\emptyset = 1$.

(b) Let $v \in \tau^\bullet$, let $v_{(0)}$ be the parent of v and let $v_{(j)} = vj$, $1 \leq j \leq p-1$, be the children of v. Then for every $j \in \{0, 1, \dots, p-1\}$, $\ell_{v_{(j+1)}} \geq \ell_{v_{(j)}} - 1$, where by convention $v_{(p)} = v_{(0)}$.

A labeled p-tree is called a p-mobile if the labels satisfy the following additional condition:

(c) $\ell_v \geq 1$ for each $v \in \tau^\circ$.

The left side of Figure 3 gives an example of a p-mobile with $p = 3$. Condition (b) above means that if one lists the white vertices adjacent to a given black vertex in clockwise order, the labels of these vertices can decrease by at most one at each step.

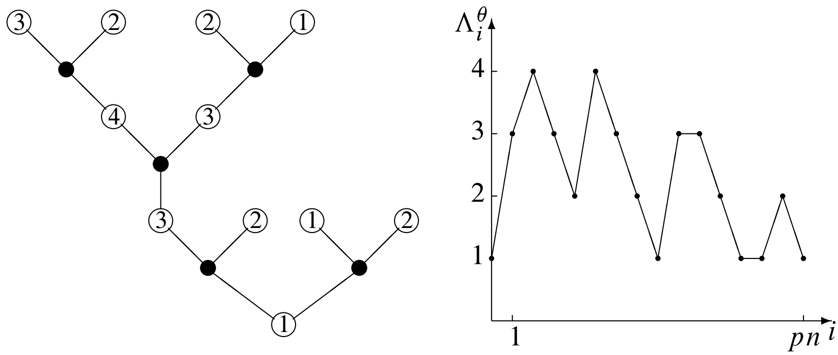

Figure 3. A 3-mobile θ with 5 black vertices and the associated spatial contour function.

Let τ be a p-tree with n black vertices and let $k = \#\tau - 1 = pn$. The depth-first search sequence of τ is the sequence $w_0, w_1, \dots, w_{2k}$ of vertices of τ which is obtained by induction as follows. First $w_0 = \emptyset$, and then for every $i \in \{0, \dots, 2k-1\}$, w_{i+1} is either the first child of w_i that has not yet appeared in the sequence $w_0, \dots, w_i$, or the parent of w_i if all children of w_i already appear in the sequence $w_0, \dots, w_i$. It is easy to verify that $w_{2k} = \emptyset$ and that all vertices of τ appear in the sequence $w_0, w_1, \dots, w_{2k}$ (of course some of them appear more than once).

Vertices w_i are white when i is even and black when i is odd. The contour sequence of τ° is by definition the sequence $v_0, \dots, v_k$ defined by $v_i = w_{2i}$ for every $i \in \{0, 1, \dots, k\}$.

Now let $\theta = (\tau, (\ell_v)_{v \in \tau^\circ})$ be a p-mobile with n black vertices. As previously, denote the contour sequence of τ° by $v_0, v_1, \dots, v_{pn}$. Suppose that the tree τ is drawn in the plane as pictured in Figure 4 and add an extra vertex ∂. We associate with θ a rooted $2p$-angulation M with n faces, whose set of vertices is

$$V(M) = \tau^\circ \cup \{\partial\}$$

and whose edges are obtained by the following device: For every $i \in \{0, 1, \dots, pn - 1\}$,

- if $\ell_{v_i} = 1$, draw an edge between v_i and ∂ ;

- if $\ell_{v_i} \geq 2$, draw an edge between v_i and v_j, where j is the first index in the sequence $i+1, i+2, \ldots, pn$ such that $\ell_{v_j} = \ell_{v_i} - 1$.

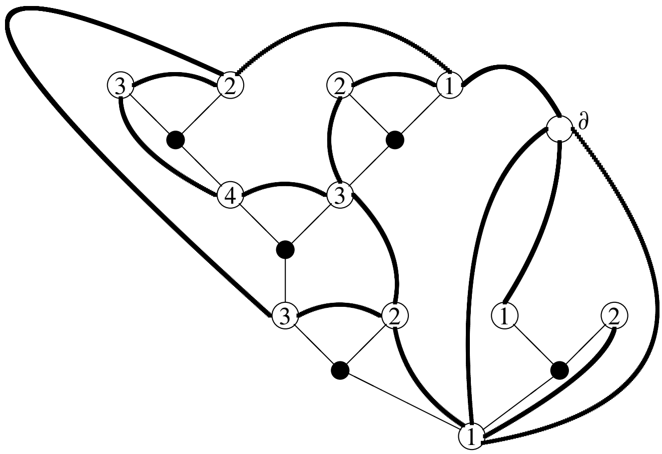

Figure 4. The Bouttier–Di Francesco–Guitter bijection: A rooted 3-mobile with 5 black vertices and the associated rooted 6-angulation with 5 faces. The root edge of the map is the rightmost edge on the figure and is oriented from the vertex ∂ to the vertex labeled 1.

Notice that $v_{pn} = v_0 = \emptyset$ and $\ell_\emptyset = 1$, and that condition (b) in the definition of a p-tree entails that $\ell_{v_{i+1}} \geq \ell_{v_i} - 1$ for every $i \in \{0, 1, \ldots, pn-1\}$. This ensures that whenever $\ell_{v_i} \geq 2$ there is at least one vertex among $v_{i+1}, v_{i+2}, \ldots, v_{pn}$ with label $\ell_{v_i} - 1$. The construction can be made in such a way that edges do not intersect, except possibly at their endpoints: For every vertex v, each index i such that $v_i = v$ corresponds to a "corner" of v, and the associated edge starts from this corner. We refer to Section 2 of Bouttier et al [BDG] for a more detailed description.

The resulting planar map M is a $2p$-angulation, which is rooted at the oriented edge between ∂ and $v_0 = \emptyset$, corresponding to $i = 0$ in the previous construction. Each black vertex of τ is associated with a face of the map M. See Figure 4 for the 6-angulation associated with the 3-mobile of Figure 3.

The preceding construction yields a bijection between the set $\mathbb{T}_n^p$ of all p-mobiles with n black vertices and the set $\mathbb{M}_n^{2p}$. This is the Bouttier–Di Francesco–Guitter bijection [BDG], called the BDG bijection in what follows.

Furthermore, this bijection enjoys the following remarkable property, which is crucial for our purposes: The graph distance in M between the root vertex ∂ and another vertex $v \in \tau^\circ$ is equal to ℓ_v. Hence knowing the labels in the tree θ already gives a lot of information about distances in the map M.

In view of our applications, it will be convenient to code a p-mobile, or more generally a labeled p-tree, by a pair a discrete functions. The *contour function* of τ°

(or of θ) is the discrete sequence $C_0^{\tau^\circ}, C_1^{\tau^\circ}, \dots, C_{pn}^{\tau^\circ}$ defined by

$$C_i^{\tau^\circ} = \frac{1}{2}\,|v_i| \quad \text{for every } 0 \le i \le pn.$$

See Figure 2 for an example with $p = n = 3$. It is easy to verify that the contour function determines τ°, which in turn determines the p-tree τ uniquely. We also introduce the *spatial contour function* of $\theta = (\tau, (\ell_v)_{v\in\tau^\circ})$, which is the discrete sequence $(\Lambda_0^\theta, \Lambda_1^\theta, \dots, \Lambda_{pn}^\theta)$ defined by

$$\Lambda_i^\theta = \ell_{v_i} \quad \text{for every } 0 \le i \le pn.$$

From property (b) of the labels and the definition of the contour sequence, it is clear that $\Lambda_{i+1}^\theta \ge \Lambda_i^\theta - 1$ for every $0 \le i \le pn - 1$ (cf. Figure 3). The pair $(C^{\tau^\circ}, \Lambda^\theta)$ determines the labeled p-tree θ uniquely.

3. Scaling limits of trees

3.1. Plane trees. Our goal is to study the scaling limits of the labeled trees that appeared in the bijections with maps. We will start with the simpler problem of obtaining the scaling limit of plane trees. We first need to recall the definition of an $\mathbb{R}$-tree.

A metric space (T, d) is an $\mathbb{R}$-tree if the following two properties hold for every $a, b \in T$.

(a) There is a unique isometric map $f_{a,b}$ from $[0, d(a,b)]$ into T such that $f_{a,b}(0) = a$ and $f_{a,b}(d(a,b)) = b$.

(b) If q is a continuous injective map from $[0,1]$ into T, such that $q(0) = a$ and $q(1) = b$, we have
$$q([0,1]) = f_{a,b}([0, d(a,b)]).$$

A rooted $\mathbb{R}$-tree is an $\mathbb{R}$-tree (T, d) with a distinguished vertex $\rho = \rho(T)$ called the root.

Informally, one should think of a (compact) $\mathbb{R}$-tree as a connected union of line segments in the plane with no loops. For any two points a and b in the tree, there is a unique arc going from a to b in the tree, which is isometric to a line segment.

The multiplicity of a point a of T is the number of connected components of $T\backslash\{a\}$. The point a is called a leaf if its multiplicity is one, and a branching point if its multiplicity is at least 3. We will be interested in compact $\mathbb{R}$-trees. Even for such trees, there can be (countably) infinitely many branching points and uncountably many leaves. This will indeed be the case for the random $\mathbb{R}$-trees that we will introduce.

We turn to the construction of (rooted) $\mathbb{R}$-trees from their contour functions. This is a continuous analogue of the well-known coding of plane trees by Dyck paths. Let $g\colon [0,1] \to \mathbb{R}_+$ be a nonnegative continuous function such that $g(0) = g(1) = 0$. We will explain how to associate with g a compact $\mathbb{R}$-tree $(\mathcal{T}_g, d_g)$.

For every $s, t \in [0,1]$, we set

$$m_g(s,t) = \inf_{r \in [s \wedge t, s \vee t]} g(r),$$

and

$$d_g(s,t) = g(s) + g(t) - 2m_g(s,t).$$

It is easy to verify that d_g is a pseudo-metric on $[0,1]$. As usual, we introduce the equivalence relation $s \sim_g t$ if and only if $d_g(s,t) = 0$ (or equivalently if and only if $g(s) = g(t) = m_g(s,t)$). The function d_g induces a distance on the quotient space $\mathcal{T}_g := [0,1] / \sim_g$, and we keep the notation d_g for this distance. We denote by $p_g\colon [0,1] \to \mathcal{T}_g$ the canonical projection. Clearly p_g is continuous (when $[0,1]$ is equipped with the Euclidean metric and $\mathcal{T}_g$ with the metric d_g), and therefore $\mathcal{T}_g = p_g([0,1])$ is a compact metric space.

By Theorem 2.1 of [DL], the metric space $(\mathcal{T}_g, d_g)$ is a compact $\mathbb{R}$-tree. Furthermore the mapping $g \to \mathcal{T}_g$ is continuous with respect to the Gromov–Hausdorff distance, if the set of continuous functions g is equipped with the supremum distance. We will always view $(\mathcal{T}_g, d_g)$ as a rooted $\mathbb{R}$-tree with root $\rho_g = p_g(0) = p_g(1)$. Note that $d_g(\rho_g, a) = g(s)$ if $a = p_g(s)$.

It is important to observe that the tree $\mathcal{T}_g$ inherits a "lexicographical order" from its coding by the function g. If $a, b \in \mathcal{T}_g$, the vertex a comes before b in lexicographical order if the smallest representative of a in $[0,1]$ is smaller than any representative of b in $[0,1]$.

By definition, the CRT is the random compact $\mathbb{R}$-tree $(\mathcal{T}_{\boldsymbol{e}}, d_{\boldsymbol{e}})$ coded in the previous sense by a normalized Brownian excursion $\boldsymbol{e} = (\boldsymbol{e}_t)_{0 \le t \le 1}$ (recall that a normalized Brownian excursion is a linear Brownian motion over the time interval $[0,1]$, conditioned to start and to end at the origin, and to remain positive over the interval $(0,1)$). The CRT appears as the scaling limit of plane trees, as shown by the following theorem, which is a reformulation of a result in Aldous [A2].

Theorem 3.1. *For every $n \geq 1$, let τ_n be a random tree that is uniformly distributed over the set of all plane trees with n edges, and denote the graph distance on τ_n by d_{gr}. Then the rescaled trees*

$$(\tau_n, (2n)^{-1/2} d_{\mathrm{gr}})$$

converge in distribution towards the CRT, in the Gromov–Hausdorff sense.

There are in fact many other classes of random discrete trees for which the scaling limit is the CRT. For instance, it is not hard to see that this holds for random trees that are uniformly distributed over the set of all p-trees with n edges (considering only those values of n for which this set is not empty). The latter fact is an immediate consequence of the convergence of first components in Proposition 3.2 below.

3.2. Labeled trees and mobiles. In view of our applications to random planar maps, we need to understand the scaling limit of the p-mobiles of Section 2. We start with the simpler case of labeled p-trees.

For technical reasons, it is more convenient to deal with convergence of the coding functions rather than with convergence of the trees themselves. We first introduce the random functions that will appear in the limit. Let g be as above a continuous function from $[0, 1]$ into $\mathbb{R}_+$ such that $g(0) = g(1) = 0$. We can consider the centered Gaussian process $(W_t^g)_{t\in[0,1]}$ whose distribution is characterized by the covariance function

$$\mathrm{cov}(W_s^g, W_t^g) = m_g(s,t),$$

for every $s, t \in [0, 1]$. Note that $E[(W_s^g - W_t^g)^2] = d_g(s,t)$. The process $(W_s^g)_{s\in[0,1]}$ is called the Brownian snake driven by g. In the usual terminology, it is in fact the "head of the snake" rather than the snake itself – see [L1] for more information about Brownian snakes.

Under mild regularity assumptions on g, which will be satisfied in our applications, one can construct $(W_s^g)_{s\in[0,1]}$ so that it has continuous sample paths. Then the property $E[(W_s^g - W_t^g)^2] = d_g(s,t)$ implies that a.s. for every $s \in [0, 1]$, W_s^g only depends on the equivalence class of s in the quotient $\mathcal{T}_g = [0, 1]/\sim_g$. So we can find a process $Z^g = (Z_a^g)_{a\in\mathcal{T}_g}$ such that $Z_a^g = W_t^g$ whenever $a = p_g(t)$. The process Z^g should be interpreted as Brownian motion indexed by $\mathcal{T}_g$, which was briefly discussed in Section 1.

As in the previous subsection, we then randomize g. Precisely, we let $\boldsymbol{e} = (\boldsymbol{e}_s)_{s\in[0,1]}$ be as above a normalized Brownian excursion and we define a random process $(W_s)_{s\in[0,1]}$ such that, conditionally given $\boldsymbol{e}$, $(W_s)_{s\in[0,1]}$ is distributed as the Brownian snake driven by $\boldsymbol{e}$. We may again define "labels" $(Z_a)_{a\in\mathcal{T}_{\boldsymbol{e}}}$ by requiring that $Z_a = W_t$ whenever $a = p_{\boldsymbol{e}}(t)$.

We can now state a first result corresponding to the scaling limit of labeled p-trees. To simplify notation, we set

$$\lambda_p = \frac{1}{2}\sqrt{\frac{p}{p-1}}, \quad \kappa_p = \left(\frac{9}{4p(p-1)}\right)^{1/4}.$$

Proposition 3.2. *For every $n \geq 1$, let $(\tau_n, (\ell_v^n)_{v\in\tau_n^\circ})$ be uniformly distributed over the set of all labeled p-trees with n edges, and let C^n and Λ^n be respectively the contour function and the spatial contour function of $(\tau_n, (\ell_v^n)_{v\in\tau_n^\circ})$. Then,*

$$\left(\lambda_p\, n^{-1/2}\, C^n_{[pnt]}, \kappa_p\, n^{-1/4} \Lambda^n_{[pnt]}\right)_{0\leq t\leq 1} \xrightarrow[n\to\infty]{(d)} (\boldsymbol{e}_t, W_t)_{0\leq t\leq 1}.$$

This proposition is a special case of results proved in [MMi]. The convergence of the processes $(n^{-1/2}\, C^n_{[pnt]})_{t\in[0,1]}$ towards the Brownian excursion is essentially a variant of Theorem 3.1 (or rather of the formulation of this theorem in terms of contour functions, as in [A2]). The convergence of labels is related to general results about convergence of "discrete snakes" towards the Brownian snake, which are proved in [JM].

Of course the previous proposition is not sufficient for our purposes, since we are interested in p-mobiles and not in labelled p-trees. This means that we need to take the positivity constraint of labels (property (c) of the definition) into account. At an intuitive level, one may guess that this positivity constraint leads to considering the limiting pair of Proposition 3.2 conditioned on the event $\{W_s \geq 0$ for every $s \in [0,1]\}$. This conditioning however requires some care, since the conditioning event clearly has probability zero.

According to [LW], this conditioned pair, which we denote by $(\bar{e}_t, \overline{W}_t)_{t\in[0,1]}$, can be constructed as follows. If s_* denotes the (almost surely unique) time in [0, 1] such that $W_{s_*} = \min\{W_s : 0 \leq s \leq 1\}$, we set for every $t \in [0,1]$,

- $\bar{e}_t = \boldsymbol{e}_{s_*} + \boldsymbol{e}_{s_*\oplus t} - 2\, m_{\boldsymbol{e}}(s_*, s_* \oplus t)$,
- $\overline{W}_t = W_{s_*\oplus t} - W_{s_*}$,

where $s_* \oplus t = s_* + t$ if $s_* + t \leq 1$ and $s_* \oplus t = s_* + t - 1$ if $s_* + t > 1$. This definition is better understood in terms of trees. First note that $\overline{W}_t$ only depends on the equivalence class of t in $\mathcal{T}_{\bar{e}} = [0,1]/\sim_{\bar{e}}$, and thus we may construct the labels $(\overline{Z}_a)_{a\in\mathcal{T}_{\bar{e}}}$ such that $\overline{Z}_a = \overline{W}_t$ if $a = p_{\bar{e}}(t)$. Then the tree $\mathcal{T}_{\bar{e}}$ is canonically identified with the tree $\mathcal{T}_{\boldsymbol{e}}$ re-rooted at the vertex $p_{\boldsymbol{e}}(s_*)$ with minimum label (see Lemma 2.2 in [DL]), and, modulo this identification, we have $\overline{Z}_a = Z_a - \min\{Z_c : c \in \mathcal{T}_{\boldsymbol{e}}\}$, meaning that the original labels are shifted to become nonnegative.

With the preceding notation we can now state the analogue of Proposition 3.2 for p-mobiles, which is proved in [We].

Theorem 3.3. *For every $n \geq 1$, let $(\bar{\tau}_n, (\bar{\ell}^n_v)_{v\in\bar{\tau}^\circ_n})$ be uniformly distributed over the set of all p-mobiles with n edges, and let $\overline{C}^n$ and $\overline{\Lambda}^n$ be respectively the contour function and the spatial contour function of $(\bar{\tau}_n, (\bar{\ell}^n_v)_{v\in\bar{\tau}^\circ_n})$. Then,*

$$\left(\lambda_p\, n^{-1/2}\, \overline{C}^n_{[pnt]}, \kappa_p\, n^{-1/4}\, \overline{\Lambda}^n_{[pnt]}\right)_{0\leq t\leq 1} \xrightarrow[n\to\infty]{(d)} (\bar{e}_t, \overline{W}_t)_{0\leq t\leq 1}.$$

The following corollary was obtained in Chassaing and Schaeffer [CS] in the case $p = 2$ of quadrangulations. The general case can be found in [We], but the same result in a slightly different setting had been derived earlier by Marckert and Miermont [MMi]. See also [Mi1] for extensions to planar maps that are not bipartite.

Corollary 3.4. *For every integer $n \geq 2$, let M_n be a random planar map that is uniformly distributed over the set $\mathbb{M}_n^{2p}$ of all rooted $2p$-angulations with n faces. Denote by ∂ the root vertex of M_n and let $R(M_n) = \max\{d_{\mathrm{gr}}(\partial, v) : v \in V(M_n)\}$ be the radius of the map. Then,*

$$\kappa_p n^{-1/4} R(M_n) \xrightarrow[n\to\infty]{(d)} \max_{0\leq t\leq 1} W_t - \min_{0\leq t\leq 1} W_t.$$

The proof is immediate from Theorem 3.3. Indeed, we know that M_n may be constructed as the image of $(\bar{\tau}_n, (\bar{\ell}_v^n)_{v\in\bar{\tau}_n^\circ})$ under the BDG bijection. Then we have

$$R(M_n) = \max_{v\in\bar{\tau}_n^\circ} \bar{\ell}_v^n = \max_{0\leq k\leq pn} \bar{\Lambda}_k^n.$$

On the other hand, Theorem 3.3 implies that

$$\kappa_p n^{-1/4} \max_{0\leq k\leq pn} \bar{\Lambda}_k^n \xrightarrow[n\to\infty]{(d)} \max_{0\leq t\leq 1} \overline{W}_t$$

and from the definition of $\overline{W}$ in terms of W, we have also

$$\max_{0\leq t\leq 1} \overline{W}_t = \max_{0\leq t\leq 1} W_t - \min_{0\leq t\leq 1} W_t.$$

Remark. Detailed information about the limiting distribution in Corollary 3.4 can be found in [De].

4. Convergence towards the Brownian map

We now turn to the discussion of the convergence (2) in the case of uniformly distributed $2p$-angulations. Our results will involve the random pair $(\bar{e}, \overline{W})$ which was introduced at the end of the previous section. This should not come as a surprise since this pair appears in the scaling limit of large p-mobiles (Theorem 3.3), and we know that $2p$-angulations are coded by p-mobiles. To simplify notation, we write $\overline{\mathcal{T}} = \mathcal{T}_{\bar{e}}$ for the tree coded by $\bar{e}$, and $\bar{\rho}$ for the root of $\overline{\mathcal{T}}$. Also recall that $(\overline{Z}_a)_{a\in\overline{\mathcal{T}}}$ are the labels induced on $\overline{\mathcal{T}}$ by the process $\overline{W}$.

In the discrete setting of $2p$-angulations, vertices of the map (except the root) are in one-to-one correspondence with vertices of the coding tree. A naive guess would be that a similar property holds in the continuous setting. It turns out that this is not correct and that one needs to identify certain vertices of the continuous random tree $\overline{\mathcal{T}}$, which plays the same role as a p-tree in the discrete setting.

Let $s, t \in [0, 1]$. By definition,

$$s \simeq t \quad \text{if and only if} \quad \overline{W}_{s\wedge t} = \overline{W}_{s\vee t} = \min_{s\wedge t\leq r\leq s\vee t} \overline{W}_r.$$

In this way we obtain a random equivalence relation on $[0,1]$. For $a, b \in \overline{\mathcal{T}}$, we then say that $a \approx b$ if and only if there exist a representative s of a in $[0,1]$ and a representative t of b in $[0,1]$ such that $s \simeq t$. It turns out that $\approx$ is also an equivalence relation on $\overline{\mathcal{T}}$, a.s. Informally, $a \approx b$ if and only if a and b have the same label ($\overline{Z}_a = \overline{Z}_b$), and when going from a to b in lexicographical order (or in reverse lexicographical order) around the tree, one encounters only vertices with greater label. The preceding definition of the equivalence relation $\approx$ can be seen to be equivalent to the more informal one given in Section 1, modulo the identification of the trees $\mathcal{T}_{\boldsymbol{e}}$ and $\overline{\mathcal{T}}$ up to re-rooting.

It is easy to understand why the equivalence relation $\approx$ should be relevant to our description of the scaling limit of random maps. Indeed consider two white vertices u and u' in a p-mobile $(\tau, (\ell_v)_{v \in \tau^\circ})$, and recall our notation $(v_0, v_1, \dots)$ for the contour sequence of τ°. Then u and u' will be connected by an edge of the associated map if and only if we can write $\{u, u'\} = \{v_i, v_j\}$, with $i < j$, in such a way that

(a) $\ell_{v_j} = \ell_{v_i} - 1$,

(b) $\ell_{v_k} \geq \ell_{v_i}$ for all $k \in \{i, i+1, \dots, j-1\}$.

Note that this may occur for vertices that are far away from each other in the tree, and that such pairs of vertices should be identified in the scaling limit of maps. Recalling that the process $\overline{W}$ is the scaling limit of the spatial contour sequence of p-mobiles (Theorem 3.3), we see that our definition of the equivalence relation $\approx$ is just a continuous analogue of properties (a) and (b).

We denote by $\boldsymbol{m}_\infty$ the quotient space $\boldsymbol{m}_\infty = \overline{\mathcal{T}} / \approx$. Notice that $\overline{Z}_a = \overline{Z}_b$ if $a \approx b$, so that the labels $\overline{Z}_x$ can be defined with no ambiguity for every $x \in \boldsymbol{m}_\infty$.

The following theorem is the main result of [L2].

Theorem 4.1. *For every integer $n \geq 2$, let M_n be a random planar map that is uniformly distributed over the set $\mathbb{M}_n^{2p}$ of all rooted $2p$-angulations with n faces. From every strictly increasing sequence of positive integers, we can extract a subsequence along which the following convergence holds:*

$$\big(V(M_n), \kappa_p n^{-1/4} d_{\mathrm{gr}}\big) \xrightarrow[n\to\infty]{(d)} (\boldsymbol{m}_\infty, D),$$

where D is a random distance on $\boldsymbol{m}_\infty$, that induces the quotient topology on this space. Furthermore, for every $x \in \boldsymbol{m}_\infty$,

$$D(\bar{\rho}, x) = \overline{Z}_x. \tag{3}$$

Remark. In (3), the root $\bar{\rho}$ of $\overline{\mathcal{T}}$ is identified with its equivalence class in $\boldsymbol{m}_\infty$, which is a singleton. We will do this identification systematically, and $\bar{\rho}$ thus appears as a distinguished point of $\boldsymbol{m}_\infty$. The property of invariance under uniform re-rooting

(Theorem 8.1 in [L3]) however shows that this distinguished point plays no special role.

The limiting random metric space $(\boldsymbol{m}_\infty, D)$ is called the Brownian map. This terminology is slightly abusive because, as we already explained in Section 1, the random distance D may depend on the choice of p and of the subsequence in the theorem. One conjectures that D does not depend on these choices and that the same limiting random metric space appears as the scaling limit of more general random planar maps, such as triangulations for instance. This conjecture justifies that the name Brownian map is used in this work to denote one of the possible limits arising in Theorem 4.1. The results that are stated in Sections 5 and 6 below hold for any of these limits.

Sketch of proof. The proof of Theorem 4.1 consists of two main steps. The first one is a compactness argument showing that sequential limits of $(V(M_n), \kappa_p n^{-1/4} d_{\mathrm{gr}})$ exist, and that any such limit can be written as a quotient space of $\overline{\mathcal{T}}$. The second step, which is the hard part of the proof, is the identification of the equivalence relation corresponding to this quotient. Let us briefly sketch the compactness argument of the first step.

The random map M_n is the image under the BDG bijection of a p-mobile $(\bar\tau_n, (\bar\ell^n_v)_{v\in\bar\tau^\circ_n})$, and we can thus identify $V(M_n)$ with $\bar\tau^\circ_n \cup \{\partial\}$. We write $v^n_0, v^n_1, \ldots, v^n_{pn}$ for the contour sequence of the tree $\bar\tau^\circ_n$. As in Theorem 3.3, let $\overline{\Lambda}^n$ be the spatial contour function of $(\bar\tau_n, (\bar\ell^n_v)_{v\in\bar\tau^\circ_n})$, so that $\overline{\Lambda}^n_i = \bar\ell^n_{v^n_i}$ by definition. For every $i, j \in \{0, 1, \ldots, pn\}$, set

$$d_n(i,j) = d_{\mathrm{gr}}(v^n_i, v^n_j).$$

Lemma 4.2. *For every $i, j \in \{0, 1, \ldots, pn\}$,*

$$d_n(i,j) \le d^\circ_n(i,j) := \overline{\Lambda}^n_i + \overline{\Lambda}^n_j - 2 \min_{i\wedge j \le k \le i\vee j} \overline{\Lambda}^n_k + 2.$$

This lemma essentially follows from the properties of the BDG bijection. Note that we can construct a discrete geodesic from v^n_i to ∂ via the following procedure. We first look for the first index $i_1 > i$ such that the vertex $v^n_{i_1}$ has label $\bar\ell^n_i - 1$. By construction $d_n(i, i_1) = 1$. Similarly, we then look for the first index $i_2 > i_1$ such that $v^n_{i_2}$ has label $\bar\ell^n_i - 2$, and we have $d_n(i_1, i_2) = 1$. We continue this way until we arrive at a vertex with label 1, which is connected to ∂. We can similarly construct a discrete geodesic from v^n_j to ∂. However the two discrete geodesics we have obtained coincide for vertices whose distance from the root is less than or equal to

$$\min_{i\wedge j \le k \le i\vee j} \overline{\Lambda}^n_k - 1.$$

The bound of the lemma follows.

We extend the definition of $d_n(s,t)$ and $d_n^\circ(s,t)$ to noninteger values $s,t \in [0,pn]$ by linear interpolation. Next we use Theorem 3.3, which gives

$$\big(\kappa_p n^{-1/4}\, d_n^\circ(pns,pnt)\big)_{0\le s,t\le 1} \xrightarrow[n\to\infty]{(d)} (D^\circ(s,t))_{0\le s,t\le 1},$$

where

$$D^\circ(s,t) = \overline{W}_s + \overline{W}_t - 2\min_{s\wedge t\le r\le s\vee t} \overline{W}_r.$$

This implies that we can find two sequences ε_k, δ_k of positive reals converging to 0, such that, with a probability tending to 1 as $k\to\infty$, we have for every $n\ge 2$, and $s,t\in[0,1]$,

$$|t-s|<\delta_k \Rightarrow n^{-1/4} d_n^\circ(pns,pnt) < \varepsilon_k \Rightarrow n^{-1/4} d_n(pns,pnt) < \varepsilon_k.$$

It follows that with probability close to one when k is large, each of the metric spaces $(V(M_n), n^{-1/4} d_n)$ can be covered by at most $[\frac{1}{\delta_k}]+1$ balls of radius ε_k. By standard compactness criteria for the Gromov–Hausdorff convergence, this gives the tightness of the sequence of distributions of the metric spaces $(V(M_n), \kappa_p n^{-1/4} d_{\mathrm{gr}})$.

Remarks. (i) The preceding argument also yields a useful bound on the limiting distance D in Theorem 4.1. We denote by $\boldsymbol{p} = \Pi\circ p_{\bar{\boldsymbol{e}}}$ the composition of the projection $p_{\bar{\boldsymbol{e}}}: [0,1]\to\overline{\mathcal{T}}$ and the canonical projection $\Pi: \overline{\mathcal{T}} \to \boldsymbol{m}_\infty$. For every $x,y\in\boldsymbol{m}_\infty$, set

$$D^\circ(x,y) = \inf\{D^\circ(s,t) : s,t\in[0,1], \boldsymbol{p}(s)=a, \boldsymbol{p}(t)=b\}.$$

Then, for every $s,y\in\boldsymbol{m}_\infty$,

$$D(x,y) \le D^\circ(x,y). \tag{4}$$

This follows as a consequence of Lemma 4.2.

(ii) One may ask whether the quotient $\boldsymbol{m}_\infty = \overline{\mathcal{T}}/\approx$ involves identifying many pairs of points. In some sense, it does not: A typical equivalence class for $\approx$ is a singleton, and non-trivial equivalence classes can contain at most three points (there are only countably many classes containing three points). It is also true that if a is a point of $\overline{\mathcal{T}}$ that is not a leaf, then the equivalence class of a is a singleton. Thus only certain leaves of $\overline{\mathcal{T}}$ are identified with certain other leaves. In a sense, getting from the CRT to the Brownian map requires identifying relatively few pairs of points. Still these identifications drastically change the topology of the space, as we will see below (Theorem 5.2).

Theorem 4.1 leads to the obvious problem of characterizing the random distance D, which would imply that there is no need for taking a subsequence in the theorem.

Provided that the characterization does not depend on p, this would also prove that the limiting space does not depend on the choice of p. Let us formulate a conjecture for D from [L2] (see also [MMo]).

Conjecture. For every $x, y \in \boldsymbol{m}_\infty$, $D(x, y) = \inf\{\sum_{i=1}^{k} D^\circ(x_{i-1}, x_i)\}$ where the infimum is over all choices of the integer k and the sequence $x_0, x_1, \dots, x_k \in \boldsymbol{m}_\infty$ such that $x_0 = x$ and $x_k = y$.

Even if the preceding questions are still open, we will see in the next sections that much can be said about the Brownian map, and that the properties of this limiting space already have interesting consequences for large random planar maps.

5. Two theorems about the Brownian map

In this section and the next one, the Brownian map $(\boldsymbol{m}_\infty, D)$ is one of the possible limits arising in the convergence of Theorem 4.1.

Theorem 5.1. *The Hausdorff dimension of the Brownian map is*

$$\dim(\boldsymbol{m}_\infty, D) = 4 \quad a.s.$$

The bound $\dim(\boldsymbol{m}_\infty, D) \leq 4$ is very easy to derive from our construction. Indeed, the bound (4) almost immediately implies that the projection $\boldsymbol{p}\colon [0,1] \to \boldsymbol{m}_\infty$ is Hölder continuous with exponent $1/4$, which gives the desired upper bound. See [L2] for a proof of the corresponding lower bound.

Note that the topological type of the Brownian map is completely characterized in Theorem 4.1: The metric D induces the quotient topology on $\boldsymbol{m}_\infty$. The following theorem, which is the main result of [LP], identifies this topological type.

Theorem 5.2. *The space $(\boldsymbol{m}_\infty, D)$ is almost surely homeomorphic to the two-dimensional sphere $\mathbb{S}^2$.*

The proof of Theorem 5.2 is based on the expression of the Brownian map as a quotient space, and on a classical theorem of Moore giving sufficient conditions for a quotient space of the sphere to be still homeomorphic to the sphere. An alternative approach has been given by Miermont [Mi3].

Theorem 5.2 implies that with a probability close to one when n is large, a typical $2p$-angulation cannot have a separating cycle of length small in comparison with the diameter of the map, and such that both sides of the cycle have a “macroscopic” size. Indeed the existence of such “bottlenecks” in the map would imply that the scaling limit is a topological space which can be disconnected by removing a single point, and this is of course not true for the sphere. We state the previous observation more precisely, recalling that the diameter of a typical $2p$-angulation with n faces is of order $n^{1/4}$ (cf. Corollary 3.4).

Corollary 5.3. *For every integer $n \geq 2$, let M_n be a random planar map that is uniformly distributed over the set $\mathbb{M}_n^{2p}$ of all rooted $2p$-angulations with n faces. Let $\alpha > 0$ and let $\psi\colon \mathbb{N} \to \mathbb{R}_+$ be a function such that $\psi(n) = o(n^{1/4})$ as $n \to \infty$. Then, with a probability tending to 1 as $n \to \infty$, there exists no injective cycle C of the map M_n with length less than $\psi(n)$, such that the set of vertices that lie in either connected component of the complement of C in the sphere has diameter at least $\alpha n^{1/4}$.*

6. Geodesics in the Brownian map

Our goal in this section is to discuss geodesics in the Brownian map, and then to apply this discussion to asymptotic properties of large planar maps. We rely on the recent paper [L3]. See Miermont [Mi2] and Bouttier and Guitter [BG1], [BG3] for other interesting results about geodesics in large random planar maps.

We start by recalling a general definition. If (E, δ) is a compact metric space and $x, y \in E$, a geodesic or shortest path from x to y is a continuous path $\gamma = (\gamma(t))_{0 \leq t \leq \delta(x,y)}$ such that $\gamma(0) = x$, $\gamma(\delta(x, y)) = y$ and $\delta(\gamma(s), \gamma(t)) = |t - s|$ for every $s, t \in [0, \delta(x, y)]$. The space (E, δ) is then called a geodesic space if any two points in E are connected by (at least) one geodesic. From the fact that Gromov–Hausdorff limits of geodesic spaces are geodesic spaces (see [BBI], Theorem 7.5.1), one gets that $(\boldsymbol{m}_\infty, D)$ is almost surely a geodesic space. We will determine explicitly the geodesics between the root $\bar{\rho}$ of $\boldsymbol{m}_\infty$ and an arbitrary point of $\boldsymbol{m}_\infty$.

We define the skeleton $\mathrm{Sk}(\overline{\mathcal{T}})$ as the set of all points of the tree $\overline{\mathcal{T}}$ that are not leaves (equivalently these are the points whose removal disconnects the tree). One can verify that the restriction of the projection $\Pi\colon \overline{\mathcal{T}} \to \boldsymbol{m}_\infty$ to $\mathrm{Sk}(\overline{\mathcal{T}})$ is a homeomorphism. Moreover, since Π is Hölder continuous with exponent $1/2 - \varepsilon$ for every $\varepsilon > 0$ (essentially by the bound (4)), and $\mathrm{Sk}(\overline{\mathcal{T}})$ has dimension one, the Hausdorff dimension of $\Pi(\mathrm{Sk}(\overline{\mathcal{T}}))$ is less than or equal to 2. One can indeed prove that $\dim \Pi(\mathrm{Sk}(\overline{\mathcal{T}})) = 2$.

We write $\mathrm{Skel}_\infty = \Pi(\mathrm{Sk}(\overline{\mathcal{T}}))$ to simplify notation. Since the Hausdorff dimension of $\boldsymbol{m}_\infty$ is equal to 4 almost surely (Theorem 5.1), the set Skel_∞ is a relatively small subset of $\boldsymbol{m}_\infty$. The set Skel_∞ is dense in $\boldsymbol{m}_\infty$ and from the previous observations it is homeomorphic to a non-compact $\mathbb{R}$-tree. Moreover, for every $x \in \mathrm{Skel}_\infty$, the set $\mathrm{Skel}_\infty \backslash \{x\}$ is not connected.

The following theorem provides a nice geometric interpretation of the set Skel_∞.

Theorem 6.1. *The following properties hold almost surely. For every $x \in \boldsymbol{m}_\infty \backslash \mathrm{Skel}_\infty$, there is a unique geodesic from $\bar{\rho}$ to x. On the other hand, for every $x \in \mathrm{Skel}_\infty$,* *the number of distinct geodesics from* $\bar{\rho}$ *to* x *is equal to the number of connected components of* $\mathrm{Skel}_\infty \backslash \{x\}$. *In particular, the maximal number of distinct geodesics from* $\bar{\rho}$ *to a point of* $\boldsymbol{m}_\infty$ *is equal to* 3*, and there are countably many points for which this number is attained.*

Remark. The invariance of the distribution of the Brownian map under uniform re-rooting (see Section 8 in [L3]) shows that results analogous to Theorem 6.1 hold if one replaces the root $\bar{\rho}$ by a point z distributed uniformly over $\boldsymbol{m}_\infty$. Here the word "uniformly" refers to the volume measure λ on $\boldsymbol{m}_\infty$, which is the image of Lebesgue measure on [0, 1] under the projection $\boldsymbol{p} = \Pi \circ p_{\bar{\boldsymbol{e}}}$.

Theorem 6.1 opens a new perspective on our construction of the Brownian map $(\boldsymbol{m}_\infty, D)$ as a quotient space of the random tree $\overline{\mathcal{T}}$ (at first, this construction may appear artificial, even though it is a continuous counterpart of the BDG bijection). Indeed, Theorem 6.1 shows that the skeleton of $\overline{\mathcal{T}}$, or rather its homeomorphic image under the canonical projection Π, has an intrinsic geometric meaning: It exactly corresponds to the cut locus of $\boldsymbol{m}_\infty$ relative to the root $\bar{\rho}$, provided we define this cut locus as the set of all points that are connected to $\bar{\rho}$ by at least two distinct geodesics (this definition of the cut locus is slightly different from the one that appears in Riemannian geometry). Remarkably enough, the assertions of Theorem 6.1 parallel the known results in the setting of differential geometry, which go back to Poincaré [Po] and Myers [My].

To give a hint of the proof of Theorem 6.1, let us introduce the notion of a simple geodesic. Let $x \in \boldsymbol{m}_\infty$, let $a \in \overline{\mathcal{T}}$ be such that $\Pi(a) = x$, and let $t \in [0, 1]$ be such that $p_{\bar{\boldsymbol{e}}}(t) = a$. Recall that we have $D(\bar{\rho}, x) = \bar{Z}_x = \bar{Z}_a = \overline{W}_t$. For every $r \in [0, D(\bar{\rho}, x)]$, set

$$\gamma_t(r) = \sup\{s \in [0, t] : \overline{W}_s = r\}.$$

By a continuity argument, $\gamma_t(r)$ is well defined and $\overline{W}_{\gamma_t(r)} = r$. Set $\Gamma_t(r) = \boldsymbol{p}(\gamma_t(r))$ for every $r \in [0, D(\bar{\rho}, x)]$. We have

$$D(\bar{\rho}, \Gamma_t(r)) = \overline{W}_{\gamma_t(r)} = r.$$

On the other hand, if $0 \leq r \leq r' \leq t$,

$$\min_{\gamma_t(r) \leq s \leq \gamma_t(r')} \overline{W}_s = r$$

by the definition of $\gamma_t(r)$. The bound (4) now gives

$$D(\Gamma_t(r), \Gamma_t(r')) \leq r' - r.$$

Since the reverse bound is just the triangle inequality, we have obtained that

$$D(\Gamma_t(r), \Gamma_t(r')) = r' - r$$

for every $0 \leq r \leq r' \leq D(\bar{\rho}, x)$. Clearly $\Gamma_t(0) = \bar{\rho}$ and $\Gamma_t(D(\bar{\rho}, x)) = \boldsymbol{p}(t) = x$. Thus we have proved that the path $(\Gamma_t(r))_{0 \leq r \leq D(\bar{\rho},x)}$ is a geodesic from $\bar{\rho}$ to x. Such a geodesic is called a *simple geodesic*.

Remark. The preceding construction of simple geodesics is just a continuous analogue of the construction of discrete geodesics that was outlined in the proof of Lemma 4.2.

The main difficulty in the proof of Theorem 6.1 is to check that *all* geodesics from the root are simple geodesics. From this, the various statements of Theorem 6.1 follow by counting how many simple geodesics can exist for a given point $x \in \boldsymbol{m}_\infty$. In order that there exist more than one, two situations can occur:

- There exist several values of a such that $\Pi(a) = x$ (these values thus lie in the same equivalence class for $\approx$, and by a previous remark they are all leaves of $\overline{\mathcal{T}}$). However, essentially from the definition of $\approx$, one can check that the simple geodesics corresponding to these different values of a are the same.
- There is only one value of a such that $\Pi(a) = x$, but there are several values of $t \in [0, 1]$ such that $p_{\bar{\boldsymbol{e}}}(t) = a$. This means that a belongs to the skeleton of $\overline{\mathcal{T}}$, and the number of values of t such that $p_{\bar{\boldsymbol{e}}}(t) = a$ is the multiplicity of a in $\overline{\mathcal{T}}$. In that case, one easily checks that the simple geodesics Γ_t, for all t such that $p_{\bar{\boldsymbol{e}}}(t) = a$, are distinct.

The statement of Theorem 6.1 is a consequence of this discussion. Note that the number of connected components of $\mathrm{Skel}_\infty \setminus \{x\}$ is at most 3 because $\overline{\mathcal{T}}$, or equivalently the CRT, has only binary branching points, as a consequence of the fact that Brownian minima are distinct.

The next corollary gives a surprising confluence property for geodesics starting from the root.

Corollary 6.2. *Almost surely, for every $\eta > 0$, there exists $\alpha \in]0, \eta[$ such that the following holds. Let $x, x' \in \boldsymbol{m}_\infty$ such that $D(\bar{\rho}, x) \geq \eta$ and $D(\bar{\rho}, x') \geq \eta$, and let ω, respectively ω', be a geodesic from $\bar{\rho}$ to x, resp. from $\bar{\rho}$ to x'. Then, $\omega(t) = \omega'(t)$ for every $t \in [0, \alpha]$.*

Since we know that all geodesics from the root are simple geodesics, this corollary easily follows from the fact that two simple geodesics must coincide near the root. We indeed used a similar property in the discrete setting in the proof of Lemma 4.2.

To conclude this section, let us give two applications of the previous results to geodesics in large planar maps. In the discrete setting, there is of course no hope to establish the uniqueness of geodesics between two vertices (see [BG1], [BG3] for asymptotic results about the number of geodesics). Still it makes sense to deal with macroscopic uniqueness, meaning that any two geodesics will be close at an order that is small in comparison with the diameter of the map.

We recall that the random planar map M_n is uniform distributed over the set $\mathbb{M}_n^{2p}$ of all rooted $2p$-angulations with n faces, and that ∂ denotes the root vertex of M_n. For every $v \in V(M_n)$, we denote by $\mathrm{Geo}_n(\partial \to v)$ the set of all discrete geodesics from ∂ to v in the map M_n.

If γ, γ' are two discrete paths with the same length k, we set

$$d(\gamma, \gamma') = \max_{0 \le i \le k} d_{\mathrm{gr}}(\gamma(i), \gamma'(i)).$$

Corollary 6.3. *Let $\varepsilon > 0$. Then,*

$$\frac{1}{n} \#\{v \in V(M_n) : \exists \gamma, \gamma' \in \mathrm{Geo}_n(\partial \to v),\ d(\gamma, \gamma') \ge \varepsilon n^{1/4}\} \xrightarrow[n\to\infty]{} 0$$

in probability.

This means that for a typical vertex v in the map M_n, the discrete geodesic from ∂ to v is "macroscopically" unique. A stronger statement can be obtained by considering approximate geodesics, i.e. discrete paths from ∂ to v whose length is bounded above by $d_{\mathrm{gr}}(\partial, v) + o(n^{1/4})$. Also note that a related uniqueness result has been obtained by Miermont in [Mi2].

Now what about exceptional vertices in the map M_n? Does there exist vertices v such that there are several macroscopically different geodesics from ∂ to v? The following corollary provides an answer to this question. Before giving the statement, we need to introduce another notation. For $v \in V(M_n)$, and $\varepsilon > 0$, we set

$$\mathrm{Mult}_\varepsilon(v) = \max\{k : \exists \gamma_1, \ldots, \gamma_k \in \mathrm{Geo}_n(\partial, v),\ d(\gamma_i, \gamma_j) \ge \varepsilon n^{1/4} \text{ if } i \ne j\}.$$

Corollary 6.4. *For every $\varepsilon > 0$,*

$$P[\exists v \in V(M_n) : \mathrm{Mult}_\varepsilon(v) \ge 4] \xrightarrow[n\to\infty]{} 0.$$

However,

$$\lim_{\varepsilon \to 0} \Big(\liminf_{n \to \infty} P[\exists v \in V(M_n) : \mathrm{Mult}_\varepsilon(v) = 3] \Big) = 1\,.$$

Loosely speaking, there can be at most 3 macroscopically different geodesics from ∂ to an arbitrary vertex of M_n.

Remark. In the last two corollaries, the root vertex ∂ can be replaced by a vertex chosen uniformly at random in M_n.

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5ECM Amsterdam 2008

Geometry and non-archimedean integrals

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Abstract. Non-archimedean integrals are ubiquitous in various parts of mathematics. Motivic integration allows to understand them geometrically and to get strong uniformity statements. In these notes, intended for a general audience, we start by giving various examples of situations where one can get new geometric results by using p-adic or motivic integrals. We then present some more recent results in this area, in particular a Transfer Principle allowing to transfer identities involving functions defined by integrals from one class of local fields to another. Orbital integrals occurring in the Fundamental Lemma of Langlands Theory form a natural family of functions falling within the range of application of this Transfer Principle.

Mathematics Subject Classification (2010). Primary 11G25, 14B05, 14E18, 32S25, 32S45; Secondary 03C10, 03C65, 03C98, 14E16.

Keywords. Motivic integration, arcs, p-adic integration, birational geometry, Milnor fiber.

1. Introduction

This paper intends to provide a leisurely introduction to recent work related to the use of non-archimedean integrals in geometry. Our aim is to convey some flavour of the topic to a general audience, without entering too much into technicalities. Interested readers will find more detailed accounts in the recent surveys [46], [8], [37] and [24].

2. Using p-adic integrals and Denef's rationality theorem

2.1. An example: counting subgroups of finite index in nilpotent groups. As a motivating example, we shall start with an example of application of p-adic integration coming from group theory. Let G be a group and let $a_n(G)$ be the number of its subgroups of index n, which we assume to be finite. This is the case when G is a finitely generated group. To study the asymptotic behaviour of $a_n(G)$, it is natural to introduce the generating function

$$\zeta_G(s) = \sum_{n=0}^{\infty} a_n(G) n^{-s}.$$

When G is nilpotent the function ζ may be expressed as an Euler product

$$\zeta_G(s) = \prod_{p \text{ prime}} \zeta_{p,G}(s)$$

of local factors

$$\zeta_{G,p}(s) = \sum_{i=0}^{\infty} a_{p^i}(G) p^{-is}.$$

For instance, if G is the subgroup of $\mathrm{GL}_3(\mathbb{Z})$ of matrices having zero entries under the diagonal and ones on the diagonal, one has

$$\zeta_{G,p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(2s-2)\zeta_p(2s-3)\zeta_p(3s-3)^{-1},$$

with $\zeta_p(s) = (1 - p^{-s})^{-1}$, so $\zeta_{G,p}(s)$ is rational in p^{-s}. This is a very special case of a general result proved by Grunewald, Segal and Smith in 1988:

2.1.1 Theorem (Grunewald–Segal–Smith [27], 1988). *If G is a finitely generated torsion free nilpotent group, then $\zeta_{G,p}(s)$ is a rational series in p^{-s}.*

How to prove such a result? The main idea is to express $\zeta_{G,p}(s)$ as a p-adic integral, and then to use a general result of Jan Denef on the rationality of such integrals we shall explain now.

2.2. *p*-adic integrals. Given a prime p, let us recall that the field $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ with respect to the non-archimedean norm $|x|_p := p^{-v_p(x)}$, with v_p the p-adic valuation. The ring $\mathbb{Z}_p$ of p-adic integers is the subring of $\mathbb{Q}_p$ consisting of elements x with $|x|_p \leq 1$. Elements of $\mathbb{Z}_p$ can be written as infinite series $\sum_{i \geq 0} a_i p^i$, with a_i in $\{0, \ldots, p-1\}$. They are added and multiplied by rounding up to the right. Similarly, elements of $\mathbb{Q}_p$ can be written as infinite series $\sum_{i \geq -\alpha} a_i p^i$, with a_i in $\{0, \ldots, p-1\}$ and $\alpha \geq 0$.

The field $\mathbb{Q}_p$ endowed with the norm $|\ |_p$ being locally compact, $\mathbb{Q}_p^n$ admits a canonical Haar measure μ_p, normalized by $\mu_p(\mathbb{Z}_p^n) = 1$.

In many cases, the p-adic volume of a subset $X \subset \mathbb{Z}_p^n$ may be computed as

$$\mu_p(X) = \lim_{r \to \infty} (\operatorname{card} X_r) p^{-(r+1)n}$$

with X_r the image of X in $(\mathbb{Z}/p^{r+1}\mathbb{Z})^n$ (a finite set).

Let k be a field. Let us denote by C_n, the smallest collection of subsets of k^n, $n \in \mathbb{N}$, such that:

(1) the zero locus of a polynomial $f \in k[x_1, \dots, x_n]$ is an element of C_n;

(2) C_n is stable by boolean operations (complement, union, intersection);

(3) if π denotes the linear projection $k^{n+1} \to k^n$ on the first n factors and A is in C_{n+1}, then $\pi(A)$ is in C_n.

Elements of C_n are called semi-algebraic subsets of k^n and a function $g\colon k^n \to k^m$ is said to be semi-algebraic if its graph is.

When k is the field of real numbers $\mathbb{R}$ one recovers the standard definitions of semi-algebraic sets and functions.

Now we can state the general result of Denef that Grunewald, Segal and Smith use in the proof of their theorem.

2.2.1 Theorem (Denef [12], 1984). *Let V be a bounded semi-algebraic subset of $\mathbb{Q}_p^n$ and let $g\colon \mathbb{Q}_p^n \to \mathbb{Q}_p$ be a semi-algebraic function bounded on V. Then the integral*

$$\int_V |g(x)|^s |dx|$$

is a rational function of p^{-s}.

To prove his theorem, Denef needs to have a firmer grasp on p-adic semi-algebraic sets than the one given by the above definition. Let us recall that over the reals, a classical result of A. Tarski (quantifier elimination) states that a subset of $\mathbb{R}^n$ is semi-algebraic if and only if it is a finite boolean combination of subsets of the form $f(x) \geq 0$ with $f \in \mathbb{R}[x_1, \dots, x_n]$. In 1976, A. Macintyre proved the following analogue of Tarski's theorem (note that over the reals the condition $f(x) \geq 0$ may be restated as $f(x)$ being a square):

2.2.2 Theorem (Macintyre [39], 1976). *A subset of $\mathbb{Q}_p^n$ is semi-algebraic if and only if it is a finite boolean combination of subsets of the form "$f(x)$ is a d-th power", for some integer d and some $f \in \mathbb{Q}_p[x_1, \dots, x_n]$.*

Recently, E. Hrushovski and B. Martin [32] proved rationality results for zeta functions counting isomorphism classes of irreducible representations of finitely generated nilpotent groups. They use a good description of quotients of p-adic semi-algebraic subsets by semi-algebraic equivalence relations which is provided by recent work of Haskell, Hrushovski and Macpherson [28], who proved elimination of imaginaries for algebraically closed valued fields.

3. Additive invariants

3.1. Algebraic varieties. Let k be a field and let F be a family of polynomials $f_1, \dots, f_r \in k[T_1, \dots, T_N]$. The set of k-points of the corresponding (affine) algebraic variety X_F is the set of points in k^N which are common zeroes of the polynomials f_i, that is,

$$X_F(k) = \{(x_1, \dots, x_n) \in k^N : f_i(x_1, \dots, x_N) = 0 \text{ for all } i\}.$$

For any ring K containing k, we can also consider the set of K-points

$$X_F(K) = \{(x_1, \dots, x_n) \in K^N : f_i(x_1, \dots, x_N) = 0 \text{ for all } i\}.$$

In particular, if $r = 0$, we get the affine space $\mathbb{A}^N$ with $\mathbb{A}^N(K) = K^N$ for every K containing k. If $F' = F \bigcup_{i \in I}\{g_i\}$, we have

$$X_{F'}(K) \subset X_F(K)$$

for all K. We write $X_{F'} \subset X_F$ and we say $X_{F'}$ is a (closed) subvariety of X_F.

General algebraic varieties are defined by gluing affine varieties and the notion of (closed) subvariety can be extended to that setting. If X' is a subvariety of X, there is a variety $X \setminus X'$ such that, for every K, $(X \setminus X')(K) = X(K) \setminus X'(K)$. There is also a natural notion of products and a natural notion of morphisms between algebraic varieties. Basically, morphisms are induced by "polynomial transformations". In particular, there is a notion of isomorphism of algebraic varieties. For instance, $T \mapsto (T^2, T^3, T^{-2})$ induces an isomorphism between $\mathbb{A}^1 \setminus \{0\}$ and the variety defined by

$$X_1^3 - X_2^2 = 0 \quad \text{and} \quad X_1 X_3 - 1 = 0.$$

3.2. Universal additive invariants. Let $K_0(\mathrm{Var}_k)$ denote the free abelian group on isomorphism classes $[S]$ of objects of Var_k mod out by the subgroup generated by the relations of the form

$$[S] = [S'] + [S \setminus S']$$

for S' a (closed) subvariety of S. Setting

$$[S] \cdot [S'] = [S \times S']$$

endows $K_0(\mathrm{Var}_k)$ with a natural ring structure. Denote by $\mathbb{L}$ the class of the affine line $\mathbb{A}^1_k$ in $K_0(\mathrm{Var}_k)$, and set

$$M_k := K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}],$$

that is, M_k is the ring obtained by inverting $\mathbb{L}$ in $K_0(\mathrm{Var}_k)$.

One may view the mapping $X \mapsto [X]$ assigning to an algebraic variety X over k its class in M_k as the universal additive and multiplicative invariant (not vanishing on $\mathbb{A}^1_k$) on the category of algebraic varieties.

3.3. Euler characteristics versus counting. Amongst all additive invariants, the most fundamental ones may well be given by the Euler characteristic with compact supports and by counting points over finite fields.

If k is a subfield of $\mathbb{C}$ and X is a k-algebraic variety, one sets $\mathrm{Eu}(X) := \mathrm{Eu}(X(\mathbb{C}))$, where Eu is the Euler characteristic with compact supports. This an additive invariant that factors through a morphism $\mathrm{Eu}\colon \mathcal{M}_k \to \mathbb{Z}$.

Also counting points over finite fields is additive. Recall that for every prime number p, and every $f \geq 1$, there exists a unique finite field $\mathbb{F}_q$ having $q = p^f$ elements. Furthermore, for every $e \geq 1$, $\mathbb{F}_{q^e}$ is the unique field extension of degree e of $\mathbb{F}_q$. If $k = \mathbb{F}_q$ and X is a k-algebraic variety, since $X(\mathbb{F}_{q^e})$ is finite, we may set

$$N_{q^e}(X) := |X(\mathbb{F}_{q^e})|.$$

Clearly, $X \mapsto N_{q^e}(X)$ is an additive invariant and factors through a morphism $N_{q^e}\colon \mathcal{M}_k \to \mathbb{Z}[p^{-1}]$.

When $k = \mathbb{Q}$, and X is a variety over k, we may at the same time view $\mathbb{Q}$ as a subfield of $\mathbb{C}$ and consider $\mathrm{Eu}(X)$, and reduce the equations of X mod p, for p not dividing the denominators of the equations of f, in order to get a variety X_p over $\mathbb{F}_p$. For such a p, we may consider, via counting, the number $N_{p^e}(X_p)$, for any $e \geq 1$.

It is a very striking fact, that these two invariants – apparently of a very different nature – are related. Indeed, it follows from important results by A. Grothendieck going back to the 60s that given an X, for almost all p,

$$\lim_{e \to 0} N_{p^e}(X_p) = \mathrm{Eu}(X),$$

so, Euler characteristics may be computed by counting in finite fields! Of course there is literally no meaning to taking the limit as $e \to 0$ of $N_{p^e}(X_p)$. Here is the technically correct statement:

3.3.1 Theorem (Grothendieck). *Given an X, for almost all p, there exists finite families of complex numbers α_i, $i \in I$, and β_j, $j \in J$, depending only on X and p, such that*

$$N_{p^e}(X_p) = \sum_I \alpha_i^e - \sum_J \beta_j^e$$

and

$$\mathrm{Eu}(X) = |I| - |J|.$$

It is difficult to give precise references for this result since it is merely a potpourri of various results scattered in the literature. The main ingredients are the rationality of zeta functions of varieties over finite fields due to B. Dwork [21], the cohomological interpretation of these zeta functions and general comparison results for étale cohomology both due to A. Grothendieck, for which we refer to [23] and [40].

4. Using p-adic integrals in birational geometry

In this section we outline some applications of p-adic integration to birational geometry.

4.1. Birational geometry. We assume $k = \mathbb{C}$. Let X and Y be two smooth and connected complex algebraic varieties (not necessarily affine). A morphism $h\colon Y \to X$ is called a modification or a birational morphism if h is proper (i.e., h^{-1}(compact) $=$ compact) and h is an isomorphism outside a subvariety F of Y, $F \neq Y$. If, moreover, F is a union of smooth connected hypersurfaces E_i, $i \in A$, of Y, which we also assume to be mutually transverse, we say h is a DNC modification (here DNC stands for "divisor with normal crossings"). To a DNC modification $h\colon Y \to X$ we assign the following combinatorics:

For $I \subset A$, we set

$$E_I^\circ := \bigcap\nolimits_{i \in I} E_i \setminus \bigcup\nolimits_{j \notin I} E_j .$$

Note that $E_\emptyset^\circ = Y \setminus F$ and Y is the disjoint union of all the E_I°'s.

For i in A, we set

$$n_i = 1 + (\text{order of vanishing of the jacobian of } h \text{ along } E_i)$$

and, for $I \subset A$, we set

$$n_I = \prod_{i \in I} n_i .$$

4.2. Euler characteristics. We can now state the following resulting, obtained in 1987 and published in 1992:

4.2.1 Theorem (Denef–Loeser [13], 1992). *For any* DNC *modification* $h\colon Y \to X$ *the relation*

$$\mathrm{Eu}(X) = \sum_{I \subset A} \frac{\mathrm{Eu}(E_I^\circ)}{n_I}$$

holds.

The proof was by no means direct. The main steps were:

(1) To reduce to data defined over a ring of finite type over $\mathbb{Z}$. For simplicity of exposition, we shall assume everything is defined over a localization of $\mathbb{Z}$.

(2) For general p, to evaluate the p-adic volume of $X(\mathbb{Q}_p)$ as a p-adic integral on $Y(\mathbb{Q}_p)$ involving the order of jacobian of h via "change of variables formula" for p-adic integrals.

(3) To express these integrals as number of points on varieties over a finite field.

(4) To conclude by using Grothendieck's result relating Eu to number of points.

Nowadays, two other proofs are available: one by motivic integration that we shall outline in 5.2, another one by direct application of the weak factorization theorem of Abramovich, Karu, Matsuki and Włodarczyk [1]. It is still a challenging problem to find a direct, geometric, proof.

4.3. Betti numbers of birational Calabi–Yau varieties. Inspired by mirror symmetry, physicists were led to conjecture the following statement: "Birational Calabi–Yau have the same Betti numbers". This was proved by V. Batyrev in [3] by using p-adic integrals in a way similar to the one just explained, with as additional ingredient the part of the Weil conjectures proved by Deligne, which allows for smooth projective varieties to recover not only Euler characteristics, but also Betti numbers, from counting in finite field. Shortly afterwards, M. Kontsevich found a direct approach to Batyrev's Theorem, avoiding the use of p-adic integrals and involving arc spaces. This was explained in his famous Orsay talk of December 7, 1995, entitled "String cohomology", which marked the official birth of motivic integration.

5. Motivic integration

5.1. The original construction. Motivic integration is a geometric analogue of p-adic integration with $\mathbb{Q}_p$ replaced by $k((t))$. Here k is a field (say of characteristic zero) and $k((t))$ denotes the field of formal Laurent series with coefficients in k. The most naive idea is to try to construct a real valued measure on a large class of subsets of $k((t))^n$ similarly as in the p-adic case. Such an attempt is doomed to fail immediately since, as soon as k is infinite, $k((t))$ is not locally compact.

When Kontsevich invented Motivic Integration in 1995, a real breakthrough was to realize that a sensible measure on subsets of $k((t))^n$ could in fact be constructed once the value group of the measure $\mathbb{R}$ is replaced by the ring M_k (or some of its completions) constructed in terms of geometric objects defined over k.

Let X be a variety over the field k. The arc space $\mathcal{L}(X)$ is defined by

$$\mathcal{L}(X)(K) := X(K[[t]])$$

for any field K containing k. If X is affine and defined by the vanishing of a family of polynomials f_i in the variables $x_1, \ldots, x_N$, one gets equations for $\mathcal{L}(X)$ by writing $x_j = \sum_{\ell \geq 0} a_{j,\ell} t^\ell$, developing $f_i(x_1, \ldots, x_N)$ into $\sum F_{i,\ell} t^\ell$, with $F_{i,\ell}$ polynomials in the variables $a_{j,\ell}$, and asking all the polynomials $F_{i,\ell}$ to be zero.

Note that, in general, $\mathcal{L}(X)$ is an infinite-dimensional variety over k since it involves an infinite number of variables. On the other hand, for $n \geq 0$, the space $\mathcal{L}_n(X)$ defined similarly as $\mathcal{L}(X)$ with $K[[t]]$ replaced by $K[[t]]/t^{n+1}$ is of finite type over k. The original construction, outlined by Kontsevich in 1995, and developed by Denef–Loeser [15] and Batyrev [4] uses a limiting process similar to the one we saw in the p-adic case:

The basic idea is to use truncation morphisms

$$\pi_n \colon \mathcal{L}(X) \to \mathcal{L}_n(X).$$

For reasonable subsets A of $\mathcal{L}(X)$,

$$\mu(A) := \lim_{n\to\infty} [\pi_n(A)]\, \mathbb{L}^{-(n+1)d}$$

with d the dimension of X, in some completion $\widehat{M}_k$ of M_k, in complete analogy with the p-adic case.

5.2. First application: birational geometry. As we already mentioned, the very first application of motivic integration was made by Kontsevich, who used it to get a proof of Batyrev's Theorem mentioned in 4.3 without p-adic integration. Similarly, one can avoid the use of p-adic integration in the proof of the Denef–Loeser Theorem 4.2.1.

Let us explain the underlying idea. If $h\colon Y \to X$ is a birational morphism, one can express the motivic volume of $\mathcal{L}(X)$ as a motivic integral on $\mathcal{L}(Y)$ involving the order of vanishing of the jacobian. This is achieved by using an analogue of the "change of variables formula" in this setting. This may work for the following reason: a modification $h\colon Y \to X$ induces an isomorphism outside a subset $F \subset Y$ of finite positive codimension (usually one), but at the level of arc spaces h induces a morphism between $\mathcal{L}(Y)$ and $\mathcal{L}(X)$ which restricts to a bijection between $\mathcal{L}(Y) \setminus \mathcal{L}(F)$ and $\mathcal{L}(X) \setminus \mathcal{L}(h(F))$, that is, between arcs in Y not completely contained in F and arcs in X not completely contained in $h(F)$. The key fact, making measure theoretic tools so well adapted to birational geometry, is that $\mathcal{L}(F)$ is of infinite codimension in $\mathcal{L}(Y)$, hence $\mathcal{L}(F)$ and $\mathcal{L}(h(F))$ have measure zero in $\mathcal{L}(Y)$ and $\mathcal{L}(X)$, respectively.

5.3. Second application: finite group actions. Let G be a finite group. A linear action of G on a complex vector space V has a canonical decomposition $\bigoplus V_\alpha$ parametrized by characters. If G acts on a complex algebraic variety X, there is of course no decomposition as above. But there exists one at the level of arc spaces!

Indeed, let x be a point of X and denote by $G(x)$ the isotropy subgroup at x, consisting of those elements of G fixing x. Denote by $\mathcal{L}(X)_x$ the space of arcs on X with origin at x. It was proved in [19] that there is a canonical decomposition

$$\mathcal{L}(X)_x = \bigsqcup_{\gamma\in \operatorname{Conj} G(x)} \mathcal{L}(X)_x^\gamma \sqcup B,$$

with Conj $G(x)$ the set of conjugacy classes in $G(x)$ and B a subset of infinite codimension in $\mathcal{L}(X)_x$ (hence of motivic measure zero).

This explains the use of motivic integration in relation with the McKay correspondence to get results relating certain resolutions of the quotient space X/G with group theoretical invariants of the action, cf. [5], [19], [48], [49].

5.4. Third application: motivic Milnor fiber. Let X be a smooth complex algebraic variety and $f : X \to \mathbb{C}$ a function (a morphism to the affine line). Let x be a singular point of $f^{-1}(0)$, that is, such that $df(x) = 0$. Fix $0 < \eta \ll \varepsilon \ll 1$. The morphism f restricts to a fibration – called the Milnor fibration –

$$B(x, \varepsilon) \cap f^{-1}(B(0, \eta) \setminus \{0\}) \longrightarrow B(0, \eta) \setminus \{0\}.$$

Here $B(a, r)$ denotes the closed ball of center a and radius r. The Milnor fiber at x,

$$F_x = f^{-1}(\eta) \cap B(x, \varepsilon),$$

has a diffeomorphism type that does not depend on η and ε, and it is endowed with an automorphism, the monodromy M_x, induced by the characteristic mapping of the fibration.

In particular, the trace of the action of the n-th iterate of the monodromy M_x on the cohomology of the Milnor fiber F_x,

$$\Lambda^n(M_x) := \sum_j (-1)^j \ \mathrm{tr}\,(M_x^n; H^j(F_x)),$$

is an invariant of the singularity. Quite surprisingly, this invariant may be expressed in purely algebraic terms using arcs. Consider the set $\mathcal{X}_n$ consisting of (truncated) arcs $\varphi(t)$ in $\mathcal{L}_n(X)$ such that $\varphi(0) = x$ and $f(\varphi(t)) = t^n +$ (higher order terms).

5.4.1 Theorem (Denef–Loeser [18], 2002). *For $n \geq 1$, we have*

$$\Lambda^n(M_x) = \mathrm{Eu}(\mathcal{X}_n).$$

The proof of this result is not very enlightening: one computes both sides of the equality on a resolution of singularities of $f = 0$ using the change of variables formula and checks that they are equal. Finding a direct, fully geometric proof, not using resolution of singularities, still represents a quite challenging problem.

Nicaise and Sebag ([44], [43]) have shown that this result may be naturally reformulated and generalized within the framework of rigid analytic geometry as a trace formula connecting the Euler characteristic of a motivic Serre invariant (cf. [38]) with the trace of the monodromy on the analytic Milnor fiber.

In fact, the spaces $\mathcal{X}_n$ do contain much more information about the Milnor fiber and the monodromy. Denef and Loeser (cf. [20]) proved that the series

$$Z(T) := \sum_{n \geq 1} [\mathcal{X}_n] \mathbb{L}^{-dn} T^n,$$

with d the dimension of X, is rational in T and has a limit $-\mathcal{S}_f$ as $T \to \infty$ in $\mathcal{M}_k$. In fact, by considering an equivariant version of $Z(T)$, one can define $\mathcal{S}_f$ as an element of an equivariant version of $\mathcal{M}_k$, cf. [20]. It is called the motivic Milnor fiber of f at x, and can be viewed as a motivic incarnation of the Milnor fiber together with the (semi-simplification of the) monodromy action on it. We refer to papers by Denef–Loeser [14], Bittner [6] and Guibert–Loeser–Merle [25] and [26] for more on this topic. Let us mention that the motivic Milnor fiber is used in an essential way by Kontsevich and Soibelman in their recent work [35] on motivic Donaldson–Thomas invariants and cluster transformations.

6. Motivic measure for definable sets and uniformity of p-adic integrals

6.1. Definable sets. A first order formula in the language of rings is a formula written with symbols $0, +, -, 1, \times, =$, logical symbols $\wedge$ (and), $\vee$ (or), $\neg$ (negation), quantifiers $\exists$, $\forall$, and variables. If k is a ring, we may extend the language by adding constants for every element of k and consider formulas in this extended language, which we call ring formulas over k. Now consider a ring formula $\varphi(x_1, \ldots, x_n)$ over k with all its free variables belonging to $\{x_1, \ldots, x_n\}$. If K is a field containing an homomorphic image of k, we may consider the set

$$X_\varphi(K) := \big\{(x_1, \ldots, x_n) \in K^n \mid \varphi(x_1, \ldots, x_n) \text{ holds}\big\}.$$

Objects of the form X_φ are called definable sets over k.

More generally one can consider natural extensions of the ring language to valued ring languages admitting symbols to express that the valuation is larger than something, or that the initial coefficient of a series is equal to something. This leads also to a notion of definable sets in the corresponding language.

6.2. General motivic measure. With Raf Cluckers we constructed in [7] a general theory of motivic integration based on cell decomposition. In our theory, motivic integrals take place in a ring N_k which is obtained from the Grothendieck ring $K_0(\mathrm{Def}_k)$ of definable sets over k (in the ring language) by inverting $\mathbb{L}$ and $1 - \mathbb{L}^i$ for $i \neq 0$ (here again $\mathbb{L}$ stands for the class of the affine line). There is a natural morphism $N_k \to \widehat{M}_k$.

Our construction assigns to a bounded definable subset A of $k((t))^n$ in the valued field language a motivic volume $\mu(A)$ in N_k compatible with the construction in 5.1.

It relies on a cell decomposition theorem due to Denef and Pas [45] (such cell decomposition results trace back to the work of P. Cohen [11]). The result by Denef and Pas tells us that one can cut a definable subset A of $k((t))^{n+1} = k((t))^n \times k((t))$ into 0-dimensional cells (graphs of functions defined on a definable subset B of $k((t))^n$) and 1-dimensional cells (relative balls over B), maybe after adding some auxiliary parameters over the residue field and the value group. This allows us to define the measure by induction on the valued field dimension. One of the main difficulties is to prove that the measure is well defined, that is, independent of the cell decomposition. In particular, one has to prove the non obvious fact that it is independent of the ordering of coordinates in the ambient affine space, which is a form of a Fubini theorem.

6.3. Constructible motivic functions. In fact, from the start our construction is relative: we define a natural class of constructible motivic functions and we prove stability of that class with respect to integration depending on parameters. Once the new framework is developed, integration of constructible motivic functions behaves very similar to the more classical Lebesgue integration with Fubini theorems, change of variable theorems, distributions, etc. Also one can extend the construction to allow motivic analogues of exponential functions and Fourier inversion [9].

6.4. Ax–Kochen–Eršov. Though they are basically different, for instance they have different characteristics, $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ look asymptotically when $p \gg 0$ very much the same:

6.4.1 Theorem (Ax–Kochen–Eršov [2], [22]). *Let φ be a first order sentence (that is, a formula with no free variables) in the language of rings. For almost all prime numbers p, the sentence φ is true in $\mathbb{Q}_p$ if and only if it is true in $\mathbb{F}_p((t))$.*

For instance, for every $d > 0$, there is a sentence S_d in the language of rings expressing that any homogeneous polynomial of degree $d^2 + 1$ with coefficients in a field k has a non trivial zero in that field. Since, by work of Tsen and Lang, S_d holds in $\mathbb{F}_p((t))$, it follows from Theorem 6.4.1 that S_d holds in $\mathbb{Q}_p$ for p large enough.

6.5. Generalization to definable sets. How can one extend the Ax–Kochen–Eršov Theorem to formulas with free variables?

6.5.1 Theorem (Denef–Loeser [17]). *Let φ be a formula in the valued ring language. Then, for almost all p, the sets $X_\varphi(\mathbb{Q}_p)$ and $X_\varphi(\mathbb{F}_p((t)))$ have the same volume. Furthermore this volume is equal to the number of points in $\mathbb{F}_p$ of a motive M_φ canonically attached to φ.*

When φ has no free variables, one recovers the original form of the Ax–Kochen–Eršov Theorem. There is a similar statement for integrals. This shows that p-adic integrals have a strongly uniform pattern as p varies: they are fully controlled by a single geometric object. On the other hand, it is a priori unclear what an Ax–Kochen–Eršov Theorem for integrals depending on parameters could be, since there seems to be no way to compare functions defined over different spaces. Before going more into that direction, let us look at an example.

6.6. An example. Let E/F be a non ramified degree two extension of non-archimedean local fields of residue characteristic different from 2. Let ψ be an additive character of F which is non trivial on $\mathcal{O}_F$ but trivial on the maximal ideal $\mathfrak{M}_F$. Let N_n be the group of upper triangular matrices with 1's on the diagonal and consider the character $\theta\colon N_n(F) \to \mathbb{C}^\times$ given by

$$\theta(u) := \psi\Big(\sum_i u_{i,i+1}\Big).$$

For a the diagonal matrix $(a_1, \dots, a_n)$ with a_i in $F^\times$, Jacquet and Ye considered the following complicated integral $I(a)$ defined in terms of F:

$$I(a) := \int_{N_n(F)\times N_n(F)} \mathbf{1}_{M_n(\mathcal{O}_F)}({}^t u_1 a u_2)\,\theta(u_1 u_2)\,du_1 du_2,$$

with the normalisation $\int_{N_n(\mathcal{O}_F)} du = 1$. They also considered a similar integral $J(a)$ defined in terms of E by replacing $N_n(F) \times N_n(F)$ by $N_n(E)$ and involving the non trivial element of the Galois group $x \mapsto \bar{x}$:

$$J(a) := \int_{N_n(E)} \mathbf{1}_{M_n(\mathcal{O}_E)\cap H_n}({}^t\bar{u} a u)\,\theta(u\bar{u})\,du,$$

with H_n the set of Hermitian matrices.

The Jacquet–Ye Conjecture asserts that

$$I(a) = \gamma(a)\,J(a) \tag{6.6.1}$$

with

$$\gamma(a) := \prod_{1\leq i\leq n-1} \eta(a_1 \dots a_i),$$

and η the multiplicative character of order 2 on $F^\times$.

When $n = 2$, the Jacquet–Ye Conjecture essentially reduces to classical Gauss sum identities, but already for $n = 3$ a proof by direct computation is quite hard. The full Jacquet–Ye Conjecture over finite field extensions of $\mathbb{F}_q((t))$ has been proved by Ngô in 1999 [41] and over any non-archimedean local field by Jacquet in 2004

[33]. Ngô's proof goes by reduction to a purely geometrical statement over algebraic varieties over $\mathbb{F}_q$ (which is not possible in the p-adic case), which he can prove by fully using the powerful machinery of ℓ-adic perverse sheaves over such varieties. This is a typical instance of the general principle "complicated identities between character sums over finite fields are better proved by geometrical tools".

Hence it is natural to ask if assuming we only know (6.6.1) holds over finite field extensions of $\mathbb{F}_q((t))$ whether it is possible to deduce it from general principles for p-adic fields. Note that it makes no sense to compare the values of the integrals themselves, since a does not run over the same space in the characteristic 0 and p cases. The answer is yes as we shall see now.

6.7. The transfer principle. The uniformity result given by Theorem 6.5.1 may be extended in the following way:

6.7.1 Theorem (Cluckers–Loeser [9]). *All p-adic integrals depending on parameters that are definable in a precise sense may be obtained by specialization of canonical motivic integrals of constructible functions for almost all p, and similarly for $\mathbb{Q}_p$ replaced by $\mathbb{F}_p((t))$.*

6.7.2 Transfer Principle (Cluckers–Loeser [9]). *A given equality between definable integrals depending on parameters holds for $\mathbb{Q}_p$ if and only if it holds for $\mathbb{F}_p((t))$, when $p \gg 0$.*

With Cluckers and Hales [10] we have recently proved that the range of application of the transfer principle contains in particular the so called Fundamental Lemma of Langlands theory. One can also check that it applies to the Jacquet–Ye Conjecture. Recall that the Fundamental Lemma was proved recently by Laumon and Ngô [36] in the unitary case and by Ngô in the general case over finite extensions of $\mathbb{F}_p((t))$ [41] by geometrical methods. Using specific techniques, Waldspurger [47] had already previously proved that one can then deduce it for p-adic fields. It is natural to expect that relations between non-archimedean integrals holding over all local fields of large residual characteristic already hold at the motivic level, as equalities between constructible motivic functions, but this seems to be presently out of reach.

6.8. Recent developments. Let us close this brief survey by mentioning some other recent applications of Model Theory to Geometry over valued fields. Hrushovski and Kazhdan [30] developed a geometric integration theory for general complete valued fields (with residue characteristic zero) based on Robinson's quantifier elimination for algebraically closed valued fields.

On the other hand Haskell, Hrushovski and Macpherson [29] recently introduced the notion of stably dominated type and studied it in great detail for algebraically closed valued fields. Such more advanced model theoretic tools seem to have very promising applications to the study of the geometry of Berkovich spaces, cf. [31].

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5ECM Amsterdam 2008

Feynman integrals and motives

Matilde Marcolli

… and Beyond the Infinite
(Stanley Kubrick, *2001 – A Space Odyssey*)

Abstract. This article gives an overview of recent results on the relation between quantum field theory and motives, with an emphasis on two different approaches: a "bottom-up" approach based on the algebraic geometry of varieties associated to Feynman graphs, and a "top-down" approach based on the comparison of the properties of associated categorical structures. This survey is mostly based on joint work of the author with Paolo Aluffi, along the lines of the first approach, and on previous work of the author with Alain Connes on the second approach.

Mathematics Subject Classification (2000). 81T15, 19E15.

Keywords. Feynman integrals, mixed Tate motives, periods, Grothendieck ring, graph hypersurfaces, motivic Galois groups, Riemann–Hilbert correspondence.

Contents

1. Introduction: quantum fields and motives, an unlikely match

This paper, based on the plenary lecture delivered by the author at the 5th European Congress of Mathematics in Amsterdam, aims at giving an overview of the current approaches to understanding the role of motives and periods of motives in perturbative quantum field theory. It is a priori surprising that there should be any relation at all between such distant fields. In fact, motives are a very abstract and sophisticated branch of algebraic and arithmetic geometry, introduced by Grothendieck as a universal cohomology theory for algebraic varieties. On the other hand, perturbative quantum field theory is a procedure for computing, by successive approximations in powers of the relevant coupling constants, values of physical observables in a quantum field theory. Perturbative quantum field theory is not entirely mathematically rigorous, though as we will see later in this paper, a lot of interesting mathematical structures arise when one tries to understand conceptually the procedure of extraction of finite values from divergent Feynman integrals known as renormalization.

The theory of motives itself has its mysteries, which make it a very active area of research in contemporary mathematics. The categorical structure of motives is still a problem very much under investigation. While one has a good abelian category of pure motives (with numerical equivalence), that is, of motives arising from smooth projective varieties, the "standard conjectures" of Grothendieck are still unsolved. Moreover, when it comes to the much more complicated setting of mixed motives, which no longer correspond to smooth projective varieties, one knows that they form a triangulated category, but in general one cannot improve that to the level of an abelian category with the same nice properties one has in the case of pure motives. See [14], [45] for an overview of the theory of mixed motives.

The unlikely interplay between motives and quantum field theory has recently become an area of growing interest at the interface of algebraic geometry, number theory, and theoretical physics. The first substantial indications of a relation between these two subjects came from extensive computations of Feynman diagrams carried out by Broadhurst and Kreimer [22], which showed the presence of multiple zeta values as results of Feynman integral calculations. From the number theoretic viewpoint, multiple zeta values are a prototype case of those very interesting classes of numbers which, although not themselves algebraic, can be realized by integrating algebraic differential forms on algebraic cycles in arithmetic varieties. Such numbers are called periods, *cf.* [43], and there are precise conjectures on the kind of operations (changes of variables, Stokes formula) one can perform at the level of the algebraic data that will correspond to relations in the algebra of periods. As one can consider periods of algebraic varieties, one can also consider periods of motives. In fact, the nature of the numbers one obtains is very much related to the motivic complexity of the part of the cohomology of the variety that is involved in the evaluation of the period.

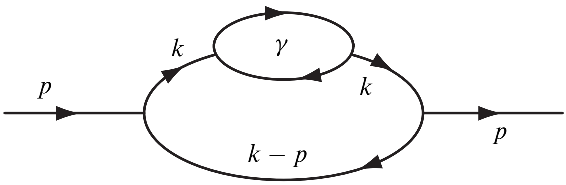
There is a special class of motives that are better understood and better behaved with respect to their categorical properties: the *mixed Tate motives*. They are also the kind of motives that are expected (see [36], [55]) to be supporting the type of periods like multiple zeta values that appear in Feynman integral computations.

At the level of pure motives the Tate motives $\mathbb{Q}(n)$ are simply motives of projective spaces and their formal inverses, but in the mixed case there are very nontrivial extensions of these objects possible. In terms of algebraic varieties, for instance, varieties that have stratifications where the successive strata are obtained by adding copies of affine spaces provide examples of mixed Tate motives. There are various conjectural geometric descriptions of such extensions (see *e.g.* [8] for one possible description in terms of hyperplane arrangements). Understanding when certain geometric objects determine motives that are or are not mixed Tate is in general a difficult question and, it turns out, one that is very much central to the relation to quantum field theory.

In fact, the main conjecture we describe here, along with an overview of some of the current approaches being developed to answer it, is whether, after a suitable subtraction of infinities, the Feynman integrals of a perturbative scalar quantum field theory always produce values that are periods of mixed Tate motives.

1.1. Feynman diagrams: graphs and integrals. We briefly introduce the main characters of our story, starting with Feynman diagrams. By these one usually means the data of a finite graph together with a prescription for assigning variables to the edges with linear relations at the vertices and a formal integral in the resulting number of independent variables.

For instance, consider a graph of the following form.



The corresponding integral gives

$$(2\pi)^{-2D} \int \frac{1}{k^4} \frac{1}{(k-p)^2} \frac{1}{(k+\ell)^2} \frac{1}{\ell^2} \, d^D k \, d^D \ell.$$

As is often the case, the resulting integral is divergent. We will explain below the regularization procedure that expresses such divergent integrals in terms of meromorphic functions. In this case one obtains

$$(4\pi)^{-D} \frac{\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-1\right)^3\Gamma(5-D)\Gamma(D-4)}{\Gamma(D-2)\Gamma\left(4-\frac{D}{2}\right)\Gamma\left(\frac{3D}{2}-5\right)} \left(p^2\right)^{D-5}$$

and one identifies the divergences with poles of the resulting function.

The renormalization problem in perturbative quantum field theory consists of removing the divergent part of such expressions by a redefinition of the running parameters (masses, coupling constants) in the Lagrangian of the theory. To avoid non-local expressions in the divergences, which cannot be canceled using the local terms in the Lagrangian, one needs a method to remove divergences from Feynman integrals that accounts for the nested structure of subdivergences inside a given Feynman graphs. Thus, the process of extracting finite values from divergent Feynman integrals is organized in two steps: *regularization*, by which one denotes a procedure that replaces a divergent integral by a function of some new regularization parameters, which is meromorphic in these parameters, and happens to have a pole at the value of the parameters that recovers the original expression; and *renormalization*, which denotes the procedure by which the polar part of the Laurent series obtained as a result of the regularization process is extracted *consistently* with the hierarchy of divergent subgraphs inside larger graphs.

1.2. Perturbative quantum field theory in a nutshell. We recall very briefly here a few notions of perturbative quantum field theory we need in the following. A detailed introduction for the use of mathematicians is given in Chapter 1 of [30].

To specify a quantum field theory, which we denote by $\mathcal{T}$ in the following, one needs to assign the Lagrangian of the theory. We restrict ourselves to the case of *scalar theories*, though it is possible that similar conjectures on number theoretic aspects of values of Feynman integrals may be formulated more generally.

A scalar field theory $\mathcal{T}$ in spacetime dimension D is determined by a classical Lagrangian density of the form

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \mathcal{L}_{\text{int}}(\phi), \tag{1}$$

in a single scalar field ϕ, with the interaction term $\mathcal{L}_{\text{int}}(\phi)$ given by a polynomial in ϕ of degree at least three. This determines the corresponding classical action as

$$S(\phi) = \int \mathcal{L}(\phi) d^D x = S_0(\phi) + S_{\text{int}}(\phi).$$

While the variational problem for the classical action gives the classical field equations, the quantum corrections are implemented by passing to the *effective action* $S_{\text{eff}}(\phi)$. The latter is not given in closed form, but in the form of an asymptotic series, the perturbative expansion parameterized by the "one-particle irreducible" (1PI) Feynman graphs. The resulting expression for the effective action is then of the form

$$S_{\text{eff}}(\phi) = S_0(\phi) + \sum_{\Gamma} \frac{\Gamma(\phi)}{\#\text{Aut}(\Gamma)} \tag{2}$$

where the contribution of a single graph is an integral on external momenta assigned to the "external edges" of the graph,

$$\Gamma(\phi) = \frac{1}{N!}\int_{\sum_i p_i = 0} \hat{\phi}(p_1)\dots\hat{\phi}(p_N) U^z_\mu(\Gamma(p_1,\dots,p_N)) dp_1 \dots dp_N.$$

In turn, the function of the external momenta that one integrates to obtain the coefficient $\Gamma(\phi)$ is an integral in momentum variables assigned to the "internal edges" of the graph Γ, with momentum conservation at each vertex. Thus, it can be expressed as an integral in a number of variables equal to the number $b_1(\Gamma)$ of loops in the graph, of the form

$$U(\Gamma(p_1,\dots,p_N)) = \int I_\Gamma(k_1,\dots,k_\ell,p_1,\dots,p_N)d^Dk_1\dots d^Dk_\ell. \tag{3}$$

The graphs involved in the expansion (2) are the 1PI Feynman graphs of the theory $\mathcal{T}$, *i.e.* those graphs that cannot be disconnected by the removal of a single edge. As Feynman graphs of a given theory, they are also subject to certain combinatorial constraints: each vertex in the graph has valence equal to the degree of one of the monomials in the Lagrangian. The edges are subdivided into internal edges connecting two vertices and external edges (or half edges) connected to a single vertex. The Feynman rules of the theory $\mathcal{T}$ specify how to assign an integral (3) to a Feynman graph, namely it specifies the form of the function $I_\Gamma(k_1,\dots,k_\ell,p_1,\dots,p_N)$ of the internal momenta. This is a product of "propagators" associated to the internal lines. These are typically of the form $1/q(k)$, where q is a quadratic form in the momentum variable of a given internal edge, which is obtained from the fundamental (distributional) solution of the associated classical field equation for the free field theory coming from the $S_0(\phi)$ part of the Lagrangian, such as the Klein–Gordon equations for the scalar case. Momentum conservations are then imposed at each vertex, and multiplied by a power of the coupling constant (the coefficient of the corresponding monomial in the Lagrangian) and a power of 2π.

As we mentioned above, the resulting integrals (3) are very often divergent. Thus, a regularization and renormalization method is used to extract a finite value. There are different regularization and renormalization schemes used in the physics literature. We concentrate here on *dimensional regularization and minimal subtraction*, which is a widely used regularization method in particle physics computations, and on the recursive procedure of Bogolyubov–Parasiuk–Hepp–Zimmermann for renormalization [20], [39], [60], see also [48]. Regularization and renormalization are two distinct steps in the process of extracting finite values from divergent Feynman integrals. The first replaces the integrals with meromorphic functions with poles that account for the divergences, while the latter organizes subdivergences in such a way that the divergent parts can be eliminated (in the case of a renormalizable theory) by readjusting finitely many parameters in the Lagrangian.

The procedure of dimensional regularization is based on the curious idea of making sense of the integrals (3) in "complexified dimension" $D - z$, with $z \in \mathbb{C}^*$, instead of working in the original dimension $D \in \mathbb{N}$. It would seem at first that, to make sense of such a procedure, one would need to make sense of geometric spaces in dimension $D - z$ and of a corresponding theory of measure and integration in such spaces. However, due to the special form of the Feynman integrals (3), a lot less is needed. In fact, it turns out that it suffices to have a *formal procedure* to define the

Gaussian integral

$$\int e^{-\lambda t^2} d^D t := \pi^{D/2} \lambda^{-D/2} \tag{4}$$

in the case where D is no longer a positive integer but a complex number. Clearly, since the right hand side of (4) continues to make sense for $D \in \mathbb{C}^*$, one can use that as the definition of the left hand side and set:

$$\int e^{-\lambda t^2} d^z t := \pi^{z/2} \lambda^{-z/2} \quad \text{for all } z \in \mathbb{C}^*. \tag{5}$$

The computations of Feynman integrals can be reformulated in terms of Gaussian integrations using the method of Schwinger parameters we return to in more detail below, hence one obtains a well defined notion of integrals in dimension $D - z$:

$$\begin{aligned} &U_\mu^z(\Gamma(p_1, \dots, p_N)) \\ &\quad = \int \mu^{z\ell} d^{D-z} k_1 \dots d^{D-z} k_\ell I_\Gamma(k_1, \dots, k_\ell, p_1, \dots, p_N). \end{aligned} \tag{6}$$

The variable μ has the physical units of a mass and appears in these integrals for dimensional reasons. It will play an important role later on, as it sets the dependence on the energy scale of the renormalized values of the Feynman integrals, hence the renormalization group flow.

It is not an easy result to show that the dimensionally regularized integrals give meromorphic functions in the variable z, with a Laurent series expansion at $z = 0$. See a detailed discussion of this point in Chapter 1 of [30]. We will not enter in details here and talk loosely about (6) as a meromorphic function of z depending on the additional parameter μ.

We return to a discussion of a possible geometric meaning of the dimensional regularization procedure in the last section of this paper.

1.3. The Feynman rules. The integrand $I_\Gamma(k_1, \dots, k_\ell, p_1, \dots, p_N)$ in the Feynman integrals (3) is determined by the *Feynman rules* of the given quantum field theory, see [40], [12]. These can be summarized as follows:

- A Feynman graph Γ of a scalar quantum field theory with Lagrangian (1) has vertices of valences equal to the degrees of the monomials in the Lagrangian, internal edges connecting pairs of vertices, and external edges connecting to a single vertex.
- To each internal edge of a Feynman graph Γ one assigns a momentum variable $k_e \in \mathbb{R}^D$ and a propagator, which is a quadratic form q_e in the variable k_e, which (in Euclidean signature) is of the form

$$q_e(k_e) = k_e^2 + m^2. \tag{7}$$

- The integrand is obtained by taking a product over all internal edges of the inverse propagators
$$\frac{1}{q_1 \dots q_n}$$
and imposing a linear relation at each vertex, which expresses the conservation law
$$\sum_{e_i \in E(\Gamma): s(e_i)=v} k_i = 0$$
for the momenta flowing through that vertex. One obtains in this way the integrand
$$\begin{aligned} I_\Gamma(k_1, \dots, k_\ell, p_1, \dots, p_N) \\ = \frac{\delta\big(\sum_{i \in E_{\mathrm{int}}(\Gamma)} \epsilon_{v,i} k_i + \sum_{j \in E_{\mathrm{ext}}(\Gamma)} \epsilon_{v,j} p_j\big)}{q_1(k_1) \dots q_n(k_n)}, \end{aligned} \tag{8}$$
where $\epsilon_{e,v}$ denotes the incidence matrix of the graph
$$\epsilon_{e,v} = \begin{cases} +1, & t(e) = v, \\ -1, & s(e) = v, \\ 0, & \text{otherwise.} \end{cases}$$
- For each vertex of Γ one also multiplies the above by a constant factor involving the coupling constants of the terms in the Lagrangian of power corresponding to the valence of the vertex and by a power of (2π), which we omit for simplicity.

There are two properties of Feynman rules that it is useful to recall for comparison with algebro-geometric settings:

(1) Reduction from graphs to connected graphs: the Feynman rules are multiplicative over disjoint unions of graphs
$$U(\Gamma, p) = U(\Gamma_1, p_1)\ U(\Gamma_2, p_2) \quad \text{for } \Gamma = \Gamma_1 \amalg \Gamma_2. \tag{9}$$

(2) Reduction from connected graphs to 1PI graphs. An arbitrary connected finite graph can be written as a tree T where some of the vertices are replaced by 1PI graphs with a number of external edges matching the valence of the vertex, $\Gamma = \bigcup_{v \in V(T)} \Gamma_v$. For these graphs the Feynman rules satisfy
$$U(\Gamma) = \prod_{v \in V(T)} U(\Gamma_v) \prod_{e \in E_{\mathrm{ext}}(\Gamma_v), e' \in E_{\mathrm{ext}}(\Gamma_{v'}), e = e' \in E_{\mathrm{int}}(\Gamma)} \frac{\delta(p_e - p_{e'})}{q_e(p_e)}. \tag{10}$$

These properties reduce the combinatorics of Feynman graphs to the 1PI case. Notice that in the particular case where $m \neq 0$ (massive theories) and the external momenta are set to zero, $p = 0$, the case (10) reduces to the simpler form

$$U(\Gamma) = U(L)^{\# E(T)} \prod_{v \in V(T)} U(\Gamma_v), \tag{11}$$

where $U(L)$ is the inverse propagator for a single edge, in this case just equal to the constant factor m^{-2}.

1.4. Parametric representation of Feynman integrals. The Feynman parameterization (also known as α-parameterization), see [12], [40], [53], reformulates the Feynman integrals (3) in such a way that they become manifestly (modulo divergences) written as the integral of an algebraic differential form on an algebraic variety, integrated over a cycle with boundary on a divisor in the variety, see [16].

One starts with the Feynman integral, written as above in the form

$$U(\Gamma) = \int \frac{\delta\left(\sum_{i=1}^{n} \epsilon_{v,i} k_i + \sum_{j=1}^{N} \epsilon_{v,j} p_j\right)}{q_1 \dots q_n} \, d^D k_1 \dots d^D k_n$$

with $n = \# E_{\mathrm{int}}(\Gamma)$ and $N = \# E_{\mathrm{ext}}(\Gamma)$ and with $\epsilon_{e,v}$ the incidence matrix.

Then, one introduces the *Schwinger parameters*. These are variables $s_i \in \mathbb{R}_+$ defined by the identity

$$\begin{aligned} &q_1^{-k_1} \dots q_n^{-k_n} \\ &\quad = \frac{1}{\Gamma(k_1) \dots \Gamma(k_n)} \int_0^\infty \cdots \int_0^\infty e^{-(s_1 q_1 + \cdots + s_n q_n)} \, s_1^{k_1 - 1} \dots s_n^{k_n - 1} \, ds_1 \dots ds_n. \end{aligned}$$

The *Feynman trick*, which consists of writing

$$\frac{1}{q_1 \dots q_n} = (n-1)! \int \frac{\delta(1 - \sum_{i=1}^{n} t_i)}{(t_1 q_1 + \cdots + t_n q_n)^n} \, dt_1 \dots dt_n,$$

is obtained from a particular case of the identity defining the Schwinger parameters, after a simple change of variables.

One then further introduces a change of variables $k_i = u_i + \sum_{k=1}^{\ell} \eta_{ik} x_k$, where η_{ik} is the matrix

$$\eta_{ik} = \begin{cases} rl + 1, & \text{edge } e_i \in \text{loop } l_k, \text{ same orientation}, \\ -1, & \text{edge } e_i \in \text{loop } l_k, \text{ reverse orientation}, \\ 0, & \text{otherwise}. \end{cases}$$

This depends on the choice of an orientation of the edges and of a basis of loops, *i.e.* a basis of $H_1(\Gamma)$. The equations imposing the conservation laws for momenta at each vertex, together with the constraint $\sum_i t_i u_i \eta_{ir} = 0$ determine uniquely u_i as functions of the external momenta p and give

$$\sum_i t_i u_i^2 = p^\dagger R_\Gamma(t) p,$$

where $R_\Gamma(t)$ is a function defined in terms of the combinatorics of the graph. Thus, one rewrites the Feynman integral after this change of coordinates in the form

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2} V_\Gamma(t, p)^{n - D\ell/2}}, \tag{12}$$

where ω_n is the volume form and the domain of integration is the simplex $\sigma_n = \{t \in \mathbb{R}_+^n | \sum_i t_i = 1\}$. In the *massless case* (with $m = 0$) the term $V_\Gamma(t, p) = p^\dagger R_\Gamma(t) p + m^2$ is of the form

$$V_\Gamma(t, p)|_{m=0} = \frac{P_\Gamma(t, p)}{\Psi_\Gamma(t)},$$

where $P_\Gamma(t, p)$ is a homogeneous polynomial of degree $b_1(\Gamma) + 1$ in t, defined in terms of the cut-sets of the graph (complements of spanning tree plus one edge),

$$P_\Gamma(t, p) = \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e,$$

with $s_C = \left(\sum_{v \in V(\Gamma_1)} P_v\right)^2$ and $P_v = \sum_{e \in E_{\mathrm{ext}}(\Gamma), t(e) = v} p_e$, where the momenta satisfy the conservation law $\sum_{e \in E_{\mathrm{ext}}(\Gamma)} p_e = 0$. The *graph polynomial* $\Psi_\Gamma(t)$ is a homogeneous polynomial of degree $b_1(\Gamma)$ given by

$$\Psi_\Gamma(t) = \det M_\Gamma(t) = \sum_T \prod_{e \notin T} t_e,$$

with the sum over spanning trees of Γ, and the matrix

$$(M_\Gamma)_{kr}(t) = \sum_{i=0}^n t_i \eta_{ik} \eta_{ir}.$$

Notice how the determinant of this matrix is independent both of the choice of an orientation of the edges and of a basis of $H_1(\Gamma)$. Similarly, in the case where $m \neq 0$ but with external momenta $p = 0$ one has

$$V_\Gamma(t, p)|_{m \neq 0, p=0} = \frac{m^2}{\Psi_\Gamma(t)}.$$

After dimensional regularization the parametric Feynman integral can be rewritten as

$$U_\mu(\Gamma)(z) = \mu^{-z\ell} \frac{\Gamma(n - \frac{(D+z)\ell}{2})}{(4\pi)^{\frac{\ell(D+z)}{2}}} \int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{\frac{(D+z)}{2}} V_\Gamma(t,p)^{n-\frac{(D+z)\ell}{2}}} .$$

Assume for simplicity that we work in the "stable range" of dimensions D such that $n \leq D\ell/2$, so that we write the integral $U(\Gamma, p)$, up to a divergent Γ-factor, in the form

$$\int_{\sigma_n} \frac{P_\Gamma(p,t)^{-n+D\ell/2}}{\Psi_\Gamma(t)^{-n+(\ell+1)D/2}} \, \omega_n. \tag{13}$$

The integrand is an algebraic differential form on the complement of the hypersurface

$$\hat{X}_\Gamma = \{t = (t_1, \ldots, t_n) \in \mathbb{A}^n \mid \Psi_\Gamma(t) = 0\}. \tag{14}$$

Since the polynomial is homogeneous, one can also consider the projective hypersurface

$$X_\Gamma = \{t = (t_1 : \cdots : t_n) \in \mathbb{P}^{n-1} \mid \Psi_\Gamma(t) = 0\}. \tag{15}$$

Moreover, the domain of integration is the simplex σ_n with boundary $\partial\sigma_n$ contained in the normal crossings divisor $\hat{\Sigma}_n = \left\{t \in \mathbb{A}^n \mid \prod_i t_i = 0\right\}$. Thus, as we discuss briefly below, if the integral converges, it defines a period of the hypersurface complement. The integral in general is still divergent, even if we have already removed a divergent Γ-factor (hence we are considering the residue of the Feynman graph $U(\Gamma)$). The divergences of (13) come from the intersections $\hat{\Sigma}_n \cap \hat{X}_\Gamma \neq \emptyset$. We discuss later how one can treat these divergences.

It is worth pointing out here that the varieties X_Γ are in general *singular* hypersurfaces, with a singularity locus that is often of low codimension. This can be seen easily by observing that the varieties defined by the derivatives of the graph polynomial are in turn cones over graph hypersurfaces of smaller graphs and that these cones do not intersect transversely. Techniques from singularity theory can be employed to estimate how singular these varieties typically are. Notice how, from the motivic viewpoint, the fact that they are highly singular is what makes it possible for many of these varieties (and possibly always for a certain part of their cohomology), to be sufficiently "simple" as motives, *i.e.* mixed Tate. This would certainly not be the case if we were dealing with smooth hypersurfaces. So the understanding of the singularities of these varieties may play a useful role in the conjectures on Feynman integrals and motives.

The parametric representation of Feynman integrals and its relation to the algebraic geometry of the graph hypersurfaces was generalized to theories with bosonic and fermionic fields in [51] where the analogous result is obtained in the form of an integration of a Berezinian on a supermanifold.

1.5. Algebraic varieties and motives. The other main objects involved in the conjecture on Feynman integrals and periods are *motives*. These are the focus of a deep chapter of arithmetic algebraic geometry, still in itself very much at the center of recent investigations in the field. Roughly speaking, motives are a universal cohomology theory for algebraic varieties, or, to say it differently, a way to embed the category of varieties into a better (triangulated, abelian, Tannakian) category.

Let $\mathcal{V}_{\mathbb{K}}$ denote the category of smooth projective algebraic varieties over a field $\mathbb{K}$. For our purposes, we may assume that $\mathbb{K}$ is $\mathbb{Q}$ or a number field. The category $\mathcal{M}_{\mathbb{K}}$ of pure motives (with the numerical equivalence relation on algebraic cycles) is defined as having objects given by triples (X, p, m) of a smooth projective variety X, a projector $p = p^2 \in \mathrm{End}(X)$, and an integer $m \in \mathbb{Z}$. The morphisms extend the usual notion of morphism of varieties, by allowing also *correspondences*, that is, algebraic cycles in the product $X \times Y$. A morphism in the usual sense is represented by the cycle given by its graph in $X \times Y$. More precisely, one has

$$\mathrm{Hom}((X, p, m), (Y, q, n)) = q\mathrm{Corr}^{m-n}_{/\sim}(X, Y)\, p,$$

for projectors $p^2 = p$, $q^2 = q$, and where $\mathrm{Corr}^{m-n}(X, Y)$ means the abelian group or vector space of cycles in $X \times Y$ of codimension equal to $\dim(X) - m + n$ and $\sim$ is the numerical equivalence relation on cycles (two cycles are the same if they have the same intersection numbers with any cycle of complementary dimension).

One defines the Tate motives $\mathbb{Q}(m)$ by formally setting $\mathbb{Q}(1) = \mathbb{L}^{-1}$, the inverse of the Lefschetz motive (the motive of an affine line) and $\mathbb{Q}(m) = \mathbb{Q}(1)^m$, with $\mathbb{Q}(0)$ the motive of a point, so that $(X, p, m) = (X, p) \otimes \mathbb{Q}(m)$. The reason for introducing these new objects in the category of motives is to allow for cycles of varying codimension: this makes it possible to have a duality $(X, p, m)^\vee = (X, p^t, -m)$ and a rigid tensor structure on the category $\mathcal{M}_{\mathbb{K}}$. It is known that, with the numerical equivalence on cycles, $\mathcal{M}_{\mathbb{K}}$ is an abelian category and it is in fact Tannakian. Since it is a semisimple category, its Tannakian Galois group (the motivic Galois group) is reductive. The subcategory generated by the $\mathbb{Q}(m)$ is the category of pure Tate motives, whose motivic Galois group is $\mathbb{G}_m$. (See [5], [41], [47].)

The situation becomes considerably more complicated when the varieties considered are not smooth projective, for instance, when one wants to include singular varieties, as is necessarily the case in relation to quantum field theory, since we have seen that the X_Γ are usually singular varieties. In this case, the theory of motives is not as well understood as in the pure case. Mixed motives, the theory of motives that accounts for these more general types of varieties, are known to form a triangulated category $\mathcal{DM}_{\mathbb{K}}$, by work of Voevodsky, Levine, Hanamura [45], [59]. Distinguished triangles in this triangulated category of motives correspond to long exact sequences in cohomology of the form

$$\mathfrak{m}(Y) \longrightarrow \mathfrak{m}(X) \longrightarrow \mathfrak{m}(X \smallsetminus Y) \longrightarrow \mathfrak{m}(Y)[1]$$

in the case of closed embeddings $Y \subset X$. Moreover, one has a homotopy invariance property expressed by the identity

$$\mathfrak{m}(X \times \mathbb{A}^1) = \mathfrak{m}(X)(1)[2].$$

However, in general one does not have an abelian category. The subcategory $\mathcal{DMT}_{\mathbb{K}} \subset \mathcal{DM}_{\mathbb{K}}$ of mixed Tate motives is the triangulated subcategory generated by the $\mathbb{Q}(m)$. In the case where $\mathbb{K}$ is a number field, it is known (see [45]) that one has a t-structure on $\mathcal{DMT}_{\mathbb{K}}$ whose heart defines an abelian category $\mathcal{MT}_{\mathbb{K}}$ of mixed Tate motives. It is in fact a Tannakian category (see [32]), whose Galois group is of the form $U \rtimes \mathbb{G}_m$, where the reductive part $\mathbb{G}_m$ accounts for the presence of the pure Tate motives among the mixed ones, while U is a pro-unipotent affine group scheme which accounts for the nontrivial extensions between pure Tate motives.

More concretely, examples of mixed Tate motives are given for instance by algebraic varieties that admit a stratification where all the strata are built out of locally trivial fibrations of affine spaces. We will discuss some explicit examples of this sort below, in the context of quantum field theory.

While explicitly constructing objects in $\mathcal{MT}_{\mathbb{K}}$ or checking whether given varieties that define objects in $\mathcal{DM}_{\mathbb{K}}$ are actually mixed Tate, *i.e.* whether they give objects in $\mathcal{DMT}_{\mathbb{K}}$ or $\mathcal{MT}_{\mathbb{K}}$, may in general be very difficult, there is an easier way to check the motivic nature of a variety X by looking at its class in the Grothendieck ring of varieties $K_0(\mathcal{V}_{\mathbb{K}})$. This is generated by isomorphism classes $[X]$, subject to the inclusion-exclusion relation $[X] = [Y] + [X \smallsetminus Y]$ for closed embeddings $Y \subset X$ and with the product given by $[X][Y] = [X \times Y]$.

The class in the Grothendieck ring can be thought of as a *universal Euler characteristic* for algebraic varieties, [11]. In fact, additive invariants of varieties, *i.e.* invariants with values in a commutative ring R which satisfy $\chi(X) = \chi(Y)$ if $X \cong Y$ are isomorphic varieties, $\chi(X) = \chi(Y) + \chi(X \smallsetminus Y)$, for closed embeddings $Y \subset X$, and are compatible with products, $\chi(X \times Y) = \chi(X)\chi(Y)$, correspond to ring homomorphisms $\chi\colon K_0(\mathcal{V}) \to R$. Examples of additive invariants are the usual Euler characteristic, or the motivic Euler characteristic of Gillet–Soulé [35], $\chi\colon K_0(\mathcal{V}_{\mathbb{K}}) \to K_0(\mathcal{M}_{\mathbb{K}})$ with values in the Grothendieck ring of the category of motives, defined on projective varieties by $\chi(X) = [(X, \mathrm{id}, 0)]$ and on more general varieties in terms of a complex in the category of complexes over $\mathcal{M}_{\mathbb{K}}$.

If one denotes by $\mathbb{L} = [\mathbb{A}^1] \in K_0(\mathcal{V}_{\mathbb{K}})$ the Lefschetz motive, then the part of $K_0(\mathcal{V}_{\mathbb{K}})$ generated by the Tate motives is a polynomial ring $\mathbb{Z}[\mathbb{L}]$ (or $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}]$ after formally inverting the Lefschetz motive in $K_0(\mathcal{M}_{\mathbb{K}})$). Checking that the class $[X]$ of a variety X lies in this subring gives strong evidence for X being a mixed Tate motive. It may seem that a lot of information is lost in passing from objects in $\mathcal{DM}_{\mathbb{K}}$ to classes in $K_0(\mathcal{V}_{\mathbb{K}})$, since this ring does not retain the information on the extensions but only keeps the rough information on scissor relations. However, at least modulo standard conjectures on motives, knowing that the class $[X]$ lies in the Tate subring

$\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}]$ of $K_0(\mathcal{M}_{\mathbb{K}})$ should in fact suffice to know that the motive is mixed Tate. In any case, computing in $K_0(\mathcal{V}_{\mathbb{K}})$ provides a lot of useful information on the motivic nature of given varieties.

One last thing that we need to recall briefly is the notion of period, as in [43]. A period is a complex number that can be obtained by pairing via integration

$$(\omega, \sigma) \longmapsto \int_\sigma \omega$$

an algebraic differential form $\omega \in \Omega^{\dim X}(X)$ on an algebraic variety X defined over a number field $\mathbb{K}$ with a cycle σ defined by semi-algebraic relations (equalities and inequalities) also defined over the same field $\mathbb{K}$. If the domain of integration σ has boundary $\partial\sigma \neq 0$, then the period should be thought of as a pairing with a relative homology group

$$\sigma \in H_{\dim X}(X(\mathbb{C}), \Sigma(\mathbb{C})),$$

where Σ is a divisor in X containing the boundary of σ. It is conjectured in [43] that the only relations between periods arise from the change of variable and Stokes formulae for integrals.

1.6. The mixed Tate mystery: supporting evidence. The main conjecture on the relation between quantum fields and motives can be formulated as follows.

Conjecture 1.1. *Are residues of Feynman integrals in scalar field theories always periods of mixed Tate motives?*

Here "residues" refers to the removal of the divergent Gamma factor in (12). Notice that, in general, the remaining integral still contains divergences that need to be removed by a renormalization procedure. Thus, implicit in the above conjecture is also an independence of the regularization and renormalization scheme used to eliminate divergences.

The supporting evidence for this conjecture starts from extensive numerical computations of Feynman integrals collected by Broadhurst and Kreimer [22], which showed the pervasive presence of zeta and multiple zeta values. This first suggested the fact that mixed Tate motives may be involved in this computation, in view of the fact that multiple zeta values are periods of mixed Tate motives, according to [36], [55].

Modulo the serious issue of divergences, the use of Schwinger and Feynman parameters expresses Feynman integrals as integrations of an algebraic differential form on the complement of a hypersurface X_Γ in affine space defined by a homogeneous polynomial depending on the combinatorics of the graph.

Kontsevich formulated the conjecture that the graph hypersurfaces X_Γ themselves may always be mixed Tate motives, which would imply Conjecture 1.1. Although numerically this conjecture was at first verified up to a large number of loops, Belkale and Brosnan [9] later disproved the conjecture in general, showing that in fact the X_Γ can be arbitrarily complicated as motives: they proved that the X_Γ generate the Grothendieck ring of varieties. This, however, does not disprove Conjecture 1.1. In fact, even though the varieties themselves may be more complicated as motives, the part of the cohomology that is involved in the computation of the period may still be a realization of a mixed Tate motive.

More evidence for the fact that the cohomology involved, that is the relative cohomology $H^{n-1}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus (\Sigma_n \cap X_\Gamma))$, where Σ_n denotes the union of the coordinate hyperplanes, is a realization of a mixed Tate motive was collected by Bloch–Esnault–Kreimer, [16], [13].

More recently, the question has been reformulated by Aluffi–Marcolli [4] in terms of a different relative cohomology involving determinant hypersurfaces and the motives of *varieties of frames*, which gives further evidence for the conjecture, as we explain below. A different kind of evidence comes from the approach followed in the work of Connes–Marcolli [27], where instead of constructing motives for specific Feynman graphs, one compares the "global" properties of the Tannakian category $\mathcal{MT}_{\mathbb{K}}$ with a similar category constructed out of the data of perturbative renormalization, the Tannakian category of *flat equisingular vector bundles*. Although one obtains in this way only a non-canonical identification between these Tannakian categories, it adds evidence to the conjectured relation between perturbative renormalization and mixed Tate motives.

We give in the following a general overview of these different methods and results.

2. A bottom-up approach to Feynman integrals and motives

With these preliminaries in place, we are now ready to discuss more closely the two different approaches to the relation of quantum field theory and motives. We first introduce what we refer to as a "bottom-up" approach, in the sense that it deals with the problem on a graph-by-graph basis and tries, for individual graphs or families of graphs sharing similar combinatorial properties, to construct explicit associated motives and periods computing the Feynman integrals. This approach was pioneered by the work of Bloch–Esnault–Kreimer [16] and further developed in [13], [17]. Here I will concentrate mostly on my recent joint work with Aluffi [2], [3], [4].

As we have mentioned above, the parametric formulation of Feynman integrals shows that, modulo divergences, they can be written as periods on the hypersurface complement $\mathbb{A}^n \smallsetminus \hat{X}_\Gamma$, with $n = \#E_{\mathrm{int}}(\Gamma)$. One can reformulate the integral in the projective setting. Then the question of whether the period so computed is a period of a mixed Tate motive can be reformulated as in [16] as the question of whether the relative cohomology

$$H^{n-1}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus X_\Gamma \cap \Sigma_n) \tag{16}$$

is the realization of a mixed Tate motive

$$\mathfrak{m}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus X_\Gamma \cap \Sigma_n), \tag{17}$$

where $\Sigma_n = \left\{t \in \mathbb{P}^{n-1} \mid \prod_i t_i = 0\right\}$ is a normal crossings divisor containing $\partial\sigma_n$, the boundary of the domain of integration.

This leads to the question of how complex, in motivic terms, the graph hypersurfaces X_Γ can be. Clearly, if it were to be the case that these would always be mixed Tate as motives, then the conjecture on the nature of the period would follow easily. However, this is known not to be the case, as we already mentioned above: it is known by [9] that the classes $[X_\Gamma]$ generate the Grothendieck ring of varieties, hence they cannot all be contained in the Tate subring $\mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V})$. The question remains, however, on whether the particular piece (16) may nonetheless be always mixed Tate even when the variety X_Γ itself may turn out to be more complicated.

One can exhibit explicit examples of computations of classes $[X_\Gamma]$ in the Grothendieck ring. A useful method to obtain information on these classes is the observation, made in [13] and used extensively in [2], [15], that the classical Cremona transformation relates the graph hypersurfaces of a planar graph and its dual graph.

In fact, if Γ is a planar graph and $\Gamma^\vee$ denotes the dual graph in a chosen embedding of Γ, the graph polynomials are related by

$$\Psi_\Gamma(t_1, \dots, t_n) = \Big(\prod_e t_e\Big)\Psi_{\Gamma^\vee}(t_1^{-1}, \dots, t_n^{-1}).$$

This means that the graph hypersurfaces have the property that

$$\mathcal{C}(X_\Gamma \cap (\mathbb{P}^{n-1} \smallsetminus \Sigma_n)) = X_{\Gamma^\vee} \cap (\mathbb{P}^{n-1} \smallsetminus \Sigma_n),$$

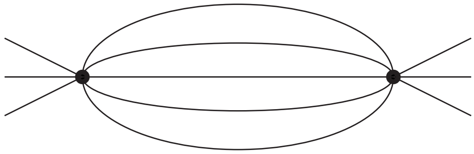
under the Cremona transformation. The latter is defined as

$$\mathcal{C} \colon (t_1 : \dots : t_n) \longmapsto \Big(\frac{1}{t_1} : \dots : \frac{1}{t_n}\Big),$$

which is well defined outside the singularity locus $\mathcal{S}_n$ of Σ_n defined by the ideal $I_{\mathcal{S}_n} = (t_1 \dots t_{n-1}, t_1 \dots t_{n-2}t_n, \dots, t_1t_3 \dots t_n)$. Notice that this relation only gives an isomorphism of the parts of X_Γ and $X_{\Gamma^\vee}$ that lie outside of Σ_n.

For example, using this method, an explicit formula for the classes $[X_{\Gamma_n}]$ of the hypersurfaces of the infinite family of so called "banana graphs" were computed in [2]. The banana graphs have graph polynomial

$$\Psi_\Gamma(t) = t_1 \dots t_n\Big(\frac{1}{t_1} + \dots + \frac{1}{t_n}\Big).$$



The parametric integral in this case is

$$\int_{\sigma_n} \frac{(t_1 \dots t_n)^{(\frac{D}{2}-1)(n-1)-1}\, \omega_n}{\Psi_\Gamma(t)^{(\frac{D}{2}-1)n}}.$$

One has in this case ([2]) that the class in the Grothendieck ring is of the form

$$[X_{\Gamma_n}] = \frac{\mathbb{L}^n - 1}{\mathbb{L} - 1} - \frac{(\mathbb{L}-1)^n - (-1)^n}{\mathbb{L}} - n\,(\mathbb{L}-1)^{n-2},$$

so it is manifestly mixed Tate. In fact, in this case the dual graph $\Gamma^\vee$ is just a polygon, so that $X_{\Gamma^\vee} = \mathcal{L}$ is a hyperplane in $\mathbb{P}^{n-1}$. One has

$$[\mathcal{L} \smallsetminus \Sigma_n] = [\mathcal{L}] - [\mathcal{L} \cap \Sigma_n] = \frac{\mathbb{T}^{n-1} - (-1)^{n-1}}{\mathbb{T}+1}$$

where $\mathbb{T} = [\mathbb{G}_m] = [\mathbb{A}^1] - [\mathbb{A}^0]$ is the class of the multiplicative group. Moreover, one finds that $X_{\Gamma_n} \cap \Sigma_n = \mathcal{S}_n$ and the scheme of singularities of Σ_n has class

$$[\mathcal{S}_n] = [\Sigma_n] - n\mathbb{T}^{n-2}.$$

This then gives

$$[X_{\Gamma_n}] = [X_{\Gamma_n} \cap \Sigma_n] + [X_{\Gamma_n} \smallsetminus \Sigma_n],$$

where one uses the Cremona transformation to identify $[X_{\Gamma_n}] = [\mathcal{S}_n] + [\mathcal{L} \smallsetminus \Sigma_n]$.

In particular this calculation yields a value for the Euler characteristic of X_{Γ_n}, of the form $\chi(X_{\Gamma_n}) = n + (-1)^n$. A different computation of the Euler characteristic based on characteristic classes of singular varieties is also given in [2].

A very interesting observation recently made in [15] is that, although individually the varieties of Feynman graphs may not be mixed Tate, as the result of [9] shows, cancellations happen when one sums over graphs and one ends up with a class in $\mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V}_{\mathbb{K}})$. More precisely, it is shown in [15] that the class

$$S_N = \sum_{\#V(\Gamma)=N} [X_\Gamma] \frac{N!}{\#\mathrm{Aut}(\Gamma)}$$

is in $\mathbb{Z}[\mathbb{L}]$. This is in agreement with the fact that in quantum field theory individual Feynman graphs do not represent observable physical processes and only sums over graphs, usually with fixed external edges and external momenta, can be physically meaningful. This result suggests that a more appropriate formulation of the conjecture on Feyman integrals and motives may perhaps be given directly in terms that involve the full expansion of perturbative quantum field theory, with sums over graphs, rather than in terms of individual graphs. As we are going to see below, this also fits in naturally with the other, "top-down" approach to relating Feynman integrals to motives that we discuss in the second half of this paper.

2.1. Feynman rules in algebraic geometry. The graph hypersurfaces have another interesting property, namely the hypersurface complements behave like Feynman rules. This was first observed and described in detail in the work [3], but we summarize it here briefly.

As we recalled above, Feynman rules have certain multiplicative properties that makes it possible to reduce the combinatorics of graphs from arbitrary finite graphs to connected and then 1PI graphs, namely the properties listed in (9) and (11). When working in affine space, one has

$$\mathbb{A}^{n_1+n_2} \smallsetminus \hat{X}_\Gamma = (\mathbb{A}^{n_1} \smallsetminus \hat{X}_{\Gamma_1}) \times (\mathbb{A}^{n_2} \smallsetminus \hat{X}_{\Gamma_2}),$$

for a graph Γ that is a disjoint union $\Gamma = \Gamma_1 \amalg \Gamma_2$. This follows immediately from the fact that the graph polynomial factors as

$$\Psi_\Gamma(t_1, \dots, t_n) = \Psi_{\Gamma_1}(t_1, \dots, t_{n_1}) \Psi_{\Gamma_2}(t_{n_1+1}, \dots, t_{n_1+n_2}).$$

In projective space, this would no longer be the case and one has a more complicated relation in terms of *joins* instead of products of varieties, which gives a fibration

$$\mathbb{P}^{n_1+n_2-1} \smallsetminus X_\Gamma \longrightarrow (\mathbb{P}^{n_1-1} \smallsetminus X_{\Gamma_1}) \times (\mathbb{P}^{n_2-1} \smallsetminus X_{\Gamma_2})$$

which is a $\mathbb{G}_m$-bundle (assuming that Γ_i not a forest, else the above map in projective spaces would not be well defined). Notice that the classes of the affine and the projective hypersurface complements are related by ([3])

$$[\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] = (\mathbb{L} - 1)[\mathbb{P}^{n-1} \smallsetminus X_\Gamma],$$

when Γ is not a forest, since $[\hat{X}_\Gamma] = (\mathbb{L} - 1)[X_\Gamma] + 1$ is the class of the affine cone $\hat{X}_\Gamma$ over X_Γ.

One can then work either with the Grothendieck ring $K_0(\mathcal{V}_\mathbb{K})$ (in which case one can talk of *motivic Feynman rules*), or with a more refined version where one does not identify varieties up to isomorphisms but only up to linear coordinate changes coming from embeddings in some ambient affine space $\mathbb{A}^N$. This version of Grothendieck ring was introduced in [3] under the name of *ring of immersed conical varieties* $\mathcal{F}_\mathbb{K}$. It is generated by classes $[V]$ of equivalence under linear coordinate changes of varieties

$V \subset \mathbb{A}^N$ (for some arbitrarily large N) defined by homogeneous ideals (hence the name "conical"), with the usual inclusion-exclusion and product relations

$$[V \cup W] = [V] + [W] - [V \cap W],$$

$$[V] \cdot [W] = [V \times W].$$

By imposing equivalence under isomorphisms one falls back on the usual Grothendieck ring $K_0(\mathcal{V})$. The reason for working with $\mathcal{F}_{\mathbb{K}}$ instead is that it allowed us in [3] to construct invariants of the graph hypersurfaces that behave like algebro-geometric Feynman rules and that measure to some extent how singular these varieties are, and which do not factor through the Grothendieck ring, since they contain specific information on how the $\hat{X}_\Gamma$ are embedded in the ambient affine space $\mathbb{A}^{\#E_{\mathrm{int}}(\Gamma)}$.

In general, one defines an $\mathcal{R}$-valued algebro-geometric Feynman rule, for a given commutative ring $\mathcal{R}$, as in [3] in terms of a ring homomorphism $I : \mathcal{F} \to \mathcal{R}$ by setting

$$\mathbb{U}(\Gamma) := I([\mathbb{A}^n]) - I([\hat{X}_\Gamma])$$

and by taking as value of the *inverse propagator*

$$\mathbb{U}(L) = I([\mathbb{A}^1]).$$

This then satisfies both (9) and (11). The ring $\mathcal{F}$ then is the receptacle of the universal algebro-geometric Feynman rule given by

$$\mathbb{U}(\Gamma) = [\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] \in \mathcal{F}.$$

A Feynman rule defined in this way is *motivic* if the homomorphism $I : \mathcal{F} \to \mathcal{R}$ factors through the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{K}})$.

An example of algebro-geometric Feynman rule that does not factor through $K_0(\mathcal{V}_{\mathbb{K}})$ was constructed in [3] using the theory of characteristic classes of singular varieties.

In the case of smooth varieties, one knows that the Chern classes of the tangent bundle can be written as a class $c(V) = c(TV) \cap [V]$ in homology whose degree of the zero dimensional component satisfies the Poincaré–Hopf theorem $\int c(TV) \cap [V] = \chi(V)$, which gives the topological Euler characteristic of the smooth variety. This was generalized to singular varieties, following two different approaches that then turned out to be equivalent, by Marie-Hélène Schwartz [54] and Robert MacPherson [46]. The approach followed by Schwartz generalized the definition of Chern classes as the homology classes of the loci where a family of $k+1$-vector fields become linearly dependent (for the lowest degree case one reads the Poincaré–Hopf theorem as saying that the Euler characteristic measures where a single vector field has zeros). In the case of singular varieties a generalization is obtained, provided that one assigns some radial conditions on the vector fields with respect to a stratification with good properties. The approach of MacPherson was instead based on functoriality: a conjecture of Grothendieck–Deligne stated that there should be a unique natural transformation c_* between the functor $\mathbb{F}(V)$ of constructible functions on a variety V, whose objects are linear combinations of characteristic classes 1_W of subvarieties $W \subset V$ and where morphisms are defined by the prescription $f_*(1_W) = \chi(W \cap f^{-1}(p))$, with χ the Euler characteristic, to the homology (or Chow group) functor, which in the smooth case agrees with $c_*(1_V) = c(TV) \cap [V]$. MacPherson constructed this natural transformation in terms of data of Mather classes and local Euler obstructions. The results of Aluffi [1] show that, in fact, it is possible to compute these classes without having to use the original definition and the local data that are usually very difficult to compute. Most notably, the resulting characteristic classes (denoted $c_{\mathrm{CSM}}(X)$ for Chern–Schwartz–MacPherson) satisfy an inclusion–exclusion formula

$$c_{\mathrm{CSM}}(X) = c_{\mathrm{CSM}}(Y) + c_{\mathrm{CSM}}(X \smallsetminus Y),$$

but are not invariant under isomorphism, hence they are naturally defined on classes in $\mathcal{F}_{\mathbb{K}}$ but not on $K_0(\mathcal{V}_{\mathbb{K}})$. This classes give a good information on the singularities of a variety: for example, in the case of hypersurfaces with isolated singularities, they can be expressed in terms of Milnor numbers, while more generally for non-isolated singularities, as observed by Aluffi, they can be expressed in terms of Euler characteristics of varieties obtained by repeatedly taking hyperplane sections.

To construct a Feynman rule out of these Chern classes, one uses the following procedure. Given a variety $\widehat{X} \subset \mathbb{A}^N$, one can view it as a locally closed locus in $\mathbb{P}^N$, hence one can apply to its characteristic function $1_{\widehat{X}}$ the natural transformation c_* that gives an element in the Chow group $A(\mathbb{P}^N)$ or in the homology $H_*(\mathbb{P}^N)$. This gives as a result a class of the form

$$c_*(1_{\widehat{X}}) = a_0[\mathbb{P}^0] + a_1[\mathbb{P}^1] + \cdots + a_N[\mathbb{P}^N].$$

One then defines an associated polynomial given by ([3])

$$G_{\widehat{X}}(T) := a_0 + a_1 T + \cdots + a_N T^N.$$

It is in fact independent of N as it stops in degree equal to $\dim \widehat{X}$. It is by construction invariant under linear changes of coordinates. It also satisfies an inclusion-exclusion property coming from the fact that the classes c_{CSM} satisfy inclusion-exclusion, namely

$$G_{\widehat{X} \cup \widehat{Y}}(T) = G_{\widehat{X}}(T) + G_{\widehat{Y}}(T) - G_{\widehat{X} \cap \widehat{Y}}(T).$$

It is a more delicate result to show that it is multiplicative,

$$G_{\widehat{X} \times \widehat{Y}}(T) = G_{\widehat{X}}(T) \cdot G_{\widehat{Y}}(T).$$

The proof of this fact is obtained in [3] using an explicit formula for the CSM classes of joins in projective spaces, where the join $J(X,Y) \subset \mathbb{P}^{m+n-1}$ of two $X \subset \mathbb{P}^{m-1}$ and $Y \subset \mathbb{P}^{n-1}$ is defined as the set of

$$(sx_1 : \cdots : sx_m : ty_1 : \cdots : ty_n), \quad \text{with } (s:t) \in \mathbb{P}^1,$$

and is related to product in affine spaces by the property that the product $\widehat{X} \times \widehat{Y}$ of the affine cones over X and Y is the affine cone over $J(X,Y)$. The resulting multiplicative property of the polynomials $G_{\widehat{X}}(T)$ shows that one has a ring homomorphism $I_{\mathrm{CSM}}\colon \mathcal{F} \to \mathbb{Z}[T]$ defined by

$$I_{\mathrm{CSM}}([\widehat{X}]) = G_{\widehat{X}}(T)$$

and an associated Feynman rule

$$\mathbb{U}_{\mathrm{CSM}}(\Gamma) = C_\Gamma(T) = I_{\mathrm{CSM}}([\mathbb{A}^n]) - I_{\mathrm{CSM}}([\widehat{X}_\Gamma]).$$

This is not motivic, *i.e.* it does not factor through the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{K}})$, as can be seen by the example given in [3] of two graphs (see the figure below) that have different $\mathbb{U}_{\mathrm{CSM}}(\Gamma)$,

$$C_{\Gamma_1}(T) = T(T+1)^2 \quad C_{\Gamma_2}(T) = T(T^2+T+1)$$

but the same hypersurface complement class in the Grothendieck ring,

$$[\mathbb{A}^n \smallsetminus \widehat{X}_{\Gamma_i}] = [\mathbb{A}^3] - [\mathbb{A}^2] \in K_0(\mathcal{V}).$$

2.2. Determinant hypersurfaces and manifolds of frames. As our excursion into the algebraic geometry of graph hypersurfaces up to this point shows, it seems very difficult to control the complexity of the motive

$$\mathfrak{m}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus X_\Gamma \cap \Sigma_n)$$

that governs the computation of the parametric Feynman integral as a period.

One way to try to estimate whether the period remains mixed Tate, as the complexity of the X_Γ grows, is to use the properties of periods, in particular the change of variable formula, which allows one to recast the computation of the same integral $\int_\sigma \omega$ associated to the data (X, D, ω, σ) of a variety X, a divisor D, a differential form ω on X, and an integration domain σ with boundary $\partial\sigma \subset D$, by mapping it via a morphism f of varieties to another set of data $(X', D', \omega', \sigma')$, with the same resulting period whenever $\omega = f^*(\omega')$ and $\sigma' = f_*(\sigma)$. In other words, we try to map the variety X_Γ inside a larger ambient variety in such a way that the part of the cohomology that is involved in the period computation will not disappear, but the motivic complexity of the new ambient space will be easier to control. This is the strategy that we followed in [4], which I will briefly describe here.

The matrix $M_\Gamma(t)$ associated to a Feynman graph Γ determines a linear map of affine spaces

$$\Upsilon\colon \mathbb{A}^n \to \mathbb{A}^{\ell^2}, \quad \Upsilon(t)_{kr} = \sum_i t_i \eta_{ik}\eta_{ir}$$

such that the affine graph hypersurface is obtained as the preimage

$$\hat{X}_\Gamma = \Upsilon^{-1}(\hat{\mathcal{D}}_\ell)$$

under this map of the determinant hypersurface

$$\hat{\mathcal{D}}_\ell = \{x = (x_{ij}) \in \mathbb{A}^{\ell^2} \mid \det(x_{ij}) = 0\}.$$

The advantage of moving the period computation via the map $\Upsilon = \Upsilon_\Gamma$ from the hypersurface complement $\mathbb{A}^n \smallsetminus \hat{X}_\Gamma$ to the complement of the determinant hypersurface $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$ is that, unlike what happens with the graph hypersurfaces, it is well known that the determinant hypersurface $\hat{\mathcal{D}}_\ell$ is a mixed Tate motive.

One can give explicit combinatorial conditions on the graph that ensure that the map Υ is an embedding. As shown in [4], for any 3-edge-connected graph with at least 3 vertices and no looping edges, which admit a closed 2-cell embedding of face width at least 3, the map Υ is injective. These combinatorial conditions are natural from a physical viewpoint. In fact, 2-edge-connected is just the usual 1PI condition, while 3-edge-connected or 2PI is the next strengthening of this condition (the 2PI effective action is often considered in quantum field theory), and the face width condition is also the next strengthening of face width 2, which a well known combinatorial conjecture on graphs [52] expects should simply follow for graphs that are 2-vertex-connected. (The latter condition is a bit more than 1PI: for graphs with at least two vertices and no looping edges it is equivalent to all the splittings of the graph at vertices also being 1PI.) The conditions that the graph has no looping edges is only a technical device for the proof. In fact, it is then easy to show (see [4]) that adding looping edge does not affect the injectivity of the map Υ.

One can then rewrite the Feynman integral (as usual up to a divergent Γ-factor) in the form

$$U(\Gamma) = \int_{\Upsilon(\sigma_n)} \frac{\mathcal{P}_\Gamma(x,p)^{-n+D\ell/2}\omega_\Gamma(x)}{\det(x)^{-n+(\ell+1)D/2}},$$

for a polynomial $\mathscr{P}_\Gamma(x,p)$ on $\mathbb{A}^{\ell^2}$ that restricts to $P_\Gamma(t,p)$, and with $\omega_\Gamma(x)$ the image of the volume form. Let then $\hat{\Sigma}_\Gamma$ be a normal crossings divisor in $\mathbb{A}^{\ell^2}$, which contains the boundary of the domain of integration, $\Upsilon(\partial\sigma_n) \subset \hat{\Sigma}_\Gamma$. The question on the motivic nature of the resulting period can then be reformulated (again modulo divergences) in this case as the question of whether the motive

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathscr{D}}_\ell, \hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathscr{D}}_\ell)) \tag{18}$$

is mixed Tate. One sees immediately that, in this reformulation of the question, the difficulty has been moved from understanding the motivic nature of the hypersurface complement to having some control on the other term of the relative cohomology, namely the normal crossings divisor $\hat{\Sigma}_\Gamma$ and the way it intersects the determinant hypersurface. One would like to have an argument showing that the motive of $\hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathscr{D}}_\ell)$ is always mixed Tate. In that case, knowing that $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathscr{D}}_\ell$ is always mixed Tate, the fact that mixed Tate motives form a triangulated subcategory of the triangulated category of mixed motives would show that the motive (18) whose realization is the relative cohomology would also be mixed Tate. A first observation in [4] is that one can use the same normal crossings divisor $\hat{\Sigma}_{\ell,g}$ for all graphs Γ with a fixed number of loops and a fixed genus (that is, the minimal genus of an orientable surface in which the graph can be embedded). This divisor is given by a union of linear spaces

$$\hat{\Sigma}_{\ell,g} = L_1 \cup \cdots \cup L_{\binom{f}{2}}$$

defined by a set of equations

$$\begin{cases} x_{ij} = 0, & 1 \le i < j \le f-1, \\ x_{i1} + \cdots + x_{i,f-1} = 0, & 1 \le i \le f-1, \end{cases}$$

where $f = \ell - 2g + 1$ is the number of faces of an embedding of the graph Γ on a surface of genus g. A second observation of [4] is then that, using inclusion-exclusion, it suffices to show that arbitrary intersections of the components L_i of $\hat{\Sigma}_{\ell,g}$ have the property that $\left(\bigcap_{i\in I} L_i\right) \smallsetminus \hat{\mathscr{D}}_\ell$ is mixed Tate. A sufficient condition is given in [4] in terms of *manifolds of frames*. These are defined as

$$\mathbb{F}(V_1,\ldots,V_\ell) := \{(v_1,\ldots,v_\ell) \in \mathbb{A}^{\ell^2} \mid v_k \in V_k\}$$

for an assigned collection of linear subspaces V_i of a given vector space $V = \mathbb{A}^{\ell^2}$. If the manifolds of frames are mixed Tate motives for arbitrary choices of the subspaces, then the desired result would follow. One can check explicitly the cases of two and three subspaces, for which one has explicit formulae for the classes $[\mathbb{F}(V_1,\ldots,V_\ell)]$ in the Grothendieck ring:

$$[\mathbb{F}(V_1,V_2)] = \mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2} - \mathbb{L}^{d_{12}+1} + \mathbb{L}^{d_{12}} + \mathbb{L},$$

with $d_i = \dim(V_i)$ and $d_{ij} = \dim(V_i \cap V_j)$, and

$$[\mathbb{F}(V_1, V_2, V_3)] = (\mathbb{L}^{d_1} - 1)(\mathbb{L}^{d_2} - 1)(\mathbb{L}^{d_3} - 1) - (\mathbb{L} - 1)((\mathbb{L}^{d_1} - \mathbb{L})(\mathbb{L}^{d_{23}} - 1) + (\mathbb{L}^{d_2} - \mathbb{L})(\mathbb{L}^{d_{13}} - 1) + (\mathbb{L}^{d_3} - \mathbb{L})(\mathbb{L}^{d_{12}} - 1) + (\mathbb{L} - 1)^2(\mathbb{L}^{d_1+d_2+d_3-D} - \mathbb{L}^{d_{123}+1}) + (\mathbb{L} - 1)^3$$

which also depends on $d_{ijk} = \dim(V_i \cap V_j \cap V_k)$ and $D = D_{ijk} = \dim(V_i + V_j + V_k)$. However, it is difficult to establish an induction argument that would take care of the cases of more subspaces, and the combinatorics of the possible subspace arrangements quickly becomes difficult to control.

A reformulation of this problem given in [4] in terms of intersections of unions of Schubert cells in flag varieties suggests a possible connection to Kazhdan–Lusztig theory [42].

2.3. Handling divergences. So far we did not discuss how one takes care of the divergences caused by the intersections of the graph hypersurface X_Γ with the domain of integration σ_n. The poles of the integrand that fall inside the integration domain happen necessarily along the boundary $\partial\sigma_n$, as in the interior the graph polynomial Ψ_Γ takes strictly positive real values. Thus, one needs to modify the integrals suitably in such a way as to eliminate, by a regularization procedure, the intersections $X_\Gamma \cap \partial\sigma_n$, or (to work in algebro-geometric terms) the intersections $X_\Gamma \cap \Sigma_n$ which contains the former. There are different possible ways to achieve such a regularization procedure. We mention here three possible approaches.

One method was developed by Belkale and Brosnan in [10] in the logarithmically divergent case where $n = D\ell/2$, that is, when the polynomial $P_\Gamma(t, p)$ is not present and only the denominator $\Psi_\Gamma(t)^{D/2}$ appears in the parametric Feynman integral. Using dimensional regularization, one can, in this case, rewrite the Feynman integral in the form of a local Igusa L-function

$$I(s) = \int_\sigma f(t)^s \omega,$$

for $f = \Psi_\Gamma$. They prove that this L-function has a Laurent series expansion where all the coefficients are periods. In this setting, the issue of eliminating divergences becomes similar to the techniques used, for instance, in the context of log canonical thresholds. The result was more recently extended to the non-log-divergent case by Bogner and Weinzierl [18], [19].

Another method, used in [16], consists of eliminating the divergences by separating Σ_n and X_Γ performing a series of blowups. This method based on iterated blowups was investigated in great detail in [17]. Yet another method was proposed in [49], based on deformations instead of resolutions. By considering the graph hypersurface X_Γ as the special fiber X_0 of a family X_s of varieties defined by the level sets

$f^{-1}(s)$, for $f = \Psi_\Gamma : \mathbb{A}^n \to \mathbb{A}^1$, one can form a tubular neighborhood

$$D_\epsilon(X) = \bigcup_{s \in \Delta_\epsilon^*} X_s,$$

for Δ_ϵ^* a punctured disk of radius ϵ, and a circle bundle $\pi_\epsilon : \partial D_\epsilon(X) \to X_\epsilon$. One can then regularize the Feynman integral by integrating "around the singularities" in the fiber $\pi_\epsilon^{-1}(\sigma \cap X_\epsilon)$. The regularized integral has a Laurent series expansion in the parameter ϵ.

In general, as we discuss at length below, a regularization procedure for Feynman integrals replaces a divergent integral with a function of some regularization parameters (such as the complexified dimension of DimReg, or the deformation parameter ϵ in the example here above) in which the resulting function has a Laurent series expansion around the pole that corresponds to the divergent integral originally considered. One then uses a procedure of extraction of finite values to eliminate the polar parts of these Laurent series in a way that is *consistent over graphs*, that is, a renormalization procedure. We therefore turn now to recalling how renormalization can be formulated geometrically, using the results of Connes–Kreimer, as this will be the step relating the "bottom-up" approach to Feynman integrals and motives discussed so far, to the top down approach developed in [27], [28], [29], [30].

3. The Connes–Kreimer theory

We give here a very brief overview of the main results of the Connes–Kreimer theory, as they form the basis upon which the "top-down" approach to understanding the relation between quantum field theory and motives rests. As we see more in detail in the next section, in this context "top-down" means that the relation between quantum fields and motives will appear in this second approach from the comparison of the formal properties of associated abstract categorical structures rather than from a direct comparison of individual objects, as in the approach we have described in the previous sections.

3.1. The BPHZ renormalization procedure. The main steps of what is known in the physics literature as the Bogolyubov–Parashchuk–Hepp–Zimmermann procedure (BPHZ) are summarized as follows. (For more details the reader is invited to look at Chapter 1 on [30]).

Step 1: *Preparation.* One replaces the Feynman integral $U(\Gamma)$ of (6) by the expression

$$\bar{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma) U(\Gamma/\gamma). \tag{19}$$

Here we suppress the dependence on z, μ and the external momenta p for simplicity of notation. The expression (19) is to be understood as a sum of Laurent series in z, depending on the extra parameter μ. The sum is over the set $\mathcal{V}(\Gamma)$ all proper subgraphs $\gamma \subset \Gamma$ with the property that the quotient graph Γ/γ, where each component of γ is shrunk to a vertex, is still a Feynman graph of the theory. The main result of BPHZ is that the coefficient of pole of $\bar{R}(\Gamma)$ is local.

Step 2: *Counterterms*. These are the expressions by which the Lagrangian needs to be modified to cancel the divergence produced by the graph Γ. They are defined as the polar part of the Laurent series $\bar{R}(\Gamma)$,

$$C(\Gamma) = -T(\bar{R}(\Gamma)).$$

Here T denotes the operator of projection onto the polar part of a Laurent series.

Step 3: *Renormalized value*. One then extracts a finite value from the integral $U(\Gamma)$ by removing the polar part, not of $U(\Gamma)$ itself but of its preparation:

$$\begin{aligned} R(\Gamma) &= \bar{R}(\Gamma) + C(\Gamma) \\ &= U(\Gamma) + C(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma) U(\Gamma/\gamma). \end{aligned}$$

A very nice conceptual understanding of the BPHZ renormalization procedure with the DimReg + MS regularization was obtained by Connes and Kreimer [25], [26], based on a reformulation of the BPHZ procedure in geometric terms.

3.2. Renormalization, Hopf algebras, Birkhoff factorization. The first step in the geometric theory of renormalization is the understanding that the combinatorics of Feynman graphs of a given theory is governed by an algebraic structure, which accounts for the bookkeeping of the hierarchy of subdivergences that occur in multi-loop Feynman integrals. The right mathematical structure that describes their interactions is a Hopf algebra. This was first formulated by Kreimer [44] as a Hopf algebra of rooted trees decorated by Feynman diagrams, and then by Connes–Kreimer [25], [26] more directly in the form of a Hopf algebra of Feynman diagrams.

The Connes–Kreimer Hopf algebra ([25]) $\mathcal{H} = \mathcal{H}(\mathcal{T})$ depends on the choice of the physical theory, in the sense that it involves only graphs that are Feynman graphs for the specified Lagrangian $\mathcal{L}(\phi)$. As an algebra it is the free commutative algebra with generators the 1PI Feynman graphs Γ of the theory. It is graded, by loop number, or by the number of internal lines,

$$\deg(\Gamma_1 \dots \Gamma_n) = \sum_i \deg(\Gamma_i), \quad \deg(1) = 0.$$

This grading corresponds to the order in the perturbative expansion.

The coproduct already reveals a close relation to the BPHZ formulae. It is given on generators by

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathcal{V}(\Gamma)} \gamma \otimes \Gamma/\gamma,$$

where the sum is over proper subgraphs $\gamma \subset \Gamma$ in a specific class $\mathcal{V}(\Gamma)$ determined by the property that the quotient graph Γ/γ is still a 1PI Feynman graph of the theory and that γ itself is a disjoint union of 1PI Feynman graphs of the theory. Unlike Γ which is assumed connected, the subgraphs γ can have multiple connected components, in which case the quotient graph Γ/γ is the one obtained by shrinking each component to a single vertex.

The antipode is defined inductively by

$$S(X) = -X - \sum S(X')X'',$$

where X is an element with coproduct $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$, where all the X' and X'' have lower degrees.

We only recalled how the Connes–Kreimer Hopf algebra is constructed for scalar field theories. Recently, van Suijlekom showed [56], [57], [58] how to extend it to gauge theories, incorporating Ward identities as Hopf ideals.

A commutative Hopf algebra $\mathcal{H}$ is dual to an *affine group scheme* G, defined by algebra homomorphisms

$$G(A) = \mathrm{Hom}(\mathcal{H}, A),$$

for any commutative unital algebra A. In the case of the Connes–Kreimer Hopf algebra this G is called the group of *diffeographisms* of the physical theory $\mathcal{T}$ and it was proved in [25] that it acts by local diffeomorphisms on the coupling constants of the theory.

The complex Lie group $G(\mathbb{C})$ of complex points of the affine group scheme G, defined as $G(\mathbb{C}) = \mathrm{Hom}(\mathcal{H}, \mathbb{C})$, is a pro-unipotent Lie group. For such groups, which are dual to graded connected Hopf algebras that are finite dimensional in each degree, Connes and Kreimer proved by a recursive formula that it is always possible to have a multiplicative Birkhoff factorization

$$\gamma(z) = \gamma_-(z)^{-1}\gamma_+(z)$$

of loops $\gamma \colon \Delta^* \to G$, defined on an infinitesimal disk Δ^* around the origin in $\mathbb{C}^*$, in terms of two holomorphic functions $\gamma_\pm(z)$ respectively defined on Δ and on $\mathbb{P}^1(\mathbb{C}) \smallsetminus \{0\}$. The factorization is unique upon fixing a normalization condition $\gamma_-(\infty) = 1$. Notice that such Birkhoff factorizations do not always exist for other kinds of complex Lie groups, as one can see in the example of $\mathrm{GL}_n(\mathbb{C})$ where the existence of holomorphic vector bundles on the Riemann sphere is an obstruction.

In Hopf algebra terms, one can describe a loop $\gamma\colon \Delta^* \to G(\mathbb{C})$ on an infinitesimal punctured disk Δ^* as an algebra homomorphism $\phi \in \mathrm{Hom}(\mathcal{H}, \mathbb{C}(\{z\}))$ with values in the field of germs of meromorphic functions (covergent Laurent series). The two terms γ_+ and γ_- of the Birkhoff factorization are, respectively, algebra homomorphisms $\phi_+ \in \mathrm{Hom}(\mathcal{H}, \mathbb{C}\{z\})$ to convergent power series, and $\phi_- \in \mathrm{Hom}(\mathcal{H}, \mathbb{C}[z^{-1}])$. The BPHZ recursive formula is then reformulated in [25] [26] as the Birkhoff factorization applied to the loop $\phi(\Gamma) = U(\Gamma)$ given by the dimensionally regularized unrenormalized Feynman integrals. In fact, the recursive formula of Connes and Kreimer for the Birkhoff factorization can be written as

$$\phi_-(X) = -T(\phi(X) + \sum \phi_-(X')\phi(X'')),$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$, and with T the projection onto the polar part of the Laurent series, and $\phi_+(X) = \phi(X) + \phi_-(X)$. The fact that the $\phi_\pm$ obtained in this way are still algebra homomorphism depends on the fact that the projection onto the polar part of Laurent series is a Rota–Baxter operator. In fact, this renormalization procedure by Birkhoff factorization was easily generalized in [33], [34] to arbitrary algebra homomorphisms $\phi \in \mathrm{Hom}(\mathcal{H}, \mathcal{A})$ from a commutative graded connected Hopf algebra to a Rota–Baxter algebra. When one applies this formula to $\phi(\Gamma) = U(\Gamma)$ one finds the BPHZ formula with $\phi_-(\Gamma) = C(\Gamma)$ the counterterms and $\phi_+(\Gamma)|_{z=0} = R(\Gamma)$ the renormalized values.

Notice how, from this point of view, the algebro-geometric Feynman rules discussed above, correspond to the data of a Hopf algebra homomorphism $\phi \in \mathrm{Hom}(\mathcal{H}, \mathcal{F}_{\mathbb{K}})$ or, in the motivic case, $\phi \in \mathrm{Hom}(\mathcal{H}, K_0(\mathcal{V}_{\mathbb{K}}))$, together with the assignment of the propagator $\mathbb{U}(L) = \mathbb{L}$. It would therefore be interesting to know if the rings $\mathcal{F}_{\mathbb{K}}$ and $K_0(\mathcal{V}_{\mathbb{K}})$ have a non-trivial Rota–Baxter structure.

4. A top-down approach via Galois theory

As we mentioned earlier, the "top-down" approach to the question of Feynman integrals and periods of mixed Tate motives consists of comparing categorical structures, instead of looking at varieties and motives associated to individual Feynman graphs. The main idea, developed in my joint work with Connes in [27], [28], [29], [30], is to show that the data of perturbative renormalization can be reformulated in terms of a Tannakian category of equivalence classes of differential systems with irregular singularities.

A neutral Tannakian category $\mathcal{C}$ is an abelian category, which is k-linear for some field k, has a rigid tensor structure and a fiber functor $\omega\colon \mathcal{C} \to \mathrm{Vect}_k$, which is a faithful exact tensor functor to the category of vector spaces over the same field k.

Tannakian categories are extremely rigid structures, namely, such a category is equivalent to a category of finite dimensional linear representations of an affine group scheme,

$$\mathcal{C} \simeq \mathrm{Rep}_G.$$

The affine group scheme G is reconstructed from the category as the invertible natural transformations of the fiber functor.

Thus, in order to relate two sets of objects of a seemingly very different nature, of which one is known (as is the case for mixed Tate motives over a number field) to form a Tannakian category, it suffices to show that the other set of objects can also be organized in a similar way, and check that the resulting affine group schemes are isomorphic: this gives then an equivalence of categories. This is precisely what is done in the results of [27].

The reason why this does not yet give an answer to the conjecture lies in the fact that one only obtains in this way a non-canonical identification, which cannot therefore be used to explicitly match Feynman integrals to mixed Tate motives. There are other mysterious aspects, for instance the category of mixed Tate motives involved in the result of [27] is not over $\mathbb{Q}$ or $\mathbb{Z}$, but over the ring $\mathbb{Z}[i][1/2]$, while all the varieties X_Γ involved in the parametric formulation of Feynman integrals are defined over $\mathbb{Z}$. Relating explicitly the top-down approach described below to the bottom-up approach is still an important missing ingredient in the geometric theory of renormalization, which may possibly provide the key to completing a proof of the main conjecture.

The main results of [27], [28], [29] are summarized as follows.

- *Step* 1: *Counterterms as iterated integrals.* One writes the negative piece $\gamma_-(z)$ of the Birkhoff factorization as an iterated integral depending on a single element β in the Lie algebra $\mathrm{Lie}(G)$ of the affine group scheme dual to the Connes–Kreimer Hopf algebra. This is a way of formulating what is known in physics as the 't Hooft–Gross relations [38], that is, the fact that counterterms only depend on the beta function of the theory (the infinitesimal generator of the renormalization group flow).
- *Step* 2: *From iterated integrals to solutions of irregular singular differential equations.* The iterated integrals obtained in the first step are uniquely solutions to certain differential equations. This makes it possible to classify the divergences of quantum field theories in terms of families of differential systems with singularities. The fact that, by dimensional analysis, counterterms are independent of the energy scale corresponds in these geometric terms to the flat singular connections describing the differential systems satisfying a certain *equisingularity* condition.
- *Step* 3: *Equisingular vector bundles.* Instead of working with equisingular connections in the context of principal G-bundles, one can formulate things equivalently in terms of linear representations and of flat connections on vector bundles. These data can then be organized in a neutral Tannakian category $\mathcal{E}$ which is independent of G and therefore universal for all physical theories.

- *Step* 4: *The Galois group.* The Tannakian category of flat equisingular connections is equivalent to a category of representations $\mathcal{E} \simeq \mathrm{Rep}_{\mathbb{U}^*}$ of an affine groups scheme $\mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$, where $\mathbb{U}$ is the prounipotent affine group scheme dual to the Hopf algebra $\mathcal{H}_{\mathbb{U}} = U(\mathcal{L})^\vee$, where $\mathcal{L} = \mathcal{F}(e_{-n}; n \in \mathbb{N})$ is the free graded Lie algebra with one generator in each degree.
- *Step* 5: *Motivic Galois group.* The same group $\mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$ is known to arise (up to a non-canonical identification) as the motivic Galois group of the category of mixed Tate motives over the scheme $S = \mathrm{Spec}(\mathbb{Z}[i][1/2])$, by a result of Deligne–Goncharov [32].

We describe briefly each of these steps below.

4.1. Counterterms as iterated integrals. In the Birkhoff factorization, there is in fact a dependence on a mass scale μ, inherited from the same dependence of the dimensionally regularized Feynman integrals $U_\mu(\Gamma)$, so that we have

$$\gamma_\mu(z) = \gamma_-(z)^{-1}\gamma_{\mu,+}(z),$$

where one knows by reasons of dimensional analysis that the negative part is independent of μ. This part is written as a time ordered exponential

$$\gamma_-(z) = Te^{-\frac{1}{z}\int_0^\infty \theta_{-t}(\beta)dt} = 1 + \sum_{n=1}^{\infty} \frac{d_n(\beta)}{z^n},$$

where

$$d_n(\beta) = \int_{s_1 \geq s_2 \geq \cdots \geq s_n \geq 0} \theta_{-s_1}(\beta) \ldots \theta_{-s_n}(\beta) ds_1 \ldots ds_n,$$

and where $\beta \in \mathrm{Lie}(G)$ is the beta function, that is, the infinitesimal generator of renormalization group flow, and the action θ_t is induced by the grading of the Hopf algebra by

$$\theta_u(X) = u^n X, \ \text{ for } u \in \mathbb{G}_m, \quad \text{and} \quad X \in \mathcal{H}, \ \text{ with } \deg(X) = n,$$

with generator the grading operator $Y(X) = nX$. This result follows from the analysis of the renormalization group in the Connes–Kreimer theory given in [25] [26], with the recursive formula for the coefficients d_n explicitly solved to give the time ordered exponential above.

The loop $\gamma_\mu(z)$ that collects all the unrenormalized values $U_\mu(\Gamma)$ of the Feynman integrals satisfies the scaling property

$$\gamma_{e^t\mu}(z) = \theta_{tz}(\gamma_\mu(z)) \tag{20}$$

in addition to the property that its negative part is independent of μ,

$$\frac{\partial}{\partial \mu}\gamma_-(z) = 0. \tag{21}$$

The Birkhoff factorization is then written in [27] in terms of iterated integrals as

$$\gamma_{\mu,+}(z) = Te^{-\frac{1}{z}\int_0^{-z\log\mu}\theta_{-t}(\beta)dt}\ \theta_{z\log\mu}(\gamma_{\rm reg}(z)).$$

Thus $\gamma_\mu(z)$ is specified by β up to an equivalence given by the regular term $\gamma_{\rm reg}(z)$. The equivalence corresponds to "having the same negative part of the Birkhoff factorization".

4.2. From iterated integrals to differential systems. The second step of the argument of [27] goes as follows. An iterated integral (or time-ordered exponential) $g(b) = Te^{\int_a^b \alpha(t)dt}$ is the unique solution of a differential equation $dg(t) = g(t)\alpha(t)dt$ with initial condition $g(a) = 1$. In particular, given the differential field $(K = \mathbb{C}(\{z\}), \delta)$ and an affine group scheme G, and the logarithmic derivative

$$G(K) \ni f \mapsto D(f) = f^{-1}\delta(f) \in \operatorname{Lie} G(K),$$

one can consider differential equations of the form $D(f) = \omega$, for a flat Lie $G(\mathbb{C})$-valued connection ω, singular at $z = 0 \in \Delta^*$. The existence of solutions is ensured by the condition of trivial monodromy on Δ^*

$$M(\omega)(\ell) = Te^{\int_0^1 \ell^*\omega} = 1, \quad \ell \in \pi_1(\Delta^*).$$

These differential systems can be considered up to the gauge equivalence relation of $D(fh) = Dh + h^{-1}Df\, h$, for a regular $h \in \mathbb{C}\{z\}$. The gauge equivalence is the same thing as the requirement considered above that the solutions have the same negative piece of the Birkhoff factorization,

$$\omega' = Dh + h^{-1}\omega h \iff f_-^\omega = f_-^{\omega'},$$

where $D(f^\omega) = \omega$ and $D(f^{\omega'}) = \omega'$.

4.3. Flat equisingular connections. The third step of [27] consists of reformulating the data of the loops $\gamma_\mu(z)$ up to the equivalence of having the same negative piece of the Birkhoff factorization in terms of gauge equivalence classes of differential systems as above. The point here is that one keeps track of the μ-dependence and of the way $\gamma_\mu(z)$ scales with μ and the fact that the negative part of the Birkhoff factorization is independent of μ, as in (20), (21). In geometric terms these conditions are reformulated in [27] as properties of connections on a principal G-bundle $P =$

$B \times G$ over a fibration $\mathbb{G}_b \to B \to \Delta$, where $z \in \Delta$ is the complexified dimension of DimReg and the fiber $\mu^z \in \mathbb{G}_m$ over z corresponds to the changing mass scale. The multiplicative group acts by

$$u(b, g) = (u(b), u^Y(g)) \quad \text{for all } u \in \mathbb{G}_m.$$

The two conditions (20) and (21) correspond to the properties that the flat connection ϖ on P^* is *equisingular*, that is, it satisfies the following:

- Under the action of $u \in \mathbb{G}_m$ the connection transforms like

$$\varpi(z, u(v)) = u^Y(\varpi(z, u)).$$

- If γ is a solution in $G(\mathbb{C}(\{z\}))$ of the equation $D\gamma = \varpi$, then the restrictions along different sections σ_1, σ_2 of B with $\sigma_1(0) = \sigma_2(0)$ have "the same type of singularities", namely
$$\sigma_1^*(\gamma) \sim \sigma_2^*(\gamma),$$
where $f_1 \sim f_2$ means that $f_1^{-1} f_2 \in G(\mathbb{C}\{z\})$, regular at zero.

4.4. Flat equisingular vector bundles. The fourth step of [27] consists of transforming the information obtained above from equivalence classes of flat equisingular connections on the principal G-bundle P to a category $\mathcal{E}$ of flat equisingular vector bundles. This is possible without losing any amount of information, since the affine group scheme G dual to the Connes–Kreimer Hopf algebra of Feynman graphs of a given physical theory is completely determined by its category Rep_G of finite dimensional linear representations. Thus, considering all possible flat equisingular vector bundles gives rise to a category that in particular contains as a subcategory the vector bundles that come from finite dimensional representations of G, for any G associated to a particular physical theory, while in itself the category $\mathcal{E}$ does not depend on any particular G, so it is therefore universal for different physical theories.

The category $\mathcal{E}$ of flat equisingular vector bundles is defined in [27] as follows.

The objects $\mathrm{Obj}(\mathcal{E})$ are pairs $\Theta = (V, [\nabla])$, where V is a finite dimensional $\mathbb{Z}$-graded vector space, out of which one forms a bundle $E = B \times V$. The vector space has a filtration $W^{-n}(V) = \oplus_{m \geq n} V_m$ induced by the grading and a $\mathbb{G}_m$ action also coming from the grading. The class $[\nabla]$ is an equivalence class of equisingular connections, which are compatible with the filtration, trivial on the induced graded spaces $\mathrm{Gr}^W_{-n}(V)$, up to the equivalence relation of W-equivalence. This is defined by $T \circ \nabla_1 = \nabla_2 \circ T$ for some $T \in \mathrm{Aut}(E)$ which is compatible with filtration and trivial on $\mathrm{Gr}^W_{-n}(V)$. Here the condition that the connections ∇ are equisingular means that they are $\mathbb{G}_m$-invariant and that restrictions of solutions to sections of B with the same $\sigma(0)$ are W-equivalent. The morphisms $\mathrm{Hom}_{\mathcal{E}}(\Theta, \Theta')$ are linear maps $T : V \to V'$

that are compatible with grading, and such that on $E \oplus E'$ the following connections are W-equivalent:

$$\begin{pmatrix} \nabla' & 0 \\ 0 & \nabla \end{pmatrix} \overset{W\text{-equiv}}{\simeq} \begin{pmatrix} \nabla' & T\nabla - \nabla' T \\ 0 & \nabla \end{pmatrix}.$$

4.5. The Riemann–Hilbert correspondence. Finally, we proved in [27] that the category $\mathcal{E}$ is a Tannakian category,

$$\mathcal{E} \simeq \mathrm{Rep}_{\mathbb{U}^*}, \quad \text{with } \mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m,$$

where $\mathbb{U}$ is dual, under the relation $\mathbb{U}(A) = \mathrm{Hom}(\mathcal{H}_{\mathbb{U}}, A)$, to the Hopf algebra $\mathcal{H}_{\mathbb{U}} = U(\mathcal{L})^\vee$ dual (as Hopf algebra) to the universal enveloping algebra of the free graded Lie algebra $\mathcal{L} = \mathcal{F}(e_{-1}, e_{-2}, e_{-3}, \ldots)$. The renormalization group $\mathbf{rg}\colon \mathbb{G}_a \to \mathbb{U}$ is a 1-parameter subgroup with generator $e = \sum_{n=1}^\infty e_{-n}$. In particular, the morphism $\mathbb{U} \to G$ that realizes the finite dimensional linear representations of G with equisingular connections as a subcategory of $\mathcal{E}$ is given by mapping the generators $e_{-n} \mapsto \beta_n$ to the n-th graded piece of the beta function of the theory, seen as an element $\beta = \sum_n \beta_n$ in the Lie algebra $\mathrm{Lie}(G)$. There are universal counterterms in $\mathbb{U}^*$ given in terms of a *universal singular frame*

$$\gamma_{\mathbb{U}}(z, v) = T e^{-\frac{1}{z}\int_0^v u^Y(e)\frac{du}{u}}.$$

For $\Theta = (V, [\nabla])$ in $\mathcal{E}$ there exists a unique $\rho \in \mathrm{Rep}_{\mathbb{U}^*}$ such that

$$D\rho(\gamma_{\mathbb{U}}) \overset{W\text{-equiv}}{\simeq} \nabla.$$

This same affine group scheme $\mathbb{U}^*$ appears in the work of Deligne–Goncharov as the motivic Galois group of the category of mixed Tate motives $\mathcal{M}_S \simeq \mathrm{Rep}_{\mathbb{U}^*}$, with $S = \mathrm{Spec}(\mathbb{Z}[i][1/2])$, albeit up to a non-canonical identification. This leads to an identification (non-canonically) of the category $\mathcal{E}$, which by the previous steps classifies the data of the counterterms in perturbative renormalization, with the category $\mathcal{M}_S$ of mixed Tate motives.

Cartier conjectured [23] the existence of a Galois group acting on the coupling constants of the physical theories and related both to the groups of diffeographisms of the Connes–Kreimer theory and to the symmetries of multiple zeta values, and he referred to it as a *cosmic Galois group*. In this sense the result of [27] is a positive answer to Cartier's conjecture, which identifies his cosmic Galois group with the affine group scheme $\mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$.

5. The geometry of Dim Reg

We end this exposition with a brief discussion on the subject of dimensional regularization. In physics this is taken to mean a *formal* extension of the rules of integration of Gaussians by setting

$$\int e^{-\lambda t^2} d^z t := \pi^{z/2} \lambda^{-z/2},$$

for $z \in \mathbb{C}^*$. This prescription can then be used to make sense of a larger set of integrations in complexified dimension z, which can be reduced to this Gaussian form by the use of Schwinger parameters. However, no attempt is made to make sense of an actual geometry in complexified dimension $z \in \mathbb{C}^*$. We argue here that there are (at least) two possible approaches that can be used to make sense of spaces in dimension z compatibly with the prescription for the Gaussian integration. One is based on noncommutative geometry and it was proposed first in the unpublished work [31] and later included in our book [30], while the second approach is based on motives and was proposed in [49]. The noncommutative geometry approach is based on the idea of taking a product, in the sense of metric noncommutative spaces (spectral triples) of the spacetime manifold over which the quantum field theory is constructed by a noncommutative space X_z whose dimension spectrum (the most sophisticated notion of dimension in noncommutative geometry) is given by a single point $z \in \mathbb{C}^*$. The motivic approach is based also on taking a product, but this time of the motive associated to an individual Feynman graph by a projective limit of logarithmic motives Log^∞.

In both cases the main idea is to deform the geometry by taking a product of the original geometry on which the computation of the un-regularized Feynman integral was performed by a new space, either noncommutative or motivic, which accounts for the shift of z in dimension. Recently there has been a considerable amount of activity in relating noncommutative geometry and motives (see [24] and [30]). It would be interesting to see if, in this context, there is a way to combine these two approaches to the geometry of dimensional regularization.

5.1. The noncommutative geometry of DimReg. The notion of metric space in noncommutative geometry is provided by *spectral triples*. These consist of data of the form $X = (\mathcal{A}, \mathcal{H}, \mathcal{D})$, with $\mathcal{A}$ an associative involutive algebra represented as an algebra of bounded operators on a Hilbert space $\mathcal{H}$, together with a self-adjoint operator $\mathcal{D}$ on $\mathcal{H}$, with compact resolvent, and with the property that the commutators $[a, \mathcal{D}]$ are bounded operators on $\mathcal{H}$, for all $a \in \mathcal{A}$. This structure generalizes the data of a compact Riemannian spin manifold, with the (commutative) algebra of smooth functions, the Hilbert space of square integrable spinors and the Dirac operator. It makes sense, however, for a wide range of examples that are not ordinary manifolds, such as quantum groups, fractals, noncommutative tori, etc. For such spectral triples there are various different notions of dimension. The most sophisticated one is the *dimension spectrum* which is not a single number but a subset of the complex plane consisting of all poles of the family of zeta functions associated to the spectral triple,

$$\mathrm{Dim} = \{s \in \mathbb{C} \mid \zeta_a(s) = \mathrm{Tr}(a|D|^{-s}) \text{ have poles}\}.$$

These are points where one has a well defined integration theory on the noncommutative space, the analog of a volume form, given in terms of a residue for the zeta functions. It is shown in [31], [30] that there exists a (type II) spectral triple X_z with the properties that the dimension spectrum is $\mathrm{Dim} = \{z\}$ and that one recovers the DimReg prescription for the Gaussian integration in the form

$$\mathrm{Tr}(e^{-\lambda D_z^2}) = \pi^{z/2}\lambda^{-z/2}.$$

The operator D_z is of the form $D_z = \rho(z)F|Z|^{1/z}$, where $Z = F|Z|$ is a self-adjoint operator affiliated to a type II_∞ von Neumann algebra $\mathcal{N}$ and $\rho(z) = \pi^{-1/2}(\Gamma(1 + z/2))^{1/z}$, with the spectral measure $\mathrm{Tr}(\chi_{[a,b]}(Z)) = \frac{1}{2}\int_{[a,b]} dt$, for the type II trace. The ordinary spacetime over which the quantum field theory is constructed can itself be modeled as a (commutative) spectral triple

$$X = (\mathcal{A}, \mathcal{H}, \mathcal{D}) = (\mathcal{C}^\infty(X), L^2(X, S), \not{D}_X)$$

and one can take a product $X \times X_z$ given by the cup product of spectral triples (adapted to type II case)

$$(\mathcal{A}, \mathcal{H}, \mathcal{D}) \cup (\mathcal{A}_z, \mathcal{H}_z, D_z) = (\mathcal{A} \otimes \mathcal{A}_z, \mathcal{H} \otimes \mathcal{H}_z, \mathcal{D} \otimes 1 + \gamma \otimes D_z).$$

This agrees with what is usually described in physics as the Breitenlohner–Maison prescription to resolve the problem of the compatibility of the chirality γ_5 operator with the DimReg procedure, [21]. The Breitenlohner–Maison prescription consists of changing the usual Dirac operator to a product, which is indeed of the form as in the cup product of spectral triples,

$$\mathcal{D} \otimes 1 + \gamma \otimes D_z.$$

It is shown in [31] and [30] that an explicit example of a space X_z that can be used to perform dimensional regularization geometrically can be constructed from the adèle class space, the noncommutative space underlying the spectral realization of the Riemann zeta function in noncommutative geometry (see *e.g.* [24]), by taking the crossed product of the partially defined action

$$\mathcal{N} = L^\infty(\hat{\mathbb{Z}} \times \mathbb{R}^*) \rtimes \mathrm{GL}_1(\mathbb{Q})$$

and the trace

$$\mathrm{Tr}(f) = \int_{\hat{\mathbb{Z}}\times\mathbb{R}^*} f(1,a)\, da,$$

with the operator

$$Z(1,\rho,\lambda) = \lambda, \quad Z(r,\rho,\lambda) = 0,\ r \neq 1 \in \mathbb{Q}^*.$$

5.2. The motivic geometry of DimReg. We now explain briefly the motivic approach to dimensional regularization proposed in [49]. The Kummer motives are simple examples of mixed Tate motives, given by the extensions

$$M = [u\colon \mathbb{Z} \to \mathbb{G}_m] \in \mathrm{Ext}^1_{\mathcal{DM}(\mathbb{K})}(\mathbb{Q}(0), \mathbb{Q}(1))$$

with $u(1) = q \in \mathbb{K}^*$ and the period matrix

$$\begin{pmatrix} 1 & 0 \\ \log q & 2\pi i \end{pmatrix}.$$

These can be combined in the form of the Kummer extension of Tate sheaves

$$\mathcal{K} \in \mathrm{Ext}^1_{\mathcal{DM}(\mathbb{G}_m)}(\mathbb{Q}_{\mathbb{G}_m}(0), \mathbb{Q}_{\mathbb{G}_m}(1)),$$

$$\mathbb{Q}_{\mathbb{G}_m}(1) \to \mathcal{K} \to \mathbb{Q}_{\mathbb{G}_m}(0) \to \mathbb{Q}_{\mathbb{G}_m}(1)[1].$$

The *logarithmic motives* $\mathrm{Log}^n = \mathrm{Sym}^n(\mathcal{K})$ are defined as symmetric products of this extension, [7] [37]. They form a projective system and one can take the limit as a pro-motive

$$\mathrm{Log}^\infty = \varprojlim_n \mathrm{Log}^n.$$

This corresponds to the period matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots \\ \log(s) & (2\pi i) & 0 & \cdots & 0 & \cdots \\ \frac{\log^2(s)}{2!} & (2\pi i)\log(s) & (2\pi i)^2 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\ \frac{\log^n(s)}{n!} & (2\pi i)\frac{\log^{n-1}(s)}{(n-1)!} & (2\pi i)^2\frac{\log^{n-2}(s)}{(n-2)!} & \cdots & (2\pi i)^{n-1} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \end{pmatrix}$$

The graph polynomials Ψ_Γ associated to Feynman graphs define motivic sheaves

$$M_\Gamma = (\Psi_\Gamma\colon \mathbb{A}^n \smallsetminus \hat{X}_\Gamma \to \mathbb{G}_m, \hat{\Sigma}_n \smallsetminus (\hat{X}_\Gamma \cap \hat{\Sigma}_n), n-1, n-1),$$

viewed as objects $(f\colon X \to S, Y, i, w)$ in Arapura's category of motivic sheaves, [6].

Then the procedure of dimensional regularization can be see as taking a product $M_\Gamma \times \mathrm{Log}^\infty$ in the Arapura category of the motivic sheaf M_Γ by the logarithmic pro-motive. The product in the Arapura category is given by the fibered product

$$(X_1 \times_S X_2 \to S, Y_1 \times_S X_2 \cup X_1 \times_S Y_2, i_1 + i_2, w_1 + w_2).$$

The reason for this identification is that period computations on a fibered products satisfy

$$\int \pi_{X_1}^*(\omega) \wedge \pi_{X_2}^*(\eta) = \int \omega \wedge f_1^*(f_2)_*(\eta),$$

where the integration takes place on $\sigma_1 \times_S \sigma_2$ with $\sigma_i \subset X_i$ with boundary $\partial\sigma_i \subset Y_i$, according to the diagram

$$\begin{tikzcd} & X_1 \times_S X_2 \arrow[dl, "\pi_{X_1}"'] \arrow[dr, "\pi_{X_2}"] & \\ X_1 \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & S & \end{tikzcd}$$

This leads to writing the dimensionally regularized parametric Feynman integrals (at least in the log-divergent case where the term $P_\Gamma(t, p)$ is absent) in the Igusa L-function form $\int_\sigma \Psi_\Gamma^z \alpha$ as a period computation on $M_\Gamma \times \mathrm{Log}^\infty$.

Acknowledgement. A large part of what I described in this paper is based on recent joint work with Paolo Aluffi and on previous joint work with Alain Connes, whom I thank for their essential contributions to the development of these ideas and results. Although parts of this article reflect the content of the lecture I delivered at the ECM in the summer of 2008, new material developed in the intervening time was included for completeness and the actual writing was carried out in the winter 2009, during a stay at MSRI: I thank the institute for the hospitality and for support. An informal discussion with Nikolai Reshetikhin and a talk I gave in Yongbin Ruan's seminar helped finalizing the form of this exposition. The topics covered in this survey will appear in extended form in the book [50] based on a course given at Caltech in the fall 2008.

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5ECM Amsterdam 2008

Topological quantum field theory: 20 years later

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Abstract. This article is an overview of the developments in topological quantum field theory, and, in particular on the progress in the Chern–Simons theory.

Mathematics Subject Classification (2010). 57R56, 81T10.

Keywords. Chern–Simons, topological quantum field theory, quantum groups.

1. Introduction

The goal of this paper is to outline the progress in topological quantum field theory and some open problems in this direction, and to emphasize the importance of the mathematical research in quantum field theory.

Quantum field theory was developed as a theory describing interactions of elementary particles. In a similar way quantum mechanics appeared as a theory describing atomic physics. Quantum mechanics stimulated the development of many areas of mathematics, such as the theories of partial differential equations, operator algebras, functional analysis, geometry etc. But mathematical complexity of quantum field theory and the sophistication of related mathematical problems are at a different magnitude.

In earlier stages of developments of quantum field theory the emphasis was on perturbation theory. Some of the main problems in this direction are the ultraviolet divergencies of Feynman diagrams, the renormalizability, and proving the unitarity and locality of the formal power series given by the sum of Feynman diagrams. Attempts to develop non-perturbative quantum field theories are usually based on a notion of path integral.

Constructive field theory appeared as an attempt to make sense of the path integral. The idea is to define the path integral analytically as a limit of finite dimensional convergent integrals. The quest for non-perturbative examples of quantum field theories continued successfully in the area of integrable quantum field theories. Many integrable classical field theories were quantized using algebraical tools from the representation theory of quantum affine algebras.

Among all quantum field theories, gauge theories are main candidates for theories of fundamental interactions and they are also particularly difficult from the mathematical point of view. Chern–Simons theory is an example of a gauge quantum field theory which is finite in the perturbation theory and has a combinatorial formulation, in terms of finite dimensional representation theory. To be more precise, whether this combinatorial theory really quantizes the classical Chern–Simons theory is still a conjecture, but with the amount of accumulated evidence, there is no doubt that it is true. These notes are focused mostly on the progress in understanding the Chern–Simons quantum field theory.

In [112] Witten proposed a path integral that formally is an invariant of 3-manifolds. He outlined the semiclassical asymptotics of this path integral and described all basic elements of the asymptotical expansion. These formulae were clarified later in [43], [62]. He also outlined the relation between the quantized Chern–Simons theory and the WZW conformal field theory on the boundary of the 3-manifold. Based on this relation he suggested to use the conformal field theory to define the invariant.

The construction of invariants of 3-manifolds based on the representation of 3-manifolds by a surgery of a handle body was proposed in [92]. The main idea of this construction was to find invariants of links which are constant on the equivalence classes of links related by Kirby moves [72]. It became clear, almost immediately that this approach gave an answer that is almost identical to the formulae for the invariant from Witten's paper written in terms of WZW data. However, there were a few differences. The most important one was that in the combinatorial approach one can actually prove that the numbers are invariants of 3-manifolds. The question of framing dependence was resolved by Atiyah, who noticed that there is a canonical framing and that the combinatorial invariant corresponds to this framing. One more difference is that the combinatorial construction in terms of representations of quantized universal enveloping algebra at roots of unity is parametrized by a primitive root of unity. Braided monoidal categories related to the WZW conformal field theory correspond to special roots of unity of the form $\exp(\frac{2\pi i}{k})$.

Perhaps the most striking part of Witten's proposal was the dichotomy between the geometry involved in the path integral description and the combinatorics involved in the CFT description. The complete understanding of this relation is an important example of how the semiclassical expansion is related to the exact solution.

In Section 2 we will outline the general framework of quantum field theory. Section 3 contains an outline of classical Lagrangian and Hamiltonian field theory. In Section 4 we outline the idea of the path integral quantization and of the semiclassical expansion in terms of Feynman diagrams. Section 5 is a digression with the brief description of quantum groups at roots of unity and their representation theory. Section 6 has an overview of the construction of invariants of tangles. Combinatorial constructions of topological quantum field theories, of invariants of tangles, and their relation to the semiclassical Chern–Simons theory are described in Section 7. The conclusion has a collection of some other important developments in the theory of invariants of manifolds and topological quantum field theory.

The author benefited from numerous discussions with many people. He is particularly grateful to J. Andersen, A. Cattaneo, L. Faddeev, G. Felder, V. Fock, S. Gukov, T. Johnson-Freyd, R. Kashaev, P. Mnev, E. Opdam, L. Rozansly, and P. Teichner. The work was supported by the NSF grant 0601912, and by the Danish Research Foundation through the Niels Bohr Visiting Professorship at the University of Aarhus.

2. Local quantum field theory

2.1. Space-time categories. The mathematical framework of a quantum field theory as we will outline it here was suggested by Atiyah for topological quantum field theory and Segal for conformal field theory.

In a nut shell, a quantum field theory is a functor (a family of functors) from a space-time category to another category, which is given and is part of the data for defining this quantum field theory.

Roughly, a space-time category is a category of cobordisms consistsing of

- *objects* which are $(d-1)$-dimensional oriented manifolds with structure (Riemannian, symplectic, etc.), and
- *morphisms* between two objects N and N' that are d-dimensional manifolds M with $\partial M = N \sqcup \overline{N'}$ with structure that agrees with the structure on N and N'.

  The composition of morphisms is the gluing along the common boundary. The precise definition of the gluing (and of objects) can be somewhat involved (as in the case of the Riemannian category). The guiding operation is opposite to the cutting procedure, which is easier to define.

An example of a d-dimensional space-time category is the category of metrized cell approximations of Riemannian manifolds. Objects are metrized cell approximations of $(d-1)$- dimensional Riemannian manifolds. Morphisms are metrized cell approximations to d-dimensional Riemannian manifolds.

Another example is the topological category. In this case objects are smooth $(d-1)$-dimensional manifolds. A morphism between two manifolds N_1 and N_2 is a homeomorphism class of a d-dimensional manifold with the boundary $N_1 \sqcup \overline{N_2}$ with respect to homeomorphisms which are trivial at the boundary.

In the smooth category objects are smooth $(d-1)$-dimensional manifolds with d-dimensional collars. A morphism M between two manifolds N_1 and N_2 is a diffeomorphism class of a smooth d-dimensional manifold with the boundary $N_1 \sqcup \overline{N_2}$ with respect to diffeomorphisms which are trivial at the boundary. The smooth structure on the collars of N_1 and N_2 should agree with the smooth structure on M. Notice that in dimensions 1, 2, 3 smooth and topological categories are equivalent.

A d-dimensional Riemannian category is another important example. *Objects* are smooth oriented $(d-1)$-dimensional Riemannian manifolds with collars. *Morphisms* between two such manifolds N_1 and N_2 are isometry classes (with respect to isometries trivial at the boundary) of oriented d-dimensional Riemannian manifolds M such that $\partial M = \overline{N_1} \sqcup N_2$. The orientation on all three manifolds should naturally agree, and the metric on M agrees with the metric on N_1 and N_2 on a collar of the boundary. The composition is the gluing of such Riemannian cobordisms. For the details see [100]. This category is important for Euclidean quantum field theory and for statistical quantum field theories.

A pseudo-Riemannian category is most interesting for physics. The difference between this category and the Riemannian category is that morphisms are pseudo-Riemannian with the signature $(d, 1)$. When $d = 4$ it represents the space-time structure of our universe.

2.2. The framework of a local quantum field theory. A local quantum field theory on a d-dimensional space-time category can be defined as a functor from the category of d-cobordisms to the category of vector spaces (or, more generally, to some 'standard', 'known' category). For more details see [10], [97]. Here we will make a brief outline of this structure for oriented manifolds.

A d-dimensional local quantum field theory assigns a vector space to each $(d-1)$-dimensional object of the space-time category in question and a vector in the vector space corresponding to ∂M to the manifold M:

$$N \mapsto H(N), \quad M \mapsto Z(M) \in H(\partial M).$$

The vector space assigned to the boundary may depend on the extra structure at the boundary, such as metric, symplectic structure, etc.

These data should satisfy natural axioms, such as $H(\emptyset) = \mathbb{C}$,

$$H(N_1 \sqcup N_2) = H(N_1) \otimes H(N_2),$$
$$Z(M_1 \sqcup M_2) = Z(M_1) \otimes Z(M_2) \in H(\partial M_1) \otimes H(\partial M_2).$$

An isomorphism $f : N_1 \to N_2$ lifts to a linear isomorphism between corresponding vector spaces. The theory is invariant with respect to a class of isomorphisms of d-dimensional space-times if their restriction to the boundary produces linear isomorphisms of spaces $H(N)$ which commute with the map $Z : M \mapsto H(\partial M)$. Topological field theories are invariant with respect to homeomorphisms, conformal field theories are invariant with respect to conformal maps, etc.

An important part of the structure of QFT is the pairing between vector spaces corresponding to $(d-1)$-objects with opposite orientations: $\langle \cdot, \cdot \rangle : H(\overline{N}) \otimes H(N) \to \mathbb{C}$. One of the most important axioms of a local quantum field theory is the gluing axiom. Assume that $N_1, N_2 \subset M$ and $f : N_1 \to \overline{N_2}$ is an orientation preserving isomorphism. Let M_f be the result of gluing N_2 to $f(N_1)$. The gluing axiom establishes the relation between $Z(M)$, and $Z(M_f)$:

$$Z(M_f) = \langle Z(M) \rangle_f.$$

Here $\langle \cdot, \cdot \rangle_f : H(N') \otimes H(N_1) \otimes H(N_2) \to H(N')$ is the composition of $\mathrm{id} \otimes \mathrm{id} \otimes f$ and of $\mathrm{id} \otimes \langle \cdot, \cdot \rangle$, and N' is the complement of $N_1 \sqcup N_2$ in ∂M.

Originally this framework was formulated by Atiyah and Segal for topological and conformal field theories. It works well in TQFT, there also substantial evidence that it works in CFT as well, though it needs further research. It is natural to ask whether this approach extends to more general and more realistic quantum field theories, including the standard model.

This framework is very natural in statistical mechanics on cell complexes (lattice models in statistical mechanics) with open boundary conditions. Observables in this setting are operators acting on vector spaces assigned to the boundary. Correlation functions (expectation values in the Euclidean formulation) are compositions of partition functions and observables followed by the gluing of boundary components where the observables are located.

The main physical concept behind this framework is the locality of the interaction. Indeed, we can cut our space-time manifold in small pieces and the resulting partition function Z in such framework is expected to be the composition of partition functions of these small pieces. In other words, such theory is determined by its structure on 'small' space-time manifolds, or at 'short distances'. This is the concept of locality.

The potential problem in extending this framework to more realistic quantum field theories is whether it agrees with scaling limits , ie. with the renormalization of local quantum field theory. We will discuss it briefly in Section 4.1.4.

3. Classical field theories

3.1. Local Lagrangian classical field theory. The basic ingredients of a d-dimensional classical field theory are:

- The space of fields is assigned to each space-time. Fields can be sections of a fiber-bundle on a space-time, connections on a fiber bundle over a space-time, etc.
- The dynamics of the theory is determined by the action functional. In local classical field theory it is determined by a local Lagrangian which assigns a volume form on M to a field configuration that depends locally on the field. This concept is illustrated below in a few examples. Classical "equations of motion" are extrema of the action functional.

- A boundary condition is a constraint on boundary values of fields. In "good cases" there is a discrete set (or a finite dimensional variety) of solutions to equations of motion satisfying the boundary conditions. Boundary conditions should come in a family if one wants to produce solutions to boundary problems on the result of the gluing of space-times from solutions to boundary problems on the components. In the Hamiltonian formulation these families are Lagrangian fibrations.

A d-dimensional classical field theory can be regarded as the functor from the space-time category to the category of sets. It assigns to a $(d-1)$-dimensional space the set of possible boundary values of fields, and to a space-time the set of possible solutions to the Euler–Lagrange equations with these boundary values.

Here are some examples of local classical field theories.

3.1.1. Scalar field. The space-time for this classical field theory is a Riemannian manifold. The fields are real valued functions on the space-time. The action function is

$$S_M[\phi] = \int_M \Big(\frac{1}{2}(d\phi, d\phi) - V(\phi)\Big) dx,$$

where $(\cdot\,,\cdot)$ is the scalar product on cotangent spaces induced by the metric on M, and dx is the Riemannian volume form. Dirichlet boundary conditions $\phi|_{\partial M} = \eta$ are the natural choice of boundary conditions in this theory. The function $V(\phi)$ describes the self-interaction of the field ϕ. Typically it is a polynomial; however in $d = 2$ there are important examples of integrable field theories with the self-interaction given by $\cos\phi, \cosh\phi, \exp\phi$ (see for example [98]).

3.1.2. Pure Euclidean d-dimensional Yang–Mills. The space-time in this case is a Riemannian d-dimensional manifold with a principal G-bundle over it, where G is a compact simple (or Abelian) Lie group (for example trivial G-bundles). Fields in the Yang–Mills theory are connections in a principal G-bundle $\pi: P \to M$ (see for example [45]) for basic definitions).

The action functional is given by the integral

$$S_M[A] = \int_M \frac{1}{2}\mathrm{tr}\langle F(A), F(A)\rangle dx,$$

where $\langle\cdot\,,\cdot\rangle$ is the scalar product of 2-forms on M induced by the metric, dx is the volume form, and $\mathrm{tr}(ab)$ is the Killing form on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$.

The Euler–Lagrange equations for the Yang–Mills action are $d_A^* F(A) = 0$.

Fix a connection A^b on $P|_{\partial M}$. The Dirichlet boundary conditions for the Yang–Mills theory require that A^b is the pull-back of A to the boundary induced by the embedding $i: \partial M \to M$. The Yang–Mills action is invariant with respect to bundle automorphisms (gauge transformations). Because of this, the space of solutions to the Euler–Lagrange equations with given Dirichlet boundary conditions is infinite dimensional. On the other hand, a generic gauge class of Dirichlet boundary conditions defines a finite dimensional moduli space of gauge classes of solutions to the Yang–Mills equations.

3.1.3. 3-dimensional Chern–Simons theory. In this case the space-time category is the category of 3-dimensional topological cobordisms (with the trivial G-bundle over spaces and space-times). Fix a smooth, compact, oriented 3-dimensional manifold M. The space of fields of the Chern–Simons theory is the space of connections on a trivial principal G-bundle P over M. We will identify such connections with 1-forms.

The Chern–Simons action is

$$\mathrm{CS}_M(A) = \int_M \mathrm{tr}\Big(\frac{1}{2}A \wedge dA - \frac{1}{3}A \wedge [A \wedge A]\Big).$$

This action is of the first order in derivatives of the fields. In this sense it is very different from the Yang–Mills theory where the action is of the second order. Also, it is a topological action, i.e., its definition does not require any additional structure on M, such as a metric in the Yang–Mills case.

The variation of the Chern–Simons action is given by the bulk and the boundary terms:

$$\delta \mathrm{CS}_M(A) = \int_M \mathrm{tr}\, F(A) \wedge \delta A + \int_{\partial M} \mathrm{tr}\, A_\tau \wedge \delta A_\tau,$$

where A_τ, δA_τ are pull-backs to the boundary of A and δA.

The Euler–Lagrange equations for this Lagrangian are

$$F(A) = 0.$$

The solutions to the Euler–Lagrange equations are flat connections on P.

The boundary term of the variation of the action can be written as the value of the 1-form Θ on the space $C_{\partial M}$ of connections on $P|_{\partial M} \to \partial M$

$$\delta \mathrm{CS}_M(A) = (\Theta, \delta A_\tau).$$

The differential of this form is the non-degenerate 2-form on this space defining the symplectic structure on $C_{\partial M}$:

$$\omega(\delta A, \delta B) = \int_{\partial M} \mathrm{tr}\, \delta A \wedge \delta B. \tag{1}$$

The solutions to the Euler–Lagrange equations (flat connections on P) define the isotropic subspace $F_{\partial M} \subset (C_{\partial M}, \omega)$ of flat connections on ∂M which continue to flat connections on M. This pair, the symplectic space $(C_{\partial M}, \omega)$ and the isotropic subspace $F_{\partial M}$, is the *non-reduced* Hamiltonian formulation of the Chern–Simons theory.

The Chern–Simons action is gauge invariant (for details see for example [42]). The action of the gauge group is Hamiltonian on $(C_{\partial M}, \omega)$. The result of the Hamiltonian reduction of this symplectic space with respect to the action of the gauge group is the finite dimensional moduli space $\mathcal{M}^G_{\partial M}$ of gauge classes of flat connections in the trivial G-bundle over Σ together with the reduced symplectic structure. The space $L_G(M)$ of gauge classes of flat connections over ∂M that continue to flat connections on M is a Lagrangian subvariety in $\mathcal{M}^G_{\partial M}$. The pair $L_G(M) \subset \mathcal{M}^G_{\partial M}$ is the *reduced* Hamiltonian formulation of the Chern–Simons theory. A boundary condition in the reduced Chern–Simons theory is a choice of a Lagrangian submanifold in $\mathcal{M}^G_{\partial M}$.

3.2. Hamiltonian classical field theory. A d-dimensional Hamiltonian field theory in a category of space-times is an assignment of the following data to manifolds which are objects and morphisms of this category [41], [90]:

- A symplectic manifold $S(M_{d-1})$ to a $(d-1)$-dimensional manifold M_{d-1}.
- A Lagrangian submanifold $L(M_d) \subset S(\partial M_d)$ to each d-dimensional manifold M_d.

These data should satisfy the following axioms:

- $S(\emptyset) = \{0\}$ $S(M_1 \sqcup M_2) = S(M_1) \times S(M_2)$.
- $L(M_1 \sqcup M_2) = L(M_1) \times L(M_2)$ with $L(M_i) \subset S(\partial M_i)$.
- The orientation-reversing morphism $\sigma\colon M \to \overline{M}$ lifts to the symplectomorphism $s(\sigma) : (S(\overline{M}), \omega) = (S(M), -\omega)$.
- An orientation preserving morphism $f\colon M_1 \to M_2$ of $(d-1)$-dimensional manifolds (a mapping preserving structures on M) lifts to a symplectomorphism $s(f)\colon S(M_1) \to S(M_2)$.
- Assume that $\partial M = (\partial M)_1 \sqcup (\partial M)_2 \sqcup (\partial M)'$ and that there is an orientation reversing morphism $f : (\partial M)_1 \to \overline{(\partial M)_2}$. Denote by M_f the result of gluing M along $(\partial M)_1 \simeq \overline{(\partial M)_2}$ via f:

  $$M_f = M/\langle(\partial M)_1 \simeq \overline{(\partial M)_2}\rangle.$$

  Then

  $$L(M_f) = \{x \in S((\partial M)') \mid \text{there exists } y \in S(\partial M)_1 \text{ with } (y, s(f)(y), x) \in L(M)\}. \tag{2}$$

  Notice that $\partial M_f = (\partial M)'$ by definition.

The last axiom is known as the gluing axiom.

In classical mechanics the gluing axiom is the composition of the evolution at consecutive intervals of time.

A *boundary condition* in the Hamiltonian formulation is a Lagrangian subspace in the symplectic manifold assigned to the boundary of the space-time manifold. The space of boundary conditions should form a *Lagrangian fibration*. Different fibrations (real polarizations) correspond to different families of boundary conditions. Classical solutions (the Hamiltonian version of solutions to the Euler–Lagrange equations) of such a Hamiltonian field theory with boundary values in a Lagrangian subspace L of $S(\partial M)$ are intersection points of L and $L(M)$.

Remark 1. When $S(M)$ is finite dimensional, the intersection $L(M)$ with the Lagrangian submanifold of boundary conditions, generically, is a discrete set of points. When $S(M)$ is infinite dimensional, it is, generically, a finite dimensional manifold. An over-determined boundary condition is a co-isotropic submanifold, and an under-determined boundary condition is an isotropic subspace. In systems with gauge symmetries the construction described above is a reduced Hamiltonian formalism.

Boundary conditions should agree with gluing. Let $L_1 \times L_2 \times L' \subset S(\partial M) = S((\partial M)_1) \times S((\partial M)_2) \times S((\partial M)')$ be the Lagrangian submanifold defining boundary conditions for M and let $f : (\partial M)_1 \to (\partial M)_2$ be the gluing map as above. Boundary conditions $L_1 \subset S((\partial M)_1)$ and $L_2 \subset S((\partial M)_2)$ agree with the gluing if $L_2 = s(f)L_1$. But generically, the intersection $L(M_f)$ with the set $L_f = \{(y, s(f)(y), z)\} \subset L_1 \times L_2 \times L'$ will be empty, i.e., there will be no classical solutions on M_f with boundary conditions $L' \subset M_f$ that pass through L_f. The set of such solutions will be non-empty only for special choices of L_1. This is why if we want to glue solutions, we should have a Lagrangian fibration on $S(\partial M)$. In this case we can vary boundary conditions on $S((\partial M)_1)$ and select those Lagrangian submanifolds L_1 for which the intersection $L_f \cap L(M_f)$ is not empty. The collection of such intersection points will give classical solutions on M_f with boundary conditions $L' \subset \partial M_f$.

Remark 2. If the space-time category is the subcategory of a smooth category where morphisms are cylinders, the Hamiltonian classical field theory is equivalent to the classical Hamiltonian (possibly infinite dimensional) dynamical system. For more details see [90].

For the scalar Bose field theory $S(\partial M)$ is the cotangent bundle to the space of Dirichlet boundary conditions, i.e., to the space $C(\partial M)$. Taking into account the Riemannian metric on ∂M it can be regarded as $C(\partial M) \oplus C(\partial M)$. The Lagrangian subspace $L(M)$ in this case consists of pairs $(\partial_n \phi_c, \eta)$, where $\partial_n \phi_c$ is the normal derivative at ∂M of a solution to the Euler–Lagrange equation with the boundary condition $\phi_c|_{\partial M} = \eta$.

In the Yang–Mills theory, $S(\partial M)$ is the result of the Hamiltonian reduction of the cotangent bundle to the space of connections on the boundary. The Lagrangian submanifold $L(M)$ is the subspace of gauge classes of pairs $(i^*((dA)_n), \alpha)$, where A is a solution to the Euler–Lagrange equation with the boundary condition $i^*(A)|_{\partial M} = \alpha$, and $(dA)_n$ is the normal component of dA (which is locally a 2-form) and is a 1-form at the boundary.

The Hamiltonian structure of the reduced Chern–Simons theory is described at the end of Section 3.1.3.

4. Path integral quantization

4.1. Semiclassical quantization and Feynmann diagrams

4.1.1. Asymptotical expansion of non-degenerate oscillatory integrals and Feynman diagrams. Let M be a smooth compact oriented manifold with a volume form and f be a smooth function with finitely many simple critical points. Feynman diagrams appear naturally in the description of the asymptotical expansion of the oscillatory integral

$$\int_{\mathcal{M}} \exp\left(i\frac{f(x)}{h}\right)dx \simeq \sum_a (2\pi h)^{\frac{N}{2}} \frac{1}{\sqrt{|\det(B_a)|}} \exp\left(\frac{if(a)}{h} + \frac{i\pi}{4}\mathrm{sign}(B_a)\right) \sum_\Gamma \frac{(ih)^{-\chi(\Gamma)+1} F_a(\Gamma)}{|\mathrm{Aut}(\Gamma)|}, \tag{3}$$

where the sum is taken over critical points a of f, and over the graphs with vertexes of valency ≥ 3, $F(\Gamma)$ is the state sum corresponding to Γ described below, $|\mathrm{Aut}(\Gamma)|$ is the number of elements in the automorphism group of Γ, $\chi(\Gamma)$ is the Euler characteristic of the graph $\chi(\Gamma) = |V| - |E|$, where $|E|$ is the number of edges of Γ and $|V|$ is the number of vertices of Γ. The state sum $F(\Gamma)$ is defined as follows. Assign elements $1, \ldots, N$ to end points of edges of Γ. This defines an assignment of indices to endpoints of stars of vertices. The state sum is defined as

$$F_a(\Gamma) = \sum_{\{i\}} \prod_{e\in E(\Gamma)} (B_a^{-1})_{i_e, j_e} \prod_{v\in V(\Gamma)} (\text{weight of star of } v)_i.$$

Here $B_a^{ij} = \partial^i \partial^j f(a)$, the weight of the star of a vertex colored as in Figure 1 is $\partial_{i_1} \ldots \partial_{i_n} f(a)$, and the indices i_e, j_e correspond to two different endpoints of e (since B is symmetric, it does not matter that this pair is defined only up to a permutation). Local coordinates in (3) are chosen such that $dx = dx^1 \ldots dx^N$.

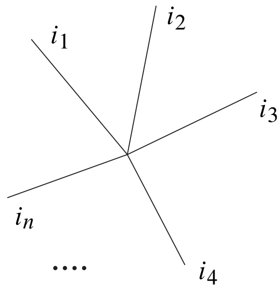

Figure 1. Weights of vertices for Feynman diagrams.

Indeed, as $h \to 0$ the leading contributions to the asymptotical expansion is determined by the integrals over small neighborhoods of critical points of f, see for example [22]. In the vicinity of a non-degenerate critical point the integral can be replaced by the formal power series of Gaussian integrals of monomials in local coordinates (see [38], [89] for more details).

Although the formulae for weights of Feynman diagrams involve the choice of local coordinates, the sum over Feynman diagrams with given Euler characteristic is defined globally.

4.1.2. Feynman diagrams in G-invariant oscillatory integrals. Because the main theme of this note is topological field theories and because one of the most interesting topological field theories, the Chern–Simons theory, is gauge invariant, we should first review the asymptotical expansions of finite dimensional G-invariant oscillating integrals. The definition of Feynman diagram expansions of such integrals goes back to the work of Faddeev and Popov [40] on quantization of gauge theories.

Let M be a smooth N-dimensional manifold with a free action of the compact (n-dimensional) Lie group G, with a G-invariant volume form dx. Let f be a G-invariant function on M with finitely many simple critical orbits (i.e., G-orbits where $df = 0$). Consider the family of integrals

$$I_h = \frac{1}{|G|} \int_M e^{\frac{i}{h} f(x)} dx \tag{4}$$

The asymptotical expansion of this integral as $h \to 0$ can be computed by the stationary phase method. It depends only on formal neighborhoods of isolated critical G-orbits.

To describe this asymptotical expansion choose a submanifold $S \in M$ such that it is a cross-section through G-orbits in a small neighborhood of each critical orbit of f (local cross-section). Assume S is defined as the level 0 surface of $n = \dim(\mathfrak{g})$

functions: $S = \{\varphi^a(x) = 0,\ a = 1, \ldots, n\}$. Choose a basis $\{e_a\}$, $a = 1, \ldots, n$ in $\mathfrak{g}$. The integral I_h can be written as the integral over M/G with induced volume measure. If S is a global cross-section, the integral over orbits can be written as the integral over S with respect to the induced measure:

$$\int_M e^{\frac{i}{h} f(x)} \det(e_a \varphi^b(x)) \delta(\varphi(x))\, dx. \tag{5}$$

Here $e_a \varphi^b(x)$ is the result of the action of $e_a \in \mathfrak{g}$ on the function φ^b.

Recall that the Grassmann algebra generated by elements $e_1, \ldots, e_n$ is the exterior algebra $\wedge^* V$ of the vector space $V = \bigoplus_{a=1}^n \mathbb{C} e_a$. Choose an orientation of V: $c = c_1 \wedge \ldots c_n \in \wedge^{\mathrm{top}} V$. The integral of $F \in \wedge^* V$ over this Grassmann algebra is, by definition, the top degree component of F, i.e., if $F = F_0 c +$lower degree terms, then $F_0 = \int F dc$. The determinant of the $n \times n$ matrix B can be written as Gaussian Grassmann integral: $\int \exp(\sum_{a,b=1}^n \bar{c}^a B_a^b c_b) dc d\bar{c} = \det(B)$. Representing the determinant in (5) as a Gaussian Grassmann integral, and taking the Fourier transform of the δ-distribution we arrive at the following formula:

$$I_h \sim \int_{M \times \mathfrak{g}_{\mathrm{odd}} \times \mathfrak{g}^*_{\mathrm{odd}} \times \mathfrak{g}^*} e^{\frac{i}{h} f_{\mathrm{FP}}(x)} dx dc d\bar{c} d\lambda,$$

where

$$f_{\mathrm{FP}} = f(x) - ih \sum_{a,b,i} \bar{c}^a e_a \varphi^b(x) c_b + \sum_a \varphi^a(x) \lambda_a.$$

The integral is taken over the super-manifold $M \times \mathfrak{g}_{\mathrm{odd}} \times \mathfrak{g}^*_{\mathrm{odd}} \times \mathfrak{g}^*$. Since f has isolated critical G-orbits, the function $f(x) + \sum_a \varphi^a(x)\lambda_a$ has isolated critical points on $M \times \mathfrak{g}^*$ (the method of Lagrange multipliers). Expanding the integral near these isolated critical points we obtain the asymptotical expansion of I_h in terms of Feynman diagrams:

$$\begin{aligned} I_h &\simeq J_h \\ &= h^{\frac{d-n}{2}} (2\pi)^{\frac{d+n}{2}} \sum_a \frac{1}{\sqrt{|\det(B(a))|}} \det(-iL(a)) e^{\frac{i}{h} f(a) + \frac{i\pi}{4} \operatorname{sign}(B(a))} \\ &\quad \left(1 + \sum_{\Gamma \neq \emptyset} \frac{(ih)^{-\chi(\Gamma)} (-1)^{c(D(\Gamma))} F_a(D(\Gamma))}{|\mathrm{Aut}(\Gamma)|}\right). \end{aligned} \tag{6}$$

The sum is taken over graphs Γ with solid (bosonic) and dashed (fermionic) edges, $\chi(\Gamma) = |\text{vertices}| - |\text{edges}|$ is its Euler characteristic, $D(\Gamma)$ is a regular projection of Γ on a plane (i.e., with only double singular points, where edges intersect). The weights $F(D(\Gamma))$ are computed according to the following rules: a) assign "states" $\alpha = (1, \ldots, d; 1, \ldots, n)$ and $a = 1, \ldots, n$ to endpoints of edges and to stars of vertices as is shown in Figure 2 and Figure 3, b) assign weights to edges (propagators) and to stars of vertices as is shown in Figure 2 and Figure 3, and take the sum of products of weights of vertices and edges of the graph over all states. It is easy to see that the product $(-1)^{c(D(\Gamma))}F(D(\Gamma))$ does not depend on the projection. The number $|\mathrm{Aut}(\Gamma)|$ is the order of the automorphism group of Γ, $c(D(\Gamma))$ is the number of intersections of fermionic (dashed) edges.

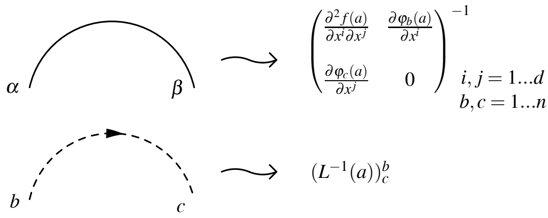

Figure 2. Weights of edges for Feynman diagrams in (6).

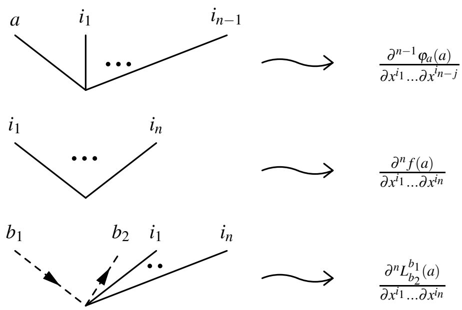

Figure 3. Weights of vertices for Feynman diagrams in (6).

When S is a *local cross section* in the vicinity of critical orbits of f, but fails to intersect each orbit once globally, the asymptotical expansion (6) of this integral is still defined and is identical to the asymptotical expansion of (4), but actual integrals may differ.

Proposition 1. *The formal power series J_h does not depend on the choice of local coordinates as long as the coordinate change is volume preserving. It does not depend on the choice of the local cross-section S.*

Remark 3. The formula (6) also works when the function f is invariant with respect to the action of the integrable distribution $L \subset TM$. If M is compact, the difference with (6) is that the power series with Feynman diagrams describes not the asymptotical expansion of I_h but of the integral[1]

$$\int_M e^{\frac{i}{h} f(x)} \frac{dx}{\mathrm{Vol}(L_x)},$$

where $\mathrm{Vol}(L_x)$ is the volume of the leaf of the distribution L through $x \in M$, so the integral is taken essentially over the space of leaves of L. In this case the functions φ^a define the local cross-section through leafs of the distribution L (a cross-section in the vicinity of critical leafs for f). The collection of volumes of leaves is the analog of the volumes of orbits with respect to the right action of G. The details will appear elsewhere.

There is a version of the Faddeev–Popov integral that works when the action of G is not free (when there are stabilizers), see for example [96].

Remark 4 (on BRST). Algebraically, the ghost fermions in the Feynman diagrams (6) can be identified with elements of the Chevalley co-chain complex for the Lie algebra $\mathfrak{g}$ with coefficients in functions on M.

Remark 5. The case when f is invariant with respect to a non-integrable distribution requires the Batalin–Vilkovisky quantization, see for example reference [28]. An important example of a field theory with such symmetry is the Poisson sigma model [76], [27].

In this case the perturbation theory describes the asymptotical expansion of an oscillatory integral over a Lagrangian submanifold of a smooth supermanifold. The paper [1] provides a powerful source of BV theories.

Remark 6. In all cases mentioned above (when f has only simple isolated critical points; when it is G-invariant with simple isolated critical orbits; when it is invariant with respect to an integrable distribution, etc.) the structure of critical points is very special. The function at a critical point in any given tangent direction is either constant, or has a simple critical point.

4.1.3. Path integrals. When path integrals are defined as mathematical objects, they provide the following construction of quantum field theory of a given classical field theory on a space-time manifold.

- The vector space $H(N)$ assigned to $(d-1)$-dimensional space is the infinite dimensional vector space of functionals on the space of boundary conditions in the corresponding classical field theory. It can also be the space of sections of a line bundle, as in the Chern–Simons theory, etc.

[1] P. Mnev and N. Reshetikhin, to appear.

- The vector $Z(M)$ in such a theory, the partition function, is the following integral:

$$Z(M|b) = \int_{\phi|_{\partial M}=b} \exp\left(i\,\frac{S_M[\phi]}{h}\right) D\phi. \tag{7}$$

  Here the integration is over all possible fields inside M with the given boundary values b, h is a real variable (Planck constant in relative units) for 'unitary' quantum field theories, and it is imaginary for Euclidean quantum field theories. In the latter case it is the partition function for the Boltzmann distribution in the space of field with the energy being given by the classical action. In this sense Euclidean quantum field theory is statistical mechanics.

This proposal makes sense only when the integral is defined, convergent, and when the spaces $H(N)$ have reasonable topology. This is the case when the space-time is the category of finite cell complexes.[2]

4.1.4. Digression: renormalization. To extend the path integral construction to theories with "continuous" space-times one should approximate the space-time by a convergent sequence of cell complexes, define a discretized classical action and the corresponding partition function Z, and then pass to the limit when the mesh of the approximation goes to zero. This is, more or less, the idea of constructive field theory [52] (see also [21] and references therein). For a given sequence of convergent approximations of the space-time, the limit of partition functions may exist only when parameters in the discretized classical action are changing as the mesh of the approximation goes to zero. Such limits are known as the scaling limits. The existence of scaling limits is closely related to the theory of critical phenomena in statistical mechanics [74].

This approach defines a "continuum" quantum field theory when such limit (i) exists, (ii) does not depend on the approximating sequence of cell complexes (within certain uniformity class), and (iii) the limiting partition functions satisfies axioms of local quantum field theory, and in particular the gluing axiom.

The discretization approach is very complicated. A "short cut" through analytical difficulties of the scaling limit is the semiclassical perturbation theory where the goal is to construct the asymptotical expansion of the partition function. The idea is to try the "wrong" order of the limits: first, take the asymptotical expansion of the discretized integral as $h \to 0$ using the asymptotical expansions in terms of Feynman diagrams outlined in the previous section, and then to pass to the limit when the mesh of the approximation in the coefficients of the asymptotical expansion goes to zero. Depending on the behavior of the coefficients in this limit several possibilities may occur:

[2] This section is a very sketchy outline of a very important chapter in modern theoretical physics. For more details see for example [60], [74], [21].

1. The coefficients converge as the mesh of the approximation goes to zero and the limit does not depend on the approximation (effectively this means that integrals defining weights of Feynman graphs in the continuum theory converge). If the asymptotical expansion of the partition function satisfies the gluing rule and factorization axioms, we have a semiclassical local quantum field theory. This is what is expected from Chern–Simons theory, and is true for semiclassical quantum mechanics [63].

2. The coefficients diverge when the mesh of the approximation goes to zero, but there exist a formal power series $h = \tilde{h} + \sum_{n \geq 1} c_n \tilde{h}^n$ and similar series for other parameters of the theory such that c_n and the coefficients in similar expansions for other parameters are mesh dependent in such a way that the coefficients of the power series in $\tilde{h}$ defining Z have finite limit. When this is possible, the field theory is called renormalizable (in perturbation theory). There is still the question of how the resulting power series depends on the choice of the approximation (known in physics literature as the regularization scheme). This is an extremely interesting mathematical question to which physics dictates a simple answer: "physically meaningful quantities should not depend (up to finite renormalization) on the regularization scheme". In a renormalizable quantum field theory change of regularization should result in a transformation on the space of parameters of the theory. Among such transformations there is the renormalization group. It is a one-dimensional subgroup corresponding to the change of scale. These transformations act on h which also means they should be closely related to the ambiguity in quantization (well known in deformation quantization).

See [79] for a detailed discussion of properties of Feynman diagrams in renormalizable field theories. One of the remarkable conjectures is that in $\mathbb{R}^n$ all of them are periods in a sense [23].

Another important issue in this case is the consistency of the assumption that $h \to 0$ with the mesh dependence of it. If we expect that the formal power series expressing h in terms of renormalized parameter $\tilde{h}$ represents the asymptotical expansion of a function, it should represent a function which vanish as the mesh goes to zero. The same should hold for other (dimensionless) parameters of the theory: corresponding power series should be represented by functions which are regular in the zero mesh limit. When this is true, the theory is called asymptotically free (assuming that h is a parameter characterizing the interaction). The Yang–Mills theory for a non-Abelian group is, conjecturally, an example of this situation [54]. Whether the partition function depends on the choice of approximating sequence is known as the question of dependence on the regularization scheme. The question of the compatibility of the gluing axiom with the renormalization was addressed in [101] but still remains an open problem.

3. The other possibility is when the theory is non-renormalizable in perturbation theory. Einstein's theory of gravity is the most known example of such theory.

However, this does not mean that a quantum theory of gravity does not exists. It only means that the perturbative "short cut" does not work there.

In the semiclassical quantization only the asymptotical behavior of the discretization of the space time when the mesh vanishes is important. Let δ be the characteristic scale of the approximation. It is a typical length of an edge assuming that valencies of cells behave regularly as $\delta \to 0$. For the scalar Bose field on the space-time which is a metrized cell complex (metrized cell approximation of a Riemannian manifold), the discretized action can be chosen as

$$S_d(\phi) = \sum_v \left(\frac{1}{2}(\partial\phi, \partial\phi) + V(\phi)\right) V(v),$$

where the first term in the brackets is the discretized Dirichlet action (see [35] for convergence of discretized Laplacians to the smooth one), and $V(v)$ is the volume of the cell v of the dual cell complex. The space of fields in a such theory are maps ϕ from vertices of the cell complex to $\mathbb{R}$.

If $\widehat{M}$ is such an approximation to M and $\partial\widehat{M}$ is its boundary, the space assigned to $\partial\widehat{M}$ is the space of maps of vertices of $\partial\widehat{M}$ to $\mathbb{R}$. The partition function is given by the integral (7), which now is finite dimensional and under simple assumptions conditionally convergent.

If ϕ is a smooth function on M, restricted to vertices on $\widehat{M}$ we have the following asymptotical expansion:

$$S_d(\phi) = \int_M \left(\frac{1}{2}(d\phi, d\phi) + V(\phi)\right) dx + \sum_{n\geq 1} \delta^n \int_M m_{n+2}(\phi,\phi)\, dx + \cdots . \quad (8)$$

Here m_n is a bidifferential operator of order n, and $\cdots$ are boundary terms of order ≥ 1 in δ. They depend on the metric on M and on the way it was approximated. The weight of any given Feynman diagram will be finite if we take enough terms in this series. If the theory is renormalizable, the resulting formal power series representing renormalized Z (and corresponding correlation functions) should depend on the choice of m_n only through transformations in the space of renormalized parameters of the action. The collection of bidifferential operators m_n is a version of the regularization by higher derivatives [60].

4.2. Semiclassical quantization of gauge theories

4.2.1. The Yang–Mills theory. In the semiclassical quantization of the Yang–Mills theory the partition function is assigned to a manifold M and to a gauge class of a connection in the principal G-bundle P restricted to the boundary. It is a formal power series, which would be the asymptotical expansion of the oscillatory path integral over all connections on P if such integral would exist, following the analogy with the finite dimensional case:

$$Z_{\mathrm{YM}}(M, A^b) = \int_{i^*(A)=A^b} e^{\frac{iS_{\mathrm{YM}}(A)}{h}} DA.$$

In a neighborhood of a classical solution A with the Dirichlet boundary condition $i^*(A) = A^b$ connections can be written as $A + \alpha$, where α is a $\mathfrak{g}$-valued 1-form on M. The Lorenz gauge condition for such connections is

$$d_A^* \alpha = 0.$$

Following the analogy with the finite dimensional case define the Faddeev–Popov action for pure Yang–Mills theory as the following action with fields $\alpha(x)$, $\bar{c}(x)$, $c(x)$:

$$\begin{aligned} S_A(\alpha) = S_{\mathrm{YM}}(A) + \int_M \frac{1}{2}\mathrm{tr}\langle F_A(\alpha), F_A(\alpha)\rangle dx \\ - \frac{ih}{2}\int_M *d_A\bar{c} \wedge d_A c - \frac{ih}{2}\int_M *d_A\bar{c} \wedge [\alpha, c]. \end{aligned} \tag{9}$$

The quadratic part in α and the quadratic part in $\bar{c}$, c of the action (9) are given by the differential operator $d_A^* d_A$, which is invertible on the space $\mathrm{Ker}(d_A^*)$ with Dirichlet boundary conditions. Other terms define weights of vertices in Feynman diagrams.

When $d \leq 4$, the Yang–Mills theory is renormalizable: there exists a renormalization procedure for Feynman diagrams which defines the power series in a "renormalized" parameter h with finite ("renormalized") coefficients [59]. The renormalization in the Yang–Mills theory is particularly remarkable because it gives an asymptotically free theory. In a nut-shell, as it was mentioned in the previous section, this means that the renormalization is consistent with the assumption $h \to 0$ and therefore with the whole idea of formal power series expansion in h [54].

Since the Yang–Mills theory is gauge invariant, natural observables are also gauge invariant. Wilson loops are examples of such observables. Let V be a finite dimensional representation of the Lie group G. The connection A defines the parallel transport in the vector bundle $V_P = P \times_G V$. The Wilson loop observable is the trace of the holonomy along a loop:

$$W_A^V(C) = \mathrm{tr}_{V_x}(h_A(C_x)), \quad h_A(C_x) = P\exp\left(\int_{C_x} A\right). \tag{10}$$

Here C_x is a path which starts and ends at $x \in M$, $h_A(C_x)$ is the holonomy of A along C_x (parallel transport), and the trace is taken over the fiber V_x of V_P over $x \in M$.

To define the "expectation value" of the Wilson loop one should make sense of

$$\int_{i^*(A)=A^b} e^{\frac{iS_{\mathrm{YM}}(A)}{h}} W_C^V(A) DA.$$

In the semiclassical (perturbative) quantization this integral is defined as a formal power series modeled after the asymptotical expansion of the finite dimensional integral (6). One of the important conjectures about the Yang–Mills theory is the *confinement conjecture*, also known as *dynamical mass generation*, which states that the expectation value of a Wilson loop (appropriately defined) should decay exponentially when the length of C increases. It is also known as dynamical mass generation. It assumes that the expectation value of Wilson loops can be defined non-perturbatively (which is a formidable mathematical problem by itself). For more details on these conjectures see [61].

Whether the renormalization is compatible with the gluing principle is more or less unknown. In order to answer this question one should investigate, first, how the metric on the space time affects the renormalization (the renormalization was developed and studied mostly in the flat space time), second, how boundary conditions affect the renormalizations, and, finally, if the renormalized theory with generic boundary conditions exists, one should check if the resulting theory satisfies the gluing/cutting principle. All these questions are open. Some results in this direction for flat space-time can be found in [101].

4.2.2. The Chern–Simons theory. The classical Chern–Simons theory with compact simple Lie group G was described in Section 3.1.3. In contrast with the Yang–Mills theory, the Chern–Simons action is a first order action. One of the implications of this is the difference in the Hamiltonian formulation and in the setting of boundary conditions for the path integral.

We want to make sense of the expressions

$$Z(M,\mathcal{L}) = \int e^{ik\mathrm{CS}(A)} DA, \quad Z_K(M,\mathcal{L}) = \int e^{ik\mathrm{CS}(A)} W_A^V(C_K) DA, \tag{11}$$

where K is a knot in M, C_K is a loop in M representing K, $W_A^V(C_K)$ is a Wilson loop observable, and k is an integer. The integral is supposed to be taken over the space of all connections on a principal G-bundle P on M. Boundary values of these connections should be pull-backs of flat connections with the gauge class in a Langrangian subspace $\mathcal{L}$ of the moduli space of flat G-connections on $P|_{\partial M}$. Because the action does not use the metric on M, the partition functions (11) for closed manifolds should depend only on the homeomorphism class of $K \subset M$, i.e., the integral should produce topological invariants of knots in 3-manifolds. This extends to links and framed graphs in 3-manifolds. The program of constructing topological invariants by quantizing the classical Chern–Simons theory was proposed and outlined by E. Witten in [112].

To define the semiclassical partition function of the Chern–Simons theory according to the rules of Section 4.1.2 we first should choose a gauge condition. Let us choose the Lorenz gauge, as in the case of Yang–Mills theory. For this we should introduce the metric (the metric in the Chern–Simons theory appear as a gauge condition). The Faddeev–Popov action for the Chern–Simons theory becomes

$$
\begin{aligned}
\mathrm{CS}_A(\alpha) = \mathrm{CS}(A) + \int_M \frac{1}{2}\mathrm{tr}(\alpha \wedge d_A\alpha - \frac{2}{3}\alpha \wedge \alpha \wedge \alpha) \\
- \frac{ih}{2}\int_M *d_A\bar{c} \wedge d_A c - \frac{ih}{2}\int_M *d_A\bar{c} \wedge [\alpha, c],
\end{aligned}
\tag{12}
$$

where h stands for $\frac{1}{k}$.

Quite remarkably [14], the field α and ghost fields in the Chern–Simons theory can be combined into one odd "super-field":

$$\Psi = c + \alpha + ih * d_A\bar{c}.$$

Here c, α, and $*d_A\bar{c}$ are 0-, 1-, and 2-forms respectively. The action (12) can be written entirely in terms of Ψ [3]:

$$\mathrm{CS}_A(\alpha) = \mathrm{CS}(A) + \frac{1}{2}\int_M \mathrm{tr}(\Psi \wedge d_A\Psi - \tfrac{2}{3}\Psi \wedge \Psi \wedge \Psi).$$

The quadratic part of this action is determined by the operator $D_A = *d_A + d_A*$ restricted to the subspace of odd forms on M (denote it by D_A^-). The operator D_A^- is invertible when there are no harmonic forms, i.e., when the Laplace–Beltrami operator $\Delta_A = D_A^2$ is invertible. An example is $\mathbb{R}^3$ with the trivial flat connection. In this case the inverse to D_A^-, restricted to the space of 1-forms, acts as

$$P \circ \alpha(x) = \frac{1}{8\pi}\sum_{ijk=1}^{3} \varepsilon^{ijk} \int_{\mathbb{R}^3} \left(\frac{(x-y)^i}{|x-y|^3}\alpha_j(x) d^3y \right) dx^k.$$

When non-zero harmonic forms for Δ_A exist, the operator D_A^- is not invertible. In this case P is a chain homotopy (parametrix) which plays the role of $(D_A^-)^{-1}$ [14], [24], [28].

The semiclassical proposal for the partition function follows the patterns of the finite dimensional case from the Section 4.1.2; however, even at the level of determinants it needs a "gravitational correction" [112], [43]. This correction is an overall factor which eliminates the metric dependence of determinants, but introduces a dependence of the partition function on the framing f on M (i.e., a section of the frame bundle over M). The final proposal for the partition function of Chern–Simons theory for a closed compact manifold M with a knot in it, in the case when the moduli space

[3]This form of the Faddeev–Popov action for the Chern–Simons theory has a simple explanation in the framework of the Batalin–Vilkovisky formalism, see for example [28].

of flat G-connections over M consists of isolated points is [112], [43], [14], [24]:

$$\begin{aligned}
&\exp\Big(c(h)\Big(\frac{i\pi}{4}\eta(g,M)+i\frac{1}{24\pi}I_M(g,f)\Big)-\frac{i\pi d(1+b^1(M))}{4}\Big)\\
&\quad\sum_{[A]}(2\pi(k+c_2(G)))^{\frac{\dim H_A^0-\dim H_A^1}{2}}\\
&\qquad\frac{1}{\mathrm{Vol}(G_A)}e^{i(\frac{1}{h}+c_2(G))\mathrm{CS}_M(A)-\frac{2\pi i I_A(M)}{4}-i\pi\frac{(\dim H_A^0+\dim H_A^1)}{2}}\\
&\qquad\quad\tau(M,A)^{1/2}\Big(W_C^V(A)+\sum_{n\geq 1}h^n F_A^{(n)}(M,C,f)\Big).
\end{aligned}\tag{13}$$

Here $c_2(G)$ is the value of the Casimir element on the adjoint representation (the dual Coxeter number), $c(h)=d/(1+hc_2(G)=k\dim(\mathfrak{g})/(k+c_2(G))$ is the central charge of the Wess–Zumino–Witten conformal field theory, $d=\dim(\mathfrak{g})$, $V(G_A)$ is the volume of the centralizer of the flat connection A (as of the representation of $\pi_1(M)$), and $b_1(M)$ is the first Betti number of M (for rational homology spheres it is zero). The eta-invariant $\eta(g,M)$ of the Riemannian manifold depends on the metric, but the combination in the exponent of the eta-invariant and the gravitational Chern–Simons term depends only on the framing f,

$$I_M(g,f)=\frac{1}{4\pi}\int_M f^*\mathrm{tr}(\omega\wedge d\omega-\tfrac{2}{3}\omega\wedge\omega\wedge\omega),\tag{14}$$

where g is the metric on M, ω is the Levi-Civita connection on M, and the integrand is the pull-back via f^* of the Chern–Simons form on TM. Other terms in (13) include the spectral flow $I_A(M)$ (see [43], [62] for details), the Ray–Singer torsion $\tau(M,A)$, and the contributions $F_A^{(n)}(M,\Gamma,f)$ from Feynman diagrams,

$$F_A^{(n)}(M,C,f)=\sum_{\Gamma,\,\mathrm{ord}\,(\Gamma)=n}\frac{I_A(D(\Gamma),M,g)(-1)^{c(D(\Gamma))}}{|\mathrm{Aut}(\Gamma)|}.\tag{15}$$

Here the sum taken over graphs with $2n$ 3-valent vertices of two types, and with two types of edges (solid and dashed), see Figures 4 and 5. The numbers $|\mathrm{Aut}(\Gamma)|$ and $c(D(\Gamma))$ are defined in (6), and $I_A(D(\Gamma),M,g)$ is the appropriate trace of the integral over M^m of the product of propagators. In other words, $I_A(D(\Gamma),M,g)$ is the contribution from the Feynman diagram $D(\Gamma)$ with weights from Figure 4. For example the first order term is

$$\int_M\int_M\sum_{\{a\},\{b\}}f_{a_1a_2a_3}f_{b_1b_2b_3}P^{a_1b_1}(x,y)P^{a_2b_2}(x,y)P^{a_3b_3}(x,y)\,dxdy.\tag{16}$$

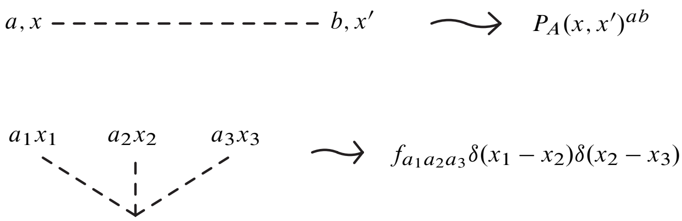

Figure 4. Weights in Feynman diagrams for the Chern–Simons theory involving propagators and vertices for the Ψ-field.

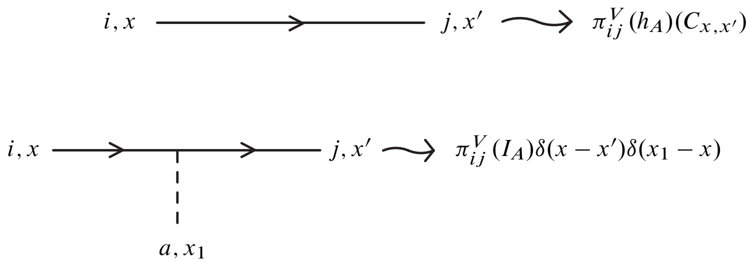

Figure 5. Weights of Feyman diagrams involving Wilson loops (solid lines). Here i, j enumerate a basis in the representation space V, $h_A(C_{x,x'})$ is the holonomy of A along $C_{x,x'} \subset C$.

Remarkably, each individual integral $I_A(D(\Gamma), M, g)$ converges, see [75], [14]. From the heuristic formula (11) we expect that the expression (13) should depend only on the homeomorphism class of M and should not depend on the choice of metric (gauge condition). But this is not obvious because in the definition of integrals we used metric on M. As it was proven in [24] for rational homology spheres (see also [75]), the dependence of the Feynman integrals on the metric has the form

$$I_A(D(\Gamma), M, g) = J_A(D(\Gamma), M, f) + c(\Gamma) I_M(g, f),$$

where f is a framing on M, $I(g, f)$ is the gravitational Chern–Simons action, and $c(\Gamma)$ is a constant. Similar formulae hold for manifolds with acyclic flat connections in which case $c(\Gamma)$ depends only on the gauge class of A and in a very special way [24].

When the moduli space of flat connections over M may have non-trival components, not only isolated points, one should expect the following formula for the asymptotical expansion of the Chern–Simons path integral [43], [62], [94]:

$$
\begin{aligned}
&\exp\left(c(h)\Big(\frac{i\pi}{4}\eta(g,M)+i\frac{1}{24}I_M(g,f)\Big)-\frac{i\pi d(1+b^1(M))}{4}\right)\\
&\sum_{[A]}(2\pi(k+h^\vee))^{\frac{\dim(H^0_A)-\dim(H^1_A)}{2}}\frac{1}{\mathrm{Vol}(G_A)}\\
&e^{i(k+h^\vee)\mathrm{CS}_M(A)-\frac{2\pi i I_A}{4}-i\pi\frac{\dim(H^0_A)+\dim(H^1_A)}{2}}\\
&\int_{M_A}\tau^{1/2}\Big(W_\Gamma(A)+\sum_{n\geq 1}F_A^{(n)}(M,C,f)\Big).
\end{aligned}
\tag{17}
$$

Here the sum is taken over connected components of the moduli space of flat connections in a principal G-bundle over M. The torsion τ is an element of $\bigotimes_i \det(H^i_A)^{\otimes(-1)^i} \simeq (\det(H^0_A)\otimes\det(H^1_A)^*)^{\otimes 2}$. The Lie algebra $\mathfrak{g}$ has an invariant scalar product and therefore H^0_A has an induced volume form. Pairing this volume form with the square root of the torsion gives a volume form on the corresponding component of the moduli space (of flat connections on the trivial bundle over M). Assuming the connected component is smooth we can integrate functions with respect to this volume form. The factor $\mathrm{Vol}(G_A)$ is the volume of the stabilizer of the flat connection. When the connection is trivial but not isolated (in particular $H^1\neq\{0\}$), Feynman diagrams were analyzed in [28].

The remarkable fact about the semiclassical quantization is that it really produces topological invariants of 3-manifolds. To be more precise, at this point we have invariants of framed 3-manifolds. But due to the existence of the canonical 2-framing on any closed 3-manifold [11], they are also invariants of 3-manifolds.

Remark 7. The inclusion $i\colon \partial M \to M$ induces the projection $i^*\colon \mathcal{M}(M) \to S(\partial M))$ from the moduli space of flat connections on $M\times G$ to the moduli space of flat connections on $\partial M\times G$. The image of this map is the Lagrangian submanifold $L(M)\subset S(\partial M)$. Sometimes this projection is a finite covering. In these cases the invariant is still given by the formula (13) but now the partition function depends on the boundary condition (a point on $L(M)$), and the square root of the torsion is a volume form on $L(M)$. This is the case when M is the complement of the 1-skeleton of a tetrahedron in S^3, or of a knot in S^3.

5. Digression: quantized universal enveloping algebras at roots of unity

In this section we will recall some basic facts about quantized universal enveloping algebras at roots of unity with the example of the quantized universal enveloping algebra of gl_2.

5.1. Quantized universal enveloping algebras. The algebra $U_t(gl_2)$ is generated over $\mathbb{C}[t,t^{-1}]$ by elements K, L, E and F with the following defining relations

$$KL = LK, \quad KE = t^2 EK, \quad KF = t^{-2} FK, \quad LE = t^2 EL, \quad LF = t^{-2} FL,$$

$$EF - FE = (t - t^{-1})(K - L^{-1}).$$

The center of $U_t(gl_2)$ is generated freely by Laurent polynomials in KL^{-1} and $C = EF + Kt^{-1} + L^{-1}t$

This is a Hopf algebra with

$$\Delta(K) = K \otimes K, \quad \Delta(L) = L \otimes L,$$

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + L^{-1} \otimes F.$$

This Hopf algebra is not quasitriangular, but because the corresponding formal deformation of $U_h gl_2$ is quasitriangular [36], there exists an outer automorphism $\mathcal{R}$ of the division algebra of $U_t(gl_2) \otimes U_t(gl_2)$ with the following properties:

$$\mathcal{R}(\Delta(a)) = \sigma \circ \Delta(a), \tag{18}$$

$$(\Delta \otimes \mathrm{id}) \circ \mathcal{R} = \mathcal{R}_{13} \circ \mathcal{R}_{23}, \quad (\mathrm{id} \otimes \Delta) \circ \mathcal{R} = \mathcal{R}_{13} \circ \mathcal{R}_{12}. \tag{19}$$

In the formal deformation when $K = \exp(\frac{hH}{2})$, $L = \exp(\frac{hG}{2})$, $t = \exp(h)$, and the algebra is completed over $\mathbb{C}[[h]]$, the automorphism $\mathcal{R}$ becomes the conjugation with the universal R-matrix. The explicit action of $\mathcal{R}$ on the generators of $U_t(gl_2)$ is given in [69].

5.2. Specialization at roots of unity. Let ε be a primitive root of 1 of an odd degree ℓ. Denote by $\mathcal{U}_\varepsilon$ the specialization of $U_t(gl_2)$ to $t = \varepsilon$. It has the following properties[4]:

- Elements E^ℓ, F^ℓ, $K^{\pm\ell}$, and $L^{\pm\ell}$ are central in $\mathcal{U}_\varepsilon$. Denote by Z_0 the central subalgebra in $\mathcal{U}_\varepsilon$ that they generate.
- Z_0 is a Hopf subalgebra with

$$\Delta(K^\ell) = K^\ell \otimes K^\ell, \quad \Delta(L^\ell) = L^\ell \otimes L^\ell,$$

$$\Delta(E^\ell) = E^\ell \otimes K^\ell + 1 \otimes E^\ell, \quad \Delta(F^\ell) = F^\ell \otimes 1 + L^{-\ell} \otimes F^\ell.$$

- The algebra $\mathcal{U}_\varepsilon$ is a free Z_0-module of dimension ℓ^4.
- The center $Z(\mathcal{U}_\varepsilon)$ is generated by Z_0 and by $C = EF + K\varepsilon - L^{-1}\varepsilon^{-1}$, KL^{-1} modulo the relation

$$\prod_{j=0}^{\ell-1} (C - K\varepsilon^{j+1} - L^{-1}\varepsilon^{-j-1}) = E^\ell F^\ell.$$

[4] Quantum groups at roots of unity with large center were studied in [30] for any simple Lie algebra.

- Let a, b, c, d be coordinates on the group $G^* \simeq B_+ \times B_-$ such that for $b_\pm \in B_\pm$ we have
$$b_+ = \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}, \quad b_- = \begin{pmatrix} d & 0 \\ c & 1 \end{pmatrix}.$$
Then the map $F^\ell \to b$, $E^\ell \to -cd^{-1}$, $K^\ell \to a$, $L^\ell \to d$ is an isomorphism of Hopf algebras $Z_0 \to C(B_+ \times B_-)$.
- $\mathcal{U}_\varepsilon$ is semisimple over a Zariski open subvariety in $\mathrm{Spec}(Z_0) \simeq B_+ \times B_-$.

Moreover, the specialization at roots of unity induces a non-trivial Poisson structure on Z_0. To define this structure choose an identification of vector spaces $U_t(gl_2) \simeq \mathcal{U}_\varepsilon$ (for example identify the PBW bases in both algebras). For $a, b \in Z(\mathcal{U}_\varepsilon)$ define

$$\{a, b\} = \frac{a \cdot b - b \cdot a}{t - \varepsilon}|_{t=\varepsilon}. \tag{20}$$

It is easy to show that this bracket does not depend on the identification, and that it is a Lie bracket acting by derivations on the commutative algebra structure on $Z(\mathcal{U}_\varepsilon)$ and therefore it defines a Poisson Lie algebra structure on this commutative algebra. The subalgebra Z_0 is a Poisson subalgebra. It is isomorphic to $C(G^*)$ as a Hopf Poisson algebra. The subalgebra Z_1 generated by C and KL^{-1} is an algebraic extension of the Poisson center $Z_0^{(c)}$ of Z_0, and $Z(\mathcal{U}_\varepsilon) = Z_0 \otimes_{Z_0^{(c)}} Z_1$.

The identification of vector spaces $U_t(gl_2) \simeq \mathcal{U}_\varepsilon$ also defines the action of the Poisson algebra $Z(\mathcal{U}_\varepsilon)$ by derivations on $\mathcal{U}_\varepsilon$ by the same formula. This action depends on the choice of the identification and changes by an inner derivation when the identification changes. The subalgebra generated by C and KL^{-1} acts trivially.

Geometrically, the algebra $\mathcal{U}_\varepsilon$ is a sheaf of finite dimensional algebras over $G^* = B_+ \times B_-$. The restriction of this sheaf to a symplectic leaf is a bundle of algebras. Over a generic symplectic leaf fibers are semisimple finite dimensional algebras [30]. The action of Z_0 by derivations on $\mathcal{U}_\varepsilon$ assigns to each element of Z_0 a connection along Hamiltonian flow lines generated by this element on G^*.

5.3. Representations at roots of unity. Geometrically, a representation of $\mathcal{U}_\varepsilon$ corresponding to a symplectic leaf $C \subset G^*$ is a vector bundle over this leaf with a Hamiltonian connection (i.e., a mapping that assigns to a function a connection along flow lines of its Hamiltonian vector field) that agrees with the similar connection on $\mathcal{U}_\varepsilon(C)$. Over a generic symplectic leaf the algebra $\mathcal{U}_\varepsilon$ is semisimple and therefore any representation decomposes into a direct sum of irreducibles. The Lie group $G^* = B_+ \times B_-$ maps naturally to $G = \mathrm{GL}_2$, $(b_+, b_-) \mapsto b_+ b_-^{-1}$. The image of a generic symplectic leave in G is a conjugation orbit where the eigenvalues are not ℓ-th roots of unity.

Let $C \subset G^*$ be a symplectic leaf, and A_x and A_y be fibers of $\mathcal{U}_\varepsilon(C)$ over two points $x, y \in C$. Because x, y are points on the same symplectic leaf, there exists a piecewise Hamiltonian path $\gamma_{x,y}$ connecting them. The corresponding Hamiltonian connections on $\mathcal{U}_\varepsilon(C)$ lift this path to an algebra isomorphism $A_x \to A_y$. If $x \in G^*$ belongs to a generic symplectic leaf, the algebra A_x is semisimple of dimension ℓ^4, and all algebras A_y are isomorphic if y belongs to the same symplectic leaf.

Generic irreducible representations of $\mathcal{U}_\varepsilon$ are determined by central characters of $\mathcal{U}_\varepsilon$. Each of them has dimension ℓ. We will denote by V_x a representation with the Z_0-central character $x \in G^*$ and by $V_{a,x}$ the irreducible $\mathcal{U}_\varepsilon$-module with the central character a which projects to x (the projection is determined by the inclusion $Z_0 \subset Z(\mathcal{U}_\varepsilon)$).

Finite dimensional representations of $\mathcal{U}_\varepsilon$ that are Z_0-irreducible form a monoidal category $\mathcal{U}_\varepsilon$-mod fibered over G^* [68]. Objects of this category are sheafs of $\mathcal{U}_\varepsilon$-modules over G^* with the fiberwise $\mathcal{U}_\varepsilon$-mod structure. Morphisms are fiberwise $\mathcal{U}_\varepsilon$-linear mappings. This category has a subcategory of maximally non-generic representations with Z_0-characters being $1 \in G^*$. It is naturally equivalent to the category of finite dimensional modules over the quotient algebra $\mathcal{U}'_\varepsilon = \mathcal{U}_\varepsilon/\langle K^\ell - 1, L^\ell - 1, E^\ell, F^\ell\rangle$. This category has a semisimple quotient category generated, as an Abelian category, by simple $\mathcal{U}'_\varepsilon$-modules.

The decomposition of the tensor product of generic representations of $\mathcal{U}_\varepsilon$ into irreducible components is very different from the gl_2 case. For simple algebras this problem was addressed in [31]. For $\mathcal{U}_\varepsilon$, when x, y, and $x \cdot y$ are generic, we have

$$V_{a,x} \times V_{b,y} \simeq \oplus_c V_{c,x\cdot y}.$$

Here the sum is taken over all irreducible representations with the Z_0-central character $x \cdot y \in G^*$.

5.4. Braiding. The quotient algebra $\mathcal{U}'_\varepsilon$ is quasitriangular, and therefore the category $\mathcal{U}'_\varepsilon$-mod is a braided monoidal category.

The category $\mathcal{U}_\varepsilon$-mod is not braided. It has a more complicated structure of a braided category fibered over a braided group, which was introduced in [68]. Recall that a group H is braided if there exists a mapping $\rho\colon H \times H \to H \times H$ with the properties

(1) $m \circ \rho = m'$,

(2) $\rho \circ (m \times \mathrm{id}) = (m \times \mathrm{id}) \circ \rho_{23} \circ \rho_{13}$,

(3) $\rho \circ (\mathrm{id} \times m) = (\mathrm{id} \times m) \circ \rho_{12} \circ \rho_{13}$,

where m is the multiplication in H and m' is the opposite multiplication. The group GL_2^* is braided with the mapping ρ defined on an open dense subset as follows. Let $I\colon \mathrm{GL}_2^* \to \mathrm{GL}_2$ be the mapping $(b_+, b_-) \to b_+ b_-^{-1}$, and $x \to (x_+, x_-)$ be its inverse, when it is defined; then

$$\rho(x, y) = (x_R, x_L),$$

where $I(x_L) = x_- I(y) x_-^{-1}$, $I(y) = (x_L)_+^{-1} I(x)(x_L)_+$, is a braiding for GL_2^*. For GL_2 these formulae define ρ because the mapping I is a bijection on an open dense subset of GL_2. The same construction also works for any simple Lie algebra (of rank r), but the mapping I in this case has degree 2^r over a generic point of G and therefore does not have an inverse. At this point one should take into account that G is a Poisson Lie group and ρ has an interpolating flow [89], [68]. Among the $2^r \times 2^r$ branches of I one should choose the one corresponding to the pair (x_R, x_L) given by the interpolating flow.

The interpolating Hamiltonian flow defines a path connecting (x, y) and (x_R, x_L) in $\mathrm{GL}_2^* \times \mathrm{GL}_2^*$ and also a connection that lifts this path to an algebra isomorphism $A_x \otimes A_y \to A_{x_R} \otimes A_{x_L}$. The outer automorphism $\mathcal{R}$ from Section 5.1 specializes to an algebra isomorphisms $\widetilde{\mathcal{R}}_{x,y} \colon A_x \otimes A_y \to A_{x_R} \otimes A_{x_L}$,

$$\widetilde{\mathcal{R}}_{x,y}(a \otimes b) = \rho_{x,y} \circ (R_1(a \otimes b) R_1^{-1}),$$

where $R_1 \in A_x \otimes A_y$ (see [89]). This mapping inherits the quasitriangular properties of $\mathcal{R}$ from Section 5.1. The mapping

$$\mathcal{R}_{x,y} = \sigma \circ \widetilde{\mathcal{R}}_{x,y}$$

defines the braiding structure on the subcategory of $\mathcal{U}_\varepsilon$-mod generated by left regular representation of this algebra. By Schur's lemma it also defines the braiding for irreducible representations of $\mathcal{U}_\varepsilon$ [69]. As a result we have a braiding mapping $R^{V,W}(x, y) \colon V_x \otimes W_y \to W_{x_L} \otimes V_{x_R}$. It is described explicitly in [69] after each irreducible representation V_x is identified with $\mathbb{C}^\ell$ using the weight basis.

6. Invariants of framed tangled graphs with flat connections in the compliment

Let t be a tangle in $C = \mathbb{R}^2 \times [0, 1]$. Representations of $\pi_1(C \backslash t)$ in a group G can be parametrized by assigning a group element g_e to each edge e of a diagram D_t of t, which is the holonomy along a path going above the diagram from a base point to e then going once around e and going back to the base point above D_t. It is well known that such collection $\{g_e\}_{e \subset D(t)}$ defines a representation of the fundamental group of the compliment with these holonomies if it satisfies relations $x_L = xyx^{-1}$, $x_R = x$ for each overcrossing (Figure 6).

A factorization on a group G is a choice of subgroups $G_\pm \subset G$ such that the multiplication mapping $G_+ \times G_- \to G$ and its opposite are bijections. When G is factorizable another parametrization of representations of the fundamental group was proposed in [68]. In this case the elements of G assigned to edges adjacent to an overcrossing (see Figure 6) satisfy the relation $x_L = x_- y x_-^{-1}$ and $x_R = (x_L)_+^{-1} x (x_L)_+$. At extremal points of the diagram of a knot the color changes $x \mapsto i(x)$ where $i(x)$

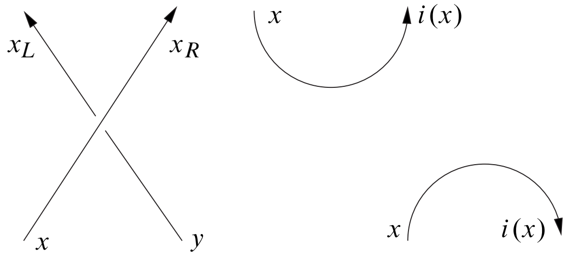

Figure 6. The Wirtinger representation of the complement to a tangle.

is the inverse to x in $G^* = G_+ \times G_-$. The Wirtinger representation corresponds to $G_- = \{0\}$ and $G_+ = G$.

Following the same strategy as in [91] one can construct invariants of tangles using the braiding for irreducible representations of $U_\varepsilon(gl_2)$ described in the previous section and the isomorphisms between a representation and its double dual $(V_x)^{\vee\vee} \simeq V_x$. For the details see [68].

For generic x the quantum dimension of V_x vanishes. This implies that corresponding invariant of tangles is zero if the tangle has a closed connected component. In particular the value of this invariant on knots is identically zero. However, an invariant can still be defined: one should cut the knot and make it into a $(1, 1)$ tangle. If the representation V is irreducible, the corresponding invariant of tangle is a multiple of the identity operator. It is easy to show that this number does not depend on where the knot has been cut. Therefore invariants of $(1, 1)$ string tangles define invariants of knots.

Such invariants of tangles and knots depend on Z_0 central characters assigned to edges of the diagram of a tangle. One can show (see [68]) that they parameterize representations of π_1 of the complement to a tangle. The corresponding invariant of a knot depends only on a conjugacy class of this representation. Thus this construction gives invariants of knots with gauge classes of flat connections in the complement.

If representations coloring connected components of links satisfy certain conditions [48], a similar construction provides invariants of links. For an Abelian gl_2-connection in the complement of a link such invariants where constructed in [48], [49]. Similarly such invariants can be constructed for tangled graphs.

7. Combinatorial TQFT

7.1. Invariants of 3-manifolds via link invariants. The combinatorial construction of invariants of 3-manifolds based on the notion of a modular category was proposed in [92]. The key idea is to use the representation of a 3-manifold as a surgery along a framed link on a handle-body [72]. Any invariant of framed links which is constant on the equivalence class of links producing the same 3-manifold is a 3-manifold invariant.

7.1.1. Modular categories. First, recall that a modular category is a semi-simple Abelian category with finitely many simple objects, which is monoidal, braided, and has additional properties such as a ribbon structure and a certain non-degeneracy condition. For the details about such categories see [92], [106], [15].

Denote by $c^{U,V}: U \otimes V \to V \otimes U$ the commutativity constraint, by I the set of isomorphism classes of simple objects, and by V_i the simple objects enumerated corresponding to $i \in I$. The ribbon structure on a monoidal, rigid, braided category is a collection of functorial isomorphisms $\tau_V : V \to V$ such that

$$\tau_{V \otimes W} = (\tau_V \otimes \tau_W) c^{-1}_{VW} c^{-1}_{WV}, \quad \tau_{V^*} = \tau_V^*, \quad \tau_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}} .$$

Here $\mathbb{1}$ is the unit object of a monoidal category, see [106] for definitions and references. In a ribbon k-linear category we have the notion of the trace over $\mathrm{Hom}(V, V)$: $\mathrm{tr}_V : \mathrm{Hom}(V, V) \to k$. Define

$$d_i = \mathrm{tr}_{V_i}(\mathrm{id}_{V_i}), \quad \tau_{V_i} = v_i \mathrm{id}_{V_i},$$

$$S_{ij} = \mathrm{tr}_{V_i \otimes V_j}(c^{V_i, V_j} c^{V_j, V_i}).$$

If $d_i \neq 0$ and if the matrix S is non-degenerate, the ribbon category is called a modular category.

The matrices S_{ij} and $T_{ij} = v_i \delta_{ij}$ define the projective representation of $\mathrm{SL}_2(\mathbb{Z})$. More generally, a modular category defines a projective representation of the mapping class group of a surface [92]. Irreducibility of these representations was studied in [9]. Most interesting examples of modular categories are obtained as quotients of the categories of finite dimensional representations of Hopf algebras $U_\varepsilon(\mathfrak{g})'$. The fact that such categories are modular was proven in [92] for sl_2 and in [2] for any simple Lie algebra.

7.1.2. Invariants of 3-manifolds and TQFT. Given a modular category $\mathcal{C}$ one can construct a TQFT as follows. Objects of the space time category are standardized surfaces (for a precise definition see [92], [106]), morphisms are 3-dimensional manifolds modulo homeomorphisms that are trivial at the boundary.

To a connected standardized surface Σ we assign the vector space

$$H(\Sigma) = \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, H^{\otimes g}),$$

where g is the genus of the surface, and $H = \bigoplus_i V_i \otimes V_i^*$. When $\Sigma \simeq \bigsqcup_{a=1}^n T^{(a)}$ this space is isomorphic to $(\mathbb{C}^k)^{\otimes k}$. The linear basis in $\mathbb{C}^k$ is given by isomorphism classes of simple objects V_i, $i = 0, \ldots, k-1$.

The construction of the corresponding combinatorial TQFT is given in [92], for more details see [106] and [15]. To the closed manifold M_L, obtained by a surgery along a framed link L on S^3, and a framed link K in the complement of $L \subset S^3$ (which defines a link in M_L) whose connected components are colored by simple object $V_{j_1}, \ldots, V_{j_m}$ we assign

$$\tau(M_L, K) = p_+^{\sigma_+(L)} p_-^{\sigma_-(L)} \sum_{i_1,\ldots,i_\ell=0}^{k-1} d_{i_1} \ldots d_{i_l} J(L_{i_1,\ldots,i_l} \sqcup K_{j_1,\ldots,j_m}), \tag{21}$$

where $\sigma_\pm(L)$ = the number of positive/negative eigenvalues of the linking matrix $\hat{L}$, $p_\pm \in \mathbb{C}$ and $J(L_{i_1,\ldots,i_l})$ is the invariant of links colored by objects of modular category $\mathcal{C}$. When $\mathcal{C}$ is the semisimple quotient of the category of finite dimensional modules of $U_\varepsilon(sl_2)'$ it is the colored Jones polynomial.

The formula (21) defines also an invariant of the complement of the tubular neighborhood of K in M_L. In particular invariants of links $J(L_{i_1,\ldots,i_l})$ can be regarded as invariants of the manifold with boundary which is the complement to L in S^3.

Remark 8. Invariants of 3-manifolds were also constructed combinatorially by Turaev and Viro [108] locally via cell decompositions and a monoidal semisimple category with finitely many objects, with some extra properties, but not necessary braided. See [50] for a detailed discussion of such categories. In particular, each modular category can be used to construct such invariants. If $\mathcal{C}$ is a modular category, such invariants are equivalent to $\tau_{\mathcal{C}}(M) \otimes \tau_{\mathcal{C}}(\overline{M})$ where $\tau_{\mathcal{C}}$ is the invariant described above.

A similar construction of invariants of 3-manifolds (with a “sufficiently linked” link inside) with gauge classes of $\mathrm{SL}_2(\mathbb{C})$-flat connections on a trivial SL_2-bundle, based on triangulation and generic representations of $U_\varepsilon(b_+)$, was proposed in [19].

7.2. Relating combinatorial and semiclassical pictures. To compare these invariants one should first choose a canonical 2-framing on M [10]. The 2-framing on M is a trivialization of $TM \oplus TM$ (the vector bundle over M with the fiber $T_xM \oplus T_xM$ over $x \in M$). The canonical 2-framing defines a branch of the gravitational Chern–Simons action with the property

$$d\frac{\pi}{4}\eta(g, M) + \frac{c(h)}{24} I_M(g, f) = 0.$$

As the dependence of higher order terms on the framing appears through $I_M(g, f)$, the canonical 2-framing defines the framing dependence of the whole asymptotical expansion.

When the moduli space of flat connection on a principal G-bundle over M is a collection of isolated points, one should expect the following asymptotics of combinatorial invariants of closed 3-manifolds with links related to $U_\varepsilon(\mathfrak{g})'$ with $\varepsilon = \exp(\frac{2\pi i}{k+h^\vee})$:

$$\tau_\varepsilon(M, L) \sim e^{-\frac{d\pi i(1+b^1(M))}{4}} \sum_{[A]} \tau_A^{1/2} (2\pi(k+h^\vee))^{\frac{\dim(H_A^0)-\dim(H_A^1)}{2}} \frac{1}{\mathrm{Vol}(G_A)} \\ e^{(k+h^\vee)\mathrm{CS}_M(A) - \frac{2i\pi I_A}{4} + i\pi\frac{\dim(H_A^0)+\dim(H_A^1)}{2}} \Big(W_L^V(A) + \sum_{n\geq 1} F_A^{(n)}(M, L)\Big), \tag{22}$$

Here $|Z(G)|$ is the number of elements in the center of G. The contribution from smooth components of non-zero dimension is expected to be

$$\tau_\varepsilon(M, L) \sim \exp\left(-\frac{d\pi i(1+b^1(M))}{4}\right) \sum_{[A]} (2\pi(k+h^\vee))^{\frac{\dim(H_A^0)-\dim(H_A^1)}{2}} \frac{1}{\mathrm{Vol}(G_A)} \\ \exp\left(i(k+h^\vee)\mathrm{CS}_M(A) - \frac{2\pi i I_A}{4} - i\pi\frac{\dim(H_A^0)+\dim(H_A^1)}{2}\right) \\ \int_{M_A} \tau_A^{1/2}\Big(W_L^V(A) + \sum_{n\geq 1} F_A^{(n)}(M, L)\Big). \tag{23}$$

This conjecture was confirmed in many cases [43], [62], [47], [94], [8].

1. When $M = S^3$, the asymptotical expansion of the expectation value of the Wilson loop is determined by the contribution from the trivial flat connection. The higher coefficients are finite type invariants. They can be computed either in terms of Feynman diagrams [17], [25] [104], or in terms of data coming from the Knizhnik–Zamolodchikov equations [16], or combinatorially [87]. Similarly, when M is a rational homology sphere, the higher order contributions for the trivial connection are known to be finite type invariants of 3-manifolds [86].

2. Freed and Gompf [43] computed the asymptotics of the invariant (21) related to quantum sl_2 for lens spaces. The results confirmed the conjecture (22). L. Jeffrey [62] extended these results to other lens spaces and to mapping tori of a torus; the results again matched (23). Rozansky [94] computed the asymptotic of the invariants for Seifert manifolds $X_g(\frac{p_1}{q_1}, \dots, \frac{p_n}{q_n})$. Andersen and Hansen [8] computed the asymptotics of the combinatorial invariant for quantum sl_2 for (p, q)-surgery on the figure 8 knot, and the results again matched the conjecture.

3. As it was mentioned in the previous section invariants of links $J(L_{i_1,\dots,i_k})$ are the simplest examples of quantum invariants of manifolds with boundary. In this case the manifold is $S^3\backslash T(L)$ where $T(L) = \bigsqcup_a T(L^a)$, and $T(L^a)$ is a solid torus with the boundary being the tubular neighborhood of L^a.

A version of the asymptotical expansion (22) for manifolds with boundary predicts the asymptotic expansion of the colored Jones polynomial, and for other invariants of links and linked graphs corresponding to irreducible representations of quantum groups, in the limit when the weight of the representation (the color) is proportional to r, and $r \to \infty$. Let $J(L_{k_1,\dots,k_\ell}, \exp(\frac{2\pi i}{r}))$ be the colored Jones polynomial of the link L with connected components colored by irreducible representations V_{k_i} of $U_\varepsilon(sl_2)'$ with $\varepsilon = \exp(\frac{2\pi i}{r})$. The semiclassical Chern–Simons theory predicts that as $k_i, r \to \infty$ with $a_i = k_i/r$ being fixed

$$J\big(L_{k_1,\dots,k_\ell}, \exp\big(\tfrac{2\pi i}{r}\big)\big) \simeq \sum_x r^{(\dim(H_x)^0 - \dim(H_x^1))/2} \sqrt{\tau_x} e^{irS(x) - \frac{2\pi i I_x}{4}} \big(1 + O\big(\tfrac{1}{r}\big)\big).$$

Here we assume that $r \to \infty$ and $k_i = a_i r$, $x \in (\pi_1(S^3\backslash L) \to \mathrm{SU}(2)/\mathrm{SU}(2))$ with the holonomy around the i-th component of L being in the conjugacy classes of $\mathrm{diag}(z_i, z_i^{-1}) \in \mathrm{SU}(2)$, and $z_j = \exp(2\pi i a_j)$.

The contribution to this asymptotical expansion from Abelian flat connections is special. For a knot, the relation between the Reidemeister torsion and the Alexander polynomial implies that one should expect

$$J\big(K_k, \exp\big(\tfrac{2\pi i}{r}\big)\big) \simeq r^{-\frac{1}{2}} \frac{\sin(\pi a)}{\Delta(K, e^{2\pi i a})}(1 + \cdots) + \cdots .$$

Here the sum is taken over gauge classes of non-Abelian flat connections on the trivial SU(2)-bundle over $S^3\backslash K$. The assumption is that the moduli space of flat connections on the trivial SU(2)-bundle with the holonomy around the meridian of $T(K)$ being conjugate to $\mathrm{diag}(e^{2\pi i a}, e^{-2\pi i a})$ consists of finitely many points. The first term is the contribution from the Abelian flat connection, and $\Delta(K, t)$ is the Alexander polynomial of the knot K. This formula, initially known as the Melvin–Morton conjecture, and its generalization, were proven in [109]. A similar formula holds for the asymptotics of colored Jones polynomials of links.

Here is an example of an explicit formula for such asymptotical expansion [95] for torus knots $K_{n,m}$:

$$\begin{aligned} &J\big(K_{n,m}(k), \exp\big(\tfrac{2\pi i}{r}\big)\big) \\ &\quad \simeq \sqrt{\frac{2}{r}} \frac{\sin(\pi n a)\sin(\pi m a)}{\sin(\pi n m a)}(1 + \cdots) \\ &\qquad + \frac{i}{2} \sum_{l \in \mathbb{Z}, 0 < l < anm} e^{i\frac{\pi}{4}\mathrm{sign}(mn) + i\pi l - \frac{i\pi r}{2}\frac{(amn - l)^2}{nm}} \frac{4 \sin \pi \frac{l}{n} \sin \pi \frac{l}{m}}{\sqrt{|nm|}}(1 + \cdots). \end{aligned} \tag{24}$$

Here the first term corresponds to the contribution of the Abelian flat connection in the complement of $K_{n,m}$, the knot is colored by the irreducible $(k + 1)$-dimensional representation of $\mathcal{U}'_\varepsilon$, and $k, r \to \infty$ such that $a = \frac{k}{r}$ is fixed. The sine function in the denominator is the value of the Alexander polynomial for $t = \exp(2i\pi a)$ for the knot $K_{n,m}$. Other terms correspond to non-Abelian flat connections in the compliment of the knot. Among higher terms in the asymptotical expansion of $J(K_k, \exp(\frac{2\pi i}{r}))$ one can identify terms which are expected to be identical to Feynman diagram contributions on the background of the Abelian flat connection in the complement with the knot (the monodromy around the meridian of the knot $\mathrm{diag}(\exp(2i\pi a), \exp(-2i\pi a))$) [95].

4. Similar asymptotical expansions should hold for invariants of framed graphs in 3-manifolds. A tetrahedron is one the simplest examples. A tetrahedron colored by $U_\varepsilon(\mathfrak{g})'$-modules is known as a $6j$-symbol. For sl_2 it is the q-analog of the Rakah–Wigner symbol. Its asymptotical expansion as $q \to 1$ ($r \to \infty$) was derived in [102]. It agrees with (22). If M is the complement of a tetrahedron in S^3, the moduli space $\mathcal{M}(M)$ of flat connections on $G \times M$ projects to $S(\partial M)$ and is a finite covering over its image $L(M) \subset S(\partial M)$. Indeed, the dimension of $S(\partial M)$ in this case is $2(6r + 2(2|\Delta_+| - 2r))$ where $|\Delta_+|$ is the number of positive roots of $\mathfrak{g}$ and r is its rank. The dimension of $\mathcal{M}(M)$ is $2\dim(\mathfrak{g})$. Since $\dim(S(\partial M)) = 2\dim(\mathcal{M}(M))$ the projection $i^*\colon \mathcal{M}(M) \to L(M)$ has zero-dimensional fibers. One can show that they consists of finitely many points. This implies that (22) describes the semiclassical asymptotic of $6j$-symbols for all $U_\varepsilon(\mathfrak{g})'$.

5. The dependence of the colored Jones polynomial of a knot on the color is very special. It involves certain q-special functions [73]. This implies that the colored Jones polynomial satisfies difference equations (with coefficients depending on the knot) [47]. The AJ-conjecture is an algebraic version of the semiclassical conjecture. It states that the ring of difference operators annihilating the colored Jones polynomial in the limit $q \to 1$ becomes the vanishing ideal describing the representation variety of $\pi_1(S^3 \backslash K)$ in $\mathrm{SL}_2(\mathbb{C})$. Its generator is known as the A-polynomial [32].

Remark 9. Another classical field theory which is closely related to the Chern–Simons theory is the $BF + B^3$ in 3d. Fields in this theory are A, a connection in a principal G-bundle over M, and B, a 1-form on M with values in $\mathfrak{g}$. The action functional is

$$S_\pm(A, B) = \int_M \mathrm{tr}(B \wedge F(A) \pm \frac{1}{3} B \wedge B \wedge B). \tag{25}$$

For the plus sign define $A_\pm = A \pm B$, and for the minus sign define $\mathcal{A} = A + iB$. Classical fields $A_\pm$ are two connection s in a principal G-bundle. It easy to check that

$$S_+(A, B) = \frac{1}{2}(\mathrm{CS}(A_+) - \mathrm{CS}(A_-)), \quad S_-(A, B) = i\,\mathrm{Im}(\mathrm{CS}(\mathcal{A}))$$

where A and B are G-connections and $\mathcal{A}$ is a $G_{\mathbb{C}}$-flat connection.

It is natural to expect that this classical topological field theory for the $+$-sign describes the semiclassical limit of the Turaev–Viro invariants. For the $-$-sign it is a natural candidate for a complex Chern–Simons theory.

7.3. Chern–Simons TQFT and geometric quantization. When the modular category is the truncated category of $U_\varepsilon(\mathfrak{g})$-modules with $\varepsilon = \exp(\frac{2\pi i}{r})$, the vector spaces $H(\Sigma)$ can be described in terms of geometric quantization of the moduli spaces of flat connections on $\Sigma \times G \to \Sigma$ [13]. In this construction the mapping class group $\Gamma(\Sigma)$ (or, rather its central extension) acts naturally on $H(\Sigma)$ [13].

Let us recall the construction of spaces $H(\Sigma)$ and of the action of the mapping class on them using geometric quantization.

The moduli space $\mathcal{M}_\Sigma^G$ is a symplectic manifold [12]. The mapping class group $\Gamma(\Sigma)$ acts naturally on it by symplectomorphisms. It also acts on the Teichmüller space $\mathcal{T}$ of complex structures on Σ. The Teichmüller space naturally parameterizes a family of Kähler structures on (M_Σ^G, ω) (with the same symplectic part), and this parametrization is $\Gamma(\Sigma)$-equivariant.

Let $(\mathcal{L}, \nabla, (\cdot,\cdot))$ by a pre-quantum line bundle over $(\mathcal{M}_\Sigma^G, \omega)$. Recall that it consists of a line bundle $\mathcal{L}$ over $\mathcal{M}_\Sigma^G$ with a connection ∇ with the curvature ω, and the Hermitian scalar product on $(\cdot,\cdot)$ on sections of $\mathcal{L}$. The prequantization line is unique up to a bundle isomorphism that preserves the connection. See [42], [88] for more details and references. Fix a Kähler structure on $\mathcal{M}_\Sigma^G$ corresponding to a $\sigma \in \mathcal{T}$. Denote the corresponding Kähler manifold M_σ, and define the Verlinde space as $\mathcal{V}_{k,\sigma} = H^0(M_\sigma, \mathcal{L}^k)$ (the space of holomorphic sections of $\mathcal{L}^k$ with respect to the complex structure induced by σ). These spaces are fibers of the *Verlinde bundle* $\mathcal{V}_k$ over $\mathcal{T}$.

One of the key properties of the bundle $\mathcal{V}_k$ is that its projectivization supports a natural flat $\Gamma(\Sigma)$-invariant connection $\tilde{\nabla}$ [13], [58], now known as the Hitchin connection. The action of the mapping class group on the covariant constant sections $\mathbb{P}(\mathcal{V}(\Sigma))$ of the projectivization of $\mathcal{V}_k$ over $\mathcal{T}$ defines the projective representation of the mapping class group $\Gamma(\Sigma)$:

$$\pi_k \colon \Gamma(\Sigma) \to \mathrm{Aut}(\mathbb{P}(\mathcal{V}(\Sigma)).$$

It was conjectured by Witten that the representation π_k of $\Gamma(\Sigma)$ is isomorphic to the one defined combinatorially on $\mathbb{P}(H(\Sigma))$. There has been done quite a lot of work on this over time. Combining the work of Laszlo [77] with the work of Tsuchiya, Ueno and Yamada, [105], and the work of Andersen and Ueno [9], one gets an explicit construction of an isomorphism between these representations.

Let Σ_f be the mapping class cylinder corresponding to the diffeomorphism $f \colon \Sigma \to \Sigma$. Taking the trace of the representation π_k define the number

$$Z_k(\Sigma_f) = \mathrm{tr}(\pi_k(f))\mathrm{Det}(f)^{-\frac{c}{2}},$$

where $c = \frac{\dim(\mathfrak{g})k}{k+h^\vee}$ is the central charge of the WZW conformal field theory. The factor $\mathrm{Det}(f)^{-\frac{c}{2}}$ is the framing correction.

This expression is expected to be the partition function of the Chern–Simons theory for the mapping class tori. It has been shown in [3] that its leading terms of the asymptotic as $k \to \infty$ agree with the semiclassical functional integral proposal.

Above we described how geometric quantization of moduli spaces $\mathcal{M}_\Sigma^G$ produces the Verlinde vector bundle $\mathcal{V}_k$. But the goal of a quantization of the moduli space as a symplectic manifold is to construct a pair: an associative algebra that quantizes the Poisson algebra of functions on $\mathcal{M}_\Sigma^G$ and a representation of this associative algebra (normally the representation is expected to have Hermitian scalar product). The geometric quantization produces the vector space. The algebra is produced by Toeplitz operators associated with smooth functions on $\mathcal{M}_\Sigma^G$. For the details on how to construct Toeplitz operators for the geometric quantization of moduli spaces see [4], [6] and references therein.

Using Toeplitz operators and their various properties, J. E. Andersen proved a number of remarkable geometric applications of the Chern–Simons TQFT. Among them are the asymptotic faithfulness of Chern–Simons representations of mapping class groups [5] and the proof of Kazhdan's property (T) for the mapping class groups [4], and that Chern–Simons representations determine the Nielsen–Thursten classification of mapping classes [7].

7.4. Complex Chern–Simons theory and volume conjectures

7.4.1. The volume conjecture. In [66] Kashaev constructed invariants of knots $\kappa(K, \varepsilon)$, which are parametrized by roots of unity ε. He also made a conjecture that these invariants have the following asymptotical behavior:

$$\lim_{k\to\infty} \frac{\ln(|\kappa(K, \exp(\frac{2\pi i}{k}))|)}{k} = \frac{1}{2\pi}\mathrm{vol}(K). \tag{26}$$

Here $\mathrm{vol}(K)$ is the hyperbolic volume of the complement of K in S^3. Murakami and Murakami [82] found that Kashaev's invariant (which was originally constructed using a triangulation of the complement to K) is the colored Jones polynomial for $q = \exp(\frac{2\pi i}{k})$ when the representation coloring the knot has the highest weight k. This representation is the quantum analog of the so-called Steinberg representation for sl_2 over a finite field, and it is the first representation (ordered by the value of the highest weight) which has zero quantum dimension. The volume conjecture (26) was verified in [70] for torus knots.

The conjecture (26) for $\kappa(K, \exp(\frac{2\pi i}{k})) = J'_k(K, \exp(\frac{2\pi i}{k}))$ is known as the *vol-*

ume conjecture. Here $J'_k(K, \exp(\frac{2\pi i}{k}))$ is the normalized colored Jones polynomial

$$J'_k\left(K, \exp\left(\tfrac{2\pi i}{k}\right)\right) = \lim_{q\to\exp(\frac{2\pi i}{k})} \frac{J_k(K,q)}{d(k)},$$

where $d(k) = (q^k - q^{-k})/(q - q^{-1})$. A good collection of references for work on this conjecture can be found on the web-site [110].

The volume conjecture was extended in [83] to describe the asymptotical behavior of the Kashaev invariant itself, not only of its absolute value. It was argued that

$$\lim_{k\to\infty} \frac{\ln(J'_k(K, \exp(\frac{2\pi i}{k})))}{k} = \frac{1}{2\pi}(\mathrm{vol}(K) + i\,\mathrm{CS}(K))$$

as $k \to \infty$, where $\mathrm{CS}(K)$ is the real part of the Chern–Simons functional evaluated at the $\mathrm{SL}_2(\mathbb{C})$ flat connection corresponding to the hyperbolic structure on the complement of K in S^3. This formula was checked in a number of examples (see [110] for references) and was extended to graphs in [111].

In [37], the asymptotical expansion of the invariant $J'_k(K, \exp(\frac{2\pi i}{k}))$ for torus knots was interpreted in terms of complex Chern–Simons and torsion factors which would match the asymptotical expansion of the path integral. It resembles the asymptotical perturbative conjecture for the Chern–Simons invariant (22), but there are some factors in the torsion part that so far have no explanation.

Gukov [55] conjectured a relation between the $\mathrm{SL}(2,\mathbb{C})$ Chern–Simons and the Jones polynomial when q is not a root of unity. In particular he conjectures that

$$\lim_{N\to\infty} \frac{a\ln(J_N(K, \exp(\frac{2\pi i a}{N})))}{N} = \frac{1}{2\pi}(\mathrm{vol}(l,m) + i\,\mathrm{CS}(l,m)),$$

where $\mathrm{vol}(l,m)$ is the Neumann–Zagier function [85], and $m = -\exp(i\pi a)$. When $0 \geq \mathrm{Re}(a) < 1$ the imaginary part of a in this formula must be non-zero. For the asymptotical analysis of the Jones polynomial for torus knots in this limits see [57].

The asymptotical behavior of invariants of tangles with $G_{\mathbb{C}}$-flat connections in the complement which were constructed in [68] (see Section 6) is more difficult to compute[5]. However, it is natural to expect that they are also related to the complex Chern–Simons theory, and to make the following *conjecture*:

$$\lim_{k\to\infty} \frac{\ln(I_k(K, \rho_{l,m}, \exp(\frac{2\pi i}{k})))}{k} = \frac{1}{2\pi}(\mathrm{vol}(l,m) + i\,\mathrm{CS}(l,m)).$$

Here k runs through positive integers. When $l, m = 1$, the invariant I_k is the Kashaev invariant and this conjecture becomes the volume conjecture. Similar limits one

[5]The invariants corresponding to the left regular representation of $U_\varepsilon(sl_2)$ where constructed in [68]. Their values for small roots of unity were computed in [81]. They turned out to be polynomials in representation variables.

should expect from invariants colored by other irreducible representations and from the invariants constructed in [19]. These invariants for knots in S^3 are conjecturally equal to invariants of knots described in Section 6 for irreducible representations of $U_\varepsilon(sl_2)$.

An integral formula for what may define invariants of knots was proposed in [56]. It uses the braiding for vertex operators in quantum Liouville theory (see for example [103] and references therein). This integral formula is very natural from the point of view of the quantization of the Teichmüller spaces [41], [67]. However, there are two problems with this formula that are not yet resolved: it is not clear whether these integrals are convergent, and also, the symmetry of the q-dilogarithms is only projectively tetrahedral. So, strictly speaking, the arguments from [56] do not add up to a theorem that the state integral proposed there is an invariant of a knot. At the same time, since this state integral is a straightforward integral version of Kashaev's formula, most likely these gaps can be fixed. The integral in [56] depends on a parameter k of a similar nature as the level in the Chern–Simons theory. The asymptotical expansion of these integrals when $k \to \infty$ was studied in [34] where it was established that (i) critical points of the logarithm of the integrant are in bijection with points in the moduli space of $\mathrm{SL}_2(\mathbb{C})$-connections in the complement of a knot (where the conjugacy class of the monodromy around a meridian is fixed by the coloring of the knot), (ii) in examples, the steepest descent contributions from critical points of the integral coincide with the complexification of the semiclassical contribution of a given flat connection to the semiclassical expansion of the Chern–Simons theory. It all indicates that the integrals from [56] should eventually define an invariant related to quantum $sl_2(\mathbb{C})$.

7.5. Complex Chern–Simons theory. The full asymptotical expansion of invariants of knots constructed in [68], [69] (Section 6) and of invariants constructed in [19] is most likely given by the semiclassical path integral over $su(2)$ 1-forms on the classical $sl_2(\mathbb{C})$ background. Such path integral can be written as

$$Z_{[A]}(M, L, b) \propto \int_{i*(a)=b} e^{ik\mathrm{CS}(A+a)} W^V_{A+a}(L) Da \tag{27}$$

The integral is understood as a semiclassical expansion. The $sl_2(\mathbb{C})$-connection A is fixed, and the integral is taken over $su(2)$-valued 1-forms. To define the asymptotical expansion of such integrals one should draw a parallel with the steepest descent method, as much as a parallel was drawn in the case of compact Chern–Simons with the oscillatory integrals.

The problem of analytical continuation of the Chern–Simons theory to $sl_2(\mathbb{C})$ is discussed recently in details in [114] where the integration is over the real slice $A = \bar{A}$ in $sl_2(\mathbb{C}) \times sl_2(\mathbb{C})$-connections.

8. Other developments

Many important developments in quantum field theory, and in particular in topological quantum field theory, were not discussed here. Here is a brief outline of some of them.

1. Dijkgraaf and Witten [33] defined a TQFT corresponding to a cocycle on a finite group. Recently there have been interesting developments in this direction, see [99], [107].

2. One of the most important developments of the last decade is the Khovanov categorification of the Jones polynomial [71] and of other invariants of knots. It is remarkable how these results are deeply related to the representation theory and in particular to Soergel's categorification. This approach provided new invariants of knots and links, however the corresponding topological quantum field theory is not understood yet.

3. Large N asymptotical behavior of the Chern–Simons theory for $G = \mathrm{SU}(N)$ is closely related to open topological strings [113] and to closed topological strings [53]. Remarkable advances took place in this directions with developing topological amplitudes, the relation between topological string theories and dimer models, matrix models etc.

4. At a deeper level topological field theories are ultimately related to n-categories. These aspects of TQFT were recently explored in [78], [44], [18].

5. One of the remarkable subjects at the interface of geometry, representation theory, and quantum field theory is the geometric Langlands program [46]. The supersymmetric $N = 4$ Yang–Mills theory is expected to have a particularly important role in this direction [65]. Among other features this theory is expected to be finite in perturbation theory: individual Feynman diagrams may diverge, but the sum of them in each order is finite. This conjecture was checked in the axial gauge. Although it is widely anticipated that it is true in any other gauge, it would be interesting to see a reliable proof of gauge independence of this result. Quantum supersymmetric Yang–Mills is also important in recent developments relating Bethe ansatz and moduli spaces of instantons [51], [84].

6. The Poisson sigma model is a topological two dimensional quantum field theory which is behind the solution to the deformation quantization problem of Poisson manifolds [76]. This quantum field theory is particularly interesting because it is an example of the gauge theory where vector fields describing the infinitesimal symmetry transformations define a non-integrable distribution on the tangent space to fields of the theory. In this theory one should really use the BV quantization [27]. The Poisson sigma model is probably the most interesting example where this technique is really important.

7. There are quantum field theories with symmetries that are so powerful that they define the QFT uniquely, or almost uniquely. In other words, the spaces $H(N)$ in such theories are representation spaces, and the vectors $Z(M)$ are invariant vectors.

In some cases there are finitely many vectors and they can be characterized entirely in terms of representation theory. Examples of such theories include all known conformal field theories (the partition function is invariant with respect to conformal mappings), integrable field theories (in this case the partition function is usually invariant with respect to the action of a quantum affine algebra), and the Chern–Simons and some other topological field theories (the partition function is invariant with respect to homeomorphisms). Such theories are a remarkable class in the sense that partition functions and correlation functions can be defined not only semiclassically/perturbatively, but in terms of the representation theory of corresponding algebraic objects. In addition, many important characteristics can be computed explicitly.

We already discussed TQFT's. Two dimensional conformal field theories [20] are probably most remarkable from an algebraic point of view among all quantum field theories. They can be described completely in terms of the representation theory of affine Kac–Moody algebras and W-algebras. There is a noteworthy relation between conformal field theories and topological field theories. Conformal field theories appear as boundary quantum field theories for a large class of topological quantum field theories. The category describing combinatorial data of a conformal field theory is modular under some natural assumptions about the theory [80]. This modular category can be used to construct a TQFT's.

Integrable models in QFT demonstrate various interesting phenomena. Algebraically, they are more complicated then conformal field theories, as they are related to the representation theory of quantized universal enveloping algebras of affine Lie algebras. They also demonstrate a number of important phenomena. The quantum Sine-Gordon model on a torus shows how solutions to the classical equations of motion can be obtained as the semiclassical limit of particle like states. Passing from soliton solutions to corresponding quantum states is known as quantization of solitons (see for example [39]). Other important examples of integrable models in QFT include non-linear O(3) σ-model, chiral Gross–Neveu models, and principal chiral field theories on simple compact Lie groups. All these models have ultraviolet divergencies in the perturbation theory, the renormalization is asymptotically free, and they have mass generation. Local correlation functions in all these models can be described explicitly [98]. For the latest developments see [64].

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Prize Lectures

5ECM Amsterdam 2008

Computational complexity and numerical stability of linear problems

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Abstract. We survey classical and recent developments in numerical linear algebra, focusing on two issues: computational complexity, or arithmetic costs, and numerical stability, or performance under roundoff error. We present a brief account of the algebraic complexity theory as well as the general error analysis for matrix multiplication and related problems. We emphasize the central role played by the matrix multiplication problem and discuss historical and modern approaches to its solution.

Mathematics Subject Classification (2010). 65Y20, 68Q17, 68Q25, 68Q05, 65F05, 65F15, 65F30, 65G50, 20C05, 20K01, 14Q20, 16S34, 16T50, 43A30, 68W30, 68W40, 65-02, 68-02.

Keywords. Arithmetic complexity, multiplicative complexity, asymptotic complexity, bilinear algorithms, tensors, tensor rank, border rank, matrix multiplication, matrix inversion, rank revealing decomposition, QR decomposition, LU decomposition, SVD, Schur form, Sylvester equation, Strassen's algorithm, group algebras, Fourier transform, wreath product, classical model of arithmetic, numerical stability, arithmetic operations, bit operations.

1. Computational complexity of linear problems

In algebraic complexity theory one is often interested in the number of arithmetic operations required to perform a given computation, modelled as a programme which receives an input (a finite set of elements of some algebra) and performs a sequence of algebra operations (addition, subtraction, multiplication, division, and scalar multiplication). This is called the total (arithmetic) complexity of the computation.[1] Moreover, it is often appropriate to count only multiplications (and divisions), but not additions or multiplications by fixed scalars. These notions can be formalized [BCS97, Definition 4.7]. For now, let us invoke

Notation. Let $\mathbb{F}$ be a field, let $\mathbb{F}[x_1, \dots, x_n] \subseteq A \subseteq \mathbb{F}(x_1, \dots, x_n)$ be an $\mathbb{F}$-algebra, and let $\Phi = \{\varphi_1, \dots, \varphi_m\}$ be a finite set of functions. The total arithmetic complexity of Φ will be denoted $L_A^{\mathrm{tot}}(\Phi)$, and its multiplicative complexity by $L_A(\Phi)$.

[1] This model only counts the operations and completely ignores storage and communications costs, limitations on precision, details of how arithmetic is implemented (in particular we are not counting *bit* operations), etc.

Intuitively, this is the minimal number of steps required to compute all of $\varphi_1, \ldots, \varphi_m$ starting from a generic input $(x_1, \ldots, x_n)$, with intermediate results in A (in all cases we consider, A will simply be the algebra of polynomials or of rational functions in the input variables, and will not always be explicitly indicated). The input and all intermediate results are understood to be stored in memory, and the simultaneous computation of a set Φ of functions means that at the end of the programme Φ is contained in the set of results.[2]

Let U, V, and W be finite-dimensional vector spaces over $\mathbb{F}$, and consider the class of bilinear functions $\varphi\colon U \times V \to W$ (which includes matrix multiplication). To define the multiplicative complexity of such a function, choose bases $\{u_i\}_{1 \le i \le m}$, $\{v_j\}_{1 \le j \le n}$, and $\{w_k\}_{1 \le k \le p}$, so that

$$\varphi\Big(\sum_{i=1}^{m} x_i u_i, \sum_{j=1}^{n} y_j v_j\Big) = \sum_{k=1}^{p} \varphi_k w_k.$$

We regard the coefficients as variables, so that each φ_k is a homogeneous polynomial of degree 2 in $x_1, \ldots, x_m, y_1, \ldots, y_n$.

Definition (Cf. [BCS97, Definition 14.2]).

$$L(\varphi) = L_{\mathbb{F}[x_1,\ldots,x_m,y_1,\ldots,y_n]}\big(\{\varphi_1, \ldots, \varphi_p\}\big).$$

Because we are considering the multiplicative complexity, this is a well-defined notion that does not depend on the choice of bases.

It turns out that the multiplicative complexity of a bilinear function $\varphi\colon U \times V \to W$ is controlled by a somewhat more well-behaved notion, the *rank* $R(\varphi)$. This is a standard notion in multilinear algebra, which generalizes that of the rank of a linear map.

Definition. Let $t \in V_1 \otimes \cdots \otimes V_n$. The rank $R(t)$ is the smallest r such that one can write $t = \sum_{i=1}^{r} t_i$ with each t_i a monomial tensor, i.e., of the form $t_i = v_1 \otimes \cdots \otimes v_n$ for some $v_i \in V_i$.

In case $\varphi\colon U \times V \to \mathbb{F}$ is the bilinear map corresponding to a linear function $\widetilde{\varphi}\colon U \to V^*$, the rank $R(\varphi)$ is the rank of $\widetilde{\varphi}$ in the usual sense. Well-known algorithms, such as Gaussian elimination, as well as the fast algorithms described in this paper (see Section 1.4), can quickly compute the rank of a matrix, but determining the rank of a tensor of order 3 already seems to be quite difficult. Computing the rank of a given tensor is a combinatorial or algebro-geometric problem [Lan08].

[2] For example, $X_0 := x$, $X_1 := X_0 \cdot X_0$, $X_2 := X_1 \cdot X_1$, $X_3 := X_2 \cdot X_0$ is a programme, using 3 operations – solely multiplications in this example, which computes x^5, as well as any subset of $\{x, x^2, x^4, x^5\}$, in $A = \mathbb{F}[x]$.

We now explain how the rank controls the complexity of a bilinear function. First, by a known result of Strassen (see [BCS97, Proposition 14.4]), if $\varphi\colon V \to W$ is a *quadratic* map between finite-dimensional vector spaces, that is,

$$\varphi\Big(\sum_{i=1}^{n} x_i v_i\Big) = \sum_{j=1}^{p} \varphi_j(x_1, \dots, x_n) w_j$$

for some bases $\{v_i\}_{1\le i\le n}$ (resp. $\{w_j\}_{1\le j\le p}$) of V (resp. W) and homogeneous polynomials $\varphi_i \in \mathbb{F}[x_1, \dots, x_n]$ of degree two, then we need not search through some rather large class of programmes to find one which computes φ optimally, for in fact $L(\varphi) = L_{\mathbb{F}[x_1,\dots,x_n]}\big(\{\varphi_1, \dots, \varphi_p\}\big)$ equals the smallest $l \ge 1$ such that

$$\varphi(v) = \sum_{i=1}^{l} f_i(v) g_i(v) w_i \tag{1}$$

for some linear functionals f_i, $g_i \in V^*$. (Note that such a formula immediately gives an obvious algorithm computing $\varphi(v)$ using only l (non-scalar) multiplications.)

Now let $\varphi\colon U \times V \to W$ be a bilinear map between finite-dimensional vector spaces. This is covered by the preceding result of Strassen, since a bilinear map $U \times V \to W$ may be regarded as a quadratic map via the isomorphism $\mathbb{F}[U \times V] = \mathbb{F}[U] \otimes \mathbb{F}[V]$. A *bilinear algorithm* for φ amounts to writing

$$\varphi(u, v) = \sum_{i=1}^{r} f_i(u) g_i(v) w_i \tag{2}$$

for certain linear functionals $f_i \in U^*$, $g_i \in V^*$, and $w_i \in W$. The minimum such r is the rank $R(\varphi)$. Note that the rank of φ is not necessarily the same as its bilinear complexity, despite the superficially similar-looking formulae (1) and (2). However, by decomposing a linear functional $f : U \times V \to \mathbb{F}$ as $f(u, v) = f(u, 0) + f(0, v)$, one can see that

$$L(\varphi) \le R(\varphi) \le 2L(\varphi).$$

It is often easier to work with the rank rather than the more subtle notion of multiplicative (or total) complexity, and the above inequality shows we do not lose much in doing so.

1.1. Algebraic complexity of matrix multiplication. The basic problem is to compute the (total or multiplicative) complexity of multiplying two $n \times n$ matrices. This is a difficult question whose answer is not at present known for $n = 3$, for instance.

Matrix multiplication is a bilinear problem (see Section 1)

$$\varphi\colon M_{n\times n}(\mathbb{F}) \times M_{n\times n}(\mathbb{F}) \to M_{n\times n}(\mathbb{F}),$$
$$(X, Y) \mapsto XY = \Big(\sum_{l=1}^{n} X_{il} Y_{lj}\Big)_{1\le i,j\le n}$$

whose corresponding tensor will be denoted

$$\langle n, n, n\rangle := \sum_{1\le i,j,k\le n} u_{ij} \otimes v_{jk} \otimes w_{ki}.$$

For $n = 2$ Winograd proved [Win71] that seven multiplications are required, so $L(\langle 2,2,2\rangle) = R(\langle 2,2,2\rangle) = 7$, but for $n = 3$ even the rank is not known at present (it is known that $19 \le R(\langle 3,3,3\rangle) \le 23$; see [BCS97, Exercise 15.3], [Lan08]).

Instead of fixing n, one considers the *asymptotic* complexity of matrix multiplication:

$$\omega(\mathbb{F}) = \inf\Big\{\tau \in \mathbb{R} \mid L^{\mathrm{tot}}_{\mathbb{F}[X_{ij},Y_{ij}]}\big(\{\textstyle\sum_{l=1}^{n} X_{il}Y_{lj} | 1 \le i,j \le n\}\big) = O(n^{\tau})\Big\} \quad (3)$$

so that $n \times n$ matrices with entries in $\mathbb{F}$ may be multiplied using $O(n^{\omega(\mathbb{F})+\eta})$ operations,[3] for every $\eta > 0$.

First of all, one can replace the total complexity in (3) by the multiplicative complexity or by the rank [BCS97, Proposition 15.1] and get the same exponent. Second, $\omega(\mathbb{F})$ is invariant under extension of scalars [BCS97, Proposition 15.18], so it does not depend on the exact choice of field $\mathbb{F}$ (e.g., $\mathbb{Q}$ versus $\mathbb{R}$ or $\mathbb{C}$), but rather only on its characteristic, which is usually taken to be zero (so ω denotes $\omega(\mathbb{C})$).

The value of ω is an important quantity in numerical linear algebra, as it determines the asymptotic complexity of not merely matrix multiplication but also matrix inversion, various matrix decompositions, evaluating determinants, etc. (see Sections 1.4 and 2.3).

An obvious bound is $2 \le \omega \le 3$, since the straightforward method of matrix multiplication uses $O(n^3)$ operations, on one hand, while on the other hand we need at least n^2 multiplications to compute n^2 independent matrix entries. The first known algorithm proving that $\omega < 3$ was Strassen's algorithm, detailed in Section 2.1, which starts with an algorithm for multiplying 2×2 matrices using seven multiplications and applies it recursively, giving $\omega \le \log_2 7$. This idea of exploiting recursion will be explored in the next section.

1.2. Asymptotic bilinear complexity via tensor ranks. The basic idea behind designing fast algorithms to multiply arbitrarily large matrices, thereby obtaining good

[3] Technically, division is not allowed, as the computation should be in $\mathbb{F}[X_{ij}, Y_{ij}]$, although this is no restriction if $\mathbb{F}$ is an infinite field (see [BCS97, Remark 15.2]).

upper bounds on ω, is to exploit recursion: multiplication of large matrices can be reduced to several smaller matrix multiplications. One obvious way to do this is to decompose the matrix into blocks, as in Strassen's original algorithm. Strassen's "laser method" [BCS97, Section 15.8] is a sophisticated version of this, where several matrix-multiplication tensors are efficiently packed into a single bilinear operation (not necessarily itself a matrix multiplication). The rank of the tensor – in fact the border rank, which will be defined below – is used to keep track of the complexity of the resulting recursive algorithm, and appears in the resulting inequality for ω. This idea of recursion is also behind the "group-theoretic" algorithms described in the next section.

We have mentioned that the exponent of matrix multiplication may be defined in terms of the rank $R\big(\langle n,n,n\rangle\big)$:

$$\omega(\mathbb{F}) = \inf\big\{\tau \in \mathbb{R} \mid R\big(\langle n,n,n\rangle\big) = O(n^\tau)\big\}.$$

The reason for dealing with the rank rather than directly with the complexity measure is that the rank is better behaved with respect to certain operations, and this will be useful for deriving bounds on the asymptotic complexity via recursion. In particular [BCS97, Proposition 14.23], we have

$$R(\varphi_1 \otimes \varphi_2) \le R(\varphi_1) \otimes R(\varphi_2)$$

for bilinear maps φ_1 and φ_2, while the corresponding inequality with L in place of R is not known to be true. Let $\langle e,h,l\rangle$ be the tensor of $M_{e\times h} \times M_{h\times l} \to M_{e\times l}$ matrix multiplication. Since $\langle e,h,l\rangle \otimes \langle e',h',l'\rangle \cong \langle ee',hh',ll'\rangle$ [BCS97, Proposition 14.26], we have $R\big(\langle ee',hh',ll'\rangle\big) = R\big(\langle e,h,l\rangle\big)R\big(\langle e',h',l'\rangle\big)$. Using properties of the rank function, it is easy to derive bounds on ω given estimates of the rank of a particular tensor.

Example. If $R\big(\langle h,h,h\rangle\big) \le r$, then $h^\omega \le r$.

The first generalization is to allow rectangular matrices, via symmetrization: we have

$$R\big(\langle e,h,l\rangle\big) = R\big(\langle h,l,e\rangle\big) = R\big(\langle l,e,h\rangle\big)$$

(another nice property of the rank not shared by the multiplicative complexity), so if $R\big(\langle e,h,l\rangle\big) \le r$, then $R\big(\langle ehl,ehl,ehl\rangle\big) \le r^3$, and therefore

$$(ehl)^{\omega/3} \le r. \tag{4}$$

The next refinement is to multiply several matrices at once. But first we need to discuss border rank. The border rank appears as follows. The idea is that one may be able to approximate a tensor t of a certain rank by a family $t_1(\varepsilon) = \sum_{i=1}^{r} u_i(\varepsilon) \otimes v_i(\varepsilon) \otimes w_i(\varepsilon)$ of tensors of possibly smaller rank, meaning

$$\varepsilon^{1-q} t_1(\varepsilon) = t + O(\varepsilon)$$

for some positive integer q. The border rank $\underline{R}(t)$ is the smallest r for which this is possible. This has a geometric interpretation, studied by Landsberg [Lan08].

The border rank is always less than or equal to the rank, and shares some of its properties, including that of being hard to determine. Landsberg [Lan06] proved that $\underline{R}(\langle 2,2,2\rangle) = R(\langle 2,2,2\rangle) = 7$, but for $n = 3$ the best result known is $14 \le \underline{R}(\langle 3,3,3\rangle) \le 21$ (to be compared with the estimate $19 \le R(\langle 3,3,3\rangle) \le 23$ mentioned before).

The border rank may be strictly less than the rank. For instance, the rank of

$$t = x_1 \otimes y_1 \otimes (z_1 + z_2) + x_1 \otimes y_2 \otimes z_1 + x_2 \otimes y_1 \otimes z_1$$

is 3, but its border rank is only 2:

$$\begin{aligned} \varepsilon^{-1} t_1(\varepsilon) &:= \varepsilon^{-1}\big[(\varepsilon - 1)x_1 \otimes y_1 \otimes z_1 + (x_1 + \varepsilon x_2) \otimes (y_1 + \varepsilon y_2) \otimes (z_1 + \varepsilon z_2)\big] \\ &= t + O(\varepsilon), \end{aligned}$$

as can be seen by expanding the left-hand side.

The importance of the border rank is that, as in this example, the original tensor may be recovered from $t_1(\varepsilon)$ by computing the coefficient of some power of ε; in other words, from such an approximate algorithm for computing t we may recover an exact one. This expansion increases the number of monomials, so this does not help to compute t itself; the magic happens when we compute $t^{\otimes N}$ for large N. Taking tensor powers corresponds to multiplying matrices recursively.

The border rank replaces the rank in a refinement of (4), so that $\underline{R}(\langle e,h,l\rangle) \le r$ implies $(ehl)^{\omega/3} \le r$. A bit of work, generalizing this to the case of several simultaneous matrix multiplications, results in Schönhage's *asymptotic sum inequality*

$$\underline{R}\Big(\bigoplus_{i=1}^{s} \langle e_i, h_i, l_i\rangle\Big) \le r \Longrightarrow \sum_{i=1}^{s} (e_i h_i l_i)^{\omega/3} \le r. \tag{5}$$

From these sorts of considerations, one can see that good bounds on the asymptotic complexity of matrix multiplication can be obtained by constructing specific tensors of small border rank which contain matrix tensors as components; this is the idea behind Strassen *et al.*'s laser method.

The principle of the laser method [BCS97, Proposition 15.41] is to look for a tensor t, of small border rank, which has a direct-sum decomposition into blocks each of which is isomorphic to a matrix tensor, and whose support is "tight", ensuring that in a large power of t one can find a sufficiently large direct sum of matrix tensors. Then one can apply (5).

This combinatorial method was used by Coppersmith and Winograd [CW90] to derive $\omega < 2.376$, the best estimate currently known.

1.3. Group-theoretic methods of fast matrix multiplication. As explained in the previous section, the general principle is to embed several simultaneous matrix multiplications in a single tensor, via some combinatorial construction to ensure that the embedding is efficient.

A rough sketch of Cohn *et al.*'s [CKSU05] "group-theoretic" algorithms is that they involve embedding matrix multiplication into multiplication in a group algebra $\mathbb{C}[G]$ of a finite group G. The embedding uses three subsets of G satisfying the "triple product property" to encode matrices as elements of the group algebra, so that the matrix product can be read off the corresponding product in $\mathbb{C}[G]$. The number of operations required to multiply two matrices is, therefore, less than or equal to the number of operations required to multiply two elements of $\mathbb{C}[G]$. As a ring, $\mathbb{C}[G] \cong M_{d_1\times d_1}(\mathbb{C}) \times \cdots \times M_{d_r\times d_r}(\mathbb{C})$, where $d_1, \ldots, d_r$ are the dimensions of the irreducible representations of G (see, for instance, [Lam01, Chapter 3]). This isomorphism may be realized as a Fourier transform on G, which can be computed efficiently. In other words, multiplication in $\mathbb{C}[G]$ is equivalent to several smaller matrix multiplications, and one can apply the algorithm recursively in order to get a bound on ω.

Cohn *et al.*'s embedding is of a very particular type, based on the following triple product property: if there are subsets $X,\ Y,\ Z \subseteq G$ such that $xx'^{-1}yy'^{-1}zz'^{-1} = 1$, then $x = x'$, $y = y'$, and $z = z'$. This realizes the $|X| \times |Y|$ by $|Y| \times |Z|$ matrix multiplication AB by sending a_{xy} to $\sum a_{xy}x^{-1}y$ and $b_{y'z}$ to $\sum b_{y'z}y'^{-1}z$; the triple product property ensures that one can extract the matrix product from the product in the group algebra by looking at the coefficients of $x^{-1}z$ for $x \in X$ and $z \in Z$.

It may be more convenient, as in the previous section, to encode *several* matrix multiplications via the *simultaneous* triple product property: for $X_i,\ Y_i,\ Z_i \subseteq H$ one should have $x_i x_j'^{-1} y_j y_k'^{-1} z_k z_i'^{-1} = 1 \longrightarrow i = j = k$ and $x_i = x_i'$, $y_i = y_i'$, $z_i = z_i'$. It follows from (5) that

$$\sum_i \big(|X_i||Y_i||Z_i|\big)^{\omega/3} \leq \sum_k d_k^{\omega}.$$

We remark that the simultaneous triple product property in H reduces to the triple product property in the wreath product $G = H^n \rtimes \mathrm{Sym}_n$, so the groups actually output by this method turn out rather large.

From this initial description it is not at all clear what kinds of groups will give good bounds. To this end, Cohn *et al.* introduce several combinatorial constructions, analogous to those of Coppersmith and Winograd, which produce subsets satisfying the simultaneous triple product property inside powers H^k of a finite Abelian group H, and hence the triple product property inside wreath products of H with the symmetric group. This reproduces the known bounds $\omega < 2.376$, etc.

The group-theoretic method therefore provides another perspective on efficiently packing several independent matrix multiplications into one. In both cases the essential problem seems to be a combinatorial one, and one can state combinatorial conjectures which would imply $\omega = 2$.

1.4. Asymptotic complexity of other linear problems. One can also use recursive "divide-and-conquer" algorithms to prove that the asymptotic complexity of other problems in linear algebra is the same as that of matrix multiplication. This justifies the emphasis placed on matrix multiplication in numerical linear algebra.

As a simple example, we will begin with

Example (matrix inversion). On one hand, we have the identity

$$\begin{pmatrix} I & A & 0 \\ 0 & I & B \\ 0 & 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -A & AB \\ 0 & I & -B \\ 0 & 0 & I \end{pmatrix},$$

which shows that two $n \times n$ matrices may be multiplied by inverting a $3n \times 3n$ matrix. This shows that if an invertible $n \times n$ matrix can be inverted in $O(n^{\omega+\eta})$ operations, then the product of two arbitrary $n \times n$ matrices can also be computed in $O(n^{\omega+\eta})$ operations.

In the other direction, consider the identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix}, \quad S := D - CA^{-1}B.$$

This shows that inversion of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in M_{2n\times 2n}(\mathbb{C})$ can be reduced to a certain (fixed) number of $n \times n$ matrix multiplications and inversions.[4] Unfortunately, the indicated inverses, e.g., A^{-1}, may not exist. This defect may be remedied by writing $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = X = X^*(XX^*)^{-1}$. Now XX^* is a positive-definite Hermitian matrix, to which the indicated algorithm may be applied (both its upper-left block and its Schur complement will be positive-definite and Hermitian). We conclude that fast multiplication implies fast inversion of positive-definite Hermitian, and therefore of arbitrary (invertible), matrices.

Example (LU decomposition). Suppose, for instance, that one wishes to decompose a matrix A as $A = LUP$, where L is lower triangular and unipotent, U is upper

[4] For instance, 2 inversions and 6 multiplications.

triangular, and P is a permutation matrix. Note that not every matrix has such a decomposition; a sufficient condition for it to exist is that A have full row rank.

One can give a recursive algorithm [BCS97, Theorem 16.4], due to Bunch and Hopcroft, for computing the decomposition in case A has full row rank, via a 2×2 block decomposition of A. This involves one inversion of a triangular matrix, two applications of the algorithm to smaller matrices, and several matrix multiplications; we elide the details. Since multiplication and inversion can be done fast, analysis of this algorithm shows that if an $n \times n$ matrix can be multiplied in $O(n^{\omega+\eta})$ operations, then the LU decomposition of an $m \times n$ matrix can be done in $O(nm^{\omega+\eta-1})$ operations, that is $O(n^{\omega+\eta})$ in the case of a square matrix.

To show, conversely, that fast LU decomposition implies fast matrix multiplication, one notes that det A may be computed from an LU decomposition of A, and that computing determinants is at least as hard as matrix multiplication (cf. [BCS97, Theorem 16.7]). This shows that the exponents of matrix multiplication, LU decomposition, and determinants coincide.[5]

Further examples involving other linear problems may be found in the literature; see [BCS97] and also Section 2.3.

2. Numerical stability of linear problems

Numerical stability is just as important for the implementation of any algorithm as computational cost, since accumulation and propagation of roundoff errors may significantly distort the output of the algorithm, making the algorithm essentially useless. On the other hand, if roundoff error bounds can be established for a given algorithm, this guarantees that its output values can be trusted to lie within the regions provided by the error bounds. Moreover, such regions can typically be made small by increasing the hardware precision appropriately. Fast matrix multiplication algorithms, from Strassen's algorithm to the recent group-theoretic algorithms of Cohn *et al.*, can be analysed in a uniform fashion from the stability point of view [DDHK07].

The roundoff-error analysis of Strassen's method was first performed by Brent ([Bre70], [Hig90], see also Chapter 23 in [Hig02]). The analysis of subsequent Strassen-like algorithms is due a number of authors, most notably by Bini and Lotti [BL80]. This latter approach was advanced in [DDHK07] to build an inclusive framework that accommodates all Strassen-like algorithms based on stationary partitioning, bilinear algorithms with non-stationary partitioning, and finally the group-theoretic algorithms of the kind developed in [CU03] and [CKSU05]. Moreover, combining this framework with a result of Raz [Raz03], one can prove that there exist

[5] Compare this result on determinants with the problem of computing the permanent, which is NP-hard!

numerically stable matrix multiplication algorithms which perform $O(n^{\omega+\eta})$ operations, for arbitrarily small $\eta > 0$, where ω is the exponent of matrix multiplication.

The starting point of the error analysis [DDHK07] is the so-called *classical model of rounded arithmetic,* where each arithmetic operation introduces a small multiplicative error, i.e., the computed value of each arithmetic operation $\mathrm{op}(a, b)$ is given by $\mathrm{op}(a, b)(1 + \theta)$ where $|\theta|$ is bounded by some fixed *machine precision* ε but is otherwise arbitrary. The arithmetic operations in classical arithmetic are $\{+, -, \cdot\}$. The roundoff errors are assumed to be introduced by *every execution* of any arithmetic operation. It is further assumed that all algorithms output the exact value in the absence of roundoff errors (i.e., when all errors θ are zero).

The error analysis can be performed with respect to various norms on the matrices $A, B, C = AB$, as will be made clear in the next section. It leads to error bounds of the form

$$\|C_{\text{comp}} - C\| \le \mu(n)\varepsilon\|A\|\,\|B\| + O(\varepsilon^2), \tag{6}$$

with $\mu(n)$ typically low-degree polynomials in the order n of the matrices involved, so that $\mu(n) = O(n^c)$ for some constant c. Switching from one norm to another is always possible, using the equivalence of norms on a finite-dimensional space, but this may incur additional factors that depend on n.

2.1. Recursive matrix multiplication: Strassen and beyond. In his breakthrough paper [Str69], Strassen observed that the multiplication of two 2×2 block matrices requires only 7 (instead of 8) block multiplications, and used that remarkable observation recursively to obtain a matrix-multiplication algorithm with running time $O(n^{\log_2 7})$. Precisely, the product

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \times \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

can be computed by calculating the submatrices

$$\begin{aligned}
M_1 &= (A_{11} + A_{22})(B_{11} + B_{22}),\\
M_2 &= (A_{21} + A_{22})B_{11},\\
M_3 &= A_{11}(B_{12} - B_{22}),\\
M_4 &= A_{22}(B_{21} - B_{11}),\\
M_5 &= (A_{11} + A_{12})B_{22},\\
M_6 &= (A_{21} - A_{11})(B_{11} + B_{12}),\\
M_7 &= (A_{12} - A_{22})(B_{21} + B_{22}),
\end{aligned}$$

and then combining them linearly as

$$\begin{aligned}
C_{11} &= M_1 + M_4 - M_5 + M_7,\\
C_{12} &= M_3 + M_5,\\
C_{21} &= M_2 + M_4,\\
C_{22} &= M_1 - M_2 + M_3 + M_6.
\end{aligned}$$

Starting with matrices of dyadic order, this algorithm can be applied by recursively partitioning each matrix into four square blocks and running these computations. This yields running time $O(n^{\log_2 7}) \approx O(n^{2.81})$. Since any matrix can be padded with zeros to achieve the nearest dyadic order, the dyadic size assumption is not restrictive at all.

The breakthrough of Strassen generated a flurry of activity in the area, leading to a number of subsequent improvements, among those by Pan [Pan78], Bini *et al.* [BCRL79], Schönhage [Sch81], Strassen [Str87], and eventually Coppersmith and Winograd [CW90]. Each of these algorithms is Strassen-like, i.e., uses recursive partitioning and a special "trick" to reduce the number of block matrix multiplications.

Such recursive algorithms for matrix multiplication can be analysed as follows. Recall that bilinear functions can be evaluated via bilinear algorithms, as in Equation (2). Since they do not use commutativity of the coordinates, these algorithms apply equally well when the input entries are elements of a non-commutative algebra; their recursive use for matrix multiplication is then straightforward. A *bilinear non-commutative algorithm* (see [BL80] or [BD78]) that computes products $C = AB$ in $M_{k\times k}(\mathbb{F})$ using t non-scalar multiplications over a subfield $\mathbb{H} \subseteq \mathbb{F}$ (not necessarily equal to $\mathbb{F}$)[6] is determined by three $k^2\times t$ matrices U, V and W with elements in $\mathbb{H}$ such that

$$c_{hl} = \sum_{s=1}^{t} w_{rs} P_s, \quad \text{where } P_s = \Big(\sum_{i=1}^{k^2} u_{is} x_i\Big)\Big(\sum_{j=1}^{k^2} v_{js} y_j\Big), \quad \begin{array}{l} r = k(h-1)+l,\\ h,l = 1,\dots,k, \end{array} \tag{7}$$

where x_i (resp. y_j) are the elements of $A = (a_{ij})$ (resp. of $B = (b_{ij})$) ordered column-wise, and $C = (c_{ij})$ is the product $C = AB$.

For an arbitrary n, the algorithm consists of recursive partitioning and applying (7) to compute products of resulting block matrices. More precisely, suppose that A and B are of size $n\times n$, where n is a power of k (which can always be achieved by padding the matrices A and B with zero columns and rows, as we already mentioned). Partition A and B into k^2 square blocks A_{ij}, B_{ij} of size $(n/k)\times(n/k)$. Then the blocks C_{hl} of the product $C = AB$ can be computed by applying (7) to the blocks of A and B, where each block A_{ij}, B_{ij} has to be again partitioned into k^2 square sub-blocks

[6] Field extensions have no effect on the asymptotic complexity, but changing $\mathbb{H}$ will affect the constants in $\mu(n)$.

to compute the t products P_s and then the blocks C_{hl}. The algorithm obtained by running this recursive procedure $\log_k n$ times computes the product $C = AB$ using at most $O(n^{\log_k t})$ multiplications.

Theorem ([DDHK07, Theorem 3.1]). *A bilinear non-commutative algorithm for matrix multiplication based on stationary partitioning is stable. It satisfies the error bound* (6) *where* $\|\cdot\|$ *is the maximum-entry norm and where*

$$\mu(n) = \bigl(1 + \max_{r,s}(\alpha_s + \beta_s + \gamma_r + 3)\log_k n\bigr) \cdot \bigl(\mathrm{emax} \cdot \|U\|\,\|V\|\,\|W\|\bigr)^{\log_k n}.$$

Here $\alpha_s = \lceil \log_2 a_s \rceil$, $\beta_s = \lceil \log_2 b_s \rceil$ *and* $\gamma_r = \lceil \log_2 c_r \rceil$ *where* a_s *and* b_s (*resp.* c_r) *are the number of non-zero entries of* U *and* V (*resp.* W) *in column* s (*resp. row* r), *while* emax *is an integer that depends* (*in a rather involved way*) *on the sparsity pattern of the matrices* U, V *and* W.

This theorem can be subsequently combined with the result of Raz [Raz03] that the exponent of matrix multiplication is achieved by bilinear non-commutative algorithms [Raz03] to produce an important corollary:

Corollary ([DDHK07, Theorem 3.3]). *For every* $\eta > 0$ *there exists an algorithm for multiplying* n*-by-*n *matrices which performs* $O(n^{\omega+\eta})$ *operations* (*where* ω *is the exponent of matrix multiplication*) *and which is numerically stable, in the sense that it satisfies the error bound* (6) *with* $\mu(n) = O(n^c)$ *for some constant* c *depending on* η *but not* n.

The analysis of stationary algorithms extends to bilinear matrix multiplication algorithms based on non-stationary partitioning. This means that the matrices $A^{[j]}_{s,\mathrm{comp}}$ and $B^{[j]}_{s,\mathrm{comp}}$ are partitioned into $k \times k$ square blocks, but k depends on the level of recursion, i.e., $k = k(j)$, and the corresponding matrices U, V and W also depend on j: $U = U(j)$, $V = V(j)$, $W = W(j)$. Otherwise the algorithm proceeds exactly like the stationary algorithms.

Finally, algorithms that combine recursive non-stationary partitioning with pre- and post-processing given by linear maps $\mathrm{P{\scriptstyle RE}}_n(\,)$ and $\mathrm{P{\scriptstyle OST}}_n(\,)$ acting on matrices of an arbitrary order n can be analysed using essentially the same approach [DDHK07]. Suppose that the matrices A and B are each (linearly) pre-processed, then partitioned into blocks, respective pairs of blocks are multiplied recursively and assembled into a large matrix, which is then (linearly) post-processed to obtain the resulting matrix C.

The analysis in [DDHK07] is performed for an arbitrary *consistent* (i.e., submultiplicative) norm $\|\cdot\|$ that in addition must be defined for matrices of all sizes and must satisfy the condition

$$\max_s \|M_s\| \le \|M\| \le \sum_s \|M_s\| \tag{8}$$

whenever the matrix M is partitioned into blocks $(M_s)_s$ (an example of such a norm is provided by the 2-norm $\|\cdot\|_2$). Note that the previously mentioned maximum-entry norm satisfies (8) but is not consistent, i.e., does not satisfy

$$\|AB\| \leq \|A\| \cdot \|B\| \quad \text{for all } A,\ B.$$

Denoting the norms of pre- and post- processing maps subordinate to the norm $\|\cdot\|$ by $\|\cdot\|_{\mathrm{op}}$, we suppose that the pre- and post-processing is performed with errors

$$\begin{aligned}
\|\mathrm{PRE}_n(M)_{\mathrm{comp}} - \mathrm{PRE}_n(M)\|_{\mathrm{op}} &\leq f_{\mathrm{pre}}(n)\varepsilon\|M\| + O(\varepsilon^2),\\
\|\mathrm{POST}_n(M)_{\mathrm{comp}} - \mathrm{POST}_n(M)\|_{\mathrm{op}} &\leq f_{\mathrm{post}}(n)\varepsilon\|M\| + O(\varepsilon^2),
\end{aligned}$$

where n is the order of the matrix M. As before, we denote by $\mu(n)$ the coefficient of ε in the final error bound (6).

Under all these assumptions, the following error estimate follows:

Theorem ([DDHK07, Theorem 3.5]). *A recursive matrix multiplication algorithm based on non-stationary partitioning with pre- and post-processing is stable. It satisfies the error bound* (6)*, with the function* μ *satisfying the recursion*

$$\begin{aligned}
\mu(n_j) = \mu(n_{j+1})t_j\|\mathrm{POST}_{n_j}\|_{\mathrm{op}}\|\mathrm{PRE}_{n_j}\|^2_{\mathrm{op}}\\
+ 2f_{\mathrm{pre}}(n_j)t_j\|\mathrm{POST}_{n_j}\|_{\mathrm{op}} + f_{\mathrm{post}}(n_j)\|\mathrm{PRE}_{n_j}\|^2_{\mathrm{op}}
\end{aligned}$$

for $j = 1, \ldots, p$.

2.2. Group-theoretic matrix multiplication. In this section we describe the group-theoretic constructions of Cohn *et al.* Our exposition closely follows the pertinent parts of [DDHK07]. To give a general idea about group-theoretic fast matrix multiplication, we must first recall some basic definitions from algebra.

Definition (semidirect product). If H is any group and Q is a group which acts (on the left) by automorphisms of H, with $q \cdot h$ denoting the action of $q \in Q$ on $h \in H$, then the *semidirect product* $H \rtimes Q$ is the set of ordered pairs (h, q) with the multiplication law

$$(h_1, q_1)(h_2, q_2) = (h_1(q_1 \cdot h_2), q_1q_2). \tag{9}$$

We will identify $H \times \{1_Q\}$ with H and $\{1_H\} \times Q$ with Q, so that an element $(h, q) \in H \rtimes Q$ may also be denoted simply by hq. Note that the multiplication law of $H \rtimes Q$ implies the relation $qh = (q \cdot h)q$.

Definition (wreath product). If H is any group, S is any finite set, and Q is a group with a left action on S, the *wreath product* $H \wr Q$ is the semidirect product $(H^S) \rtimes Q$

where Q acts on the direct product of $|S|$ copies of H by permuting the coordinates according to the action of Q on S. (To be more precise about the action of Q on H^S, if an element $h \in H^S$ is represented as a function $h\colon S \to H$, then $q \cdot h$ represents the function $s \mapsto h(q^{-1}(s))$.)

Definition (triple product property, simultaneous triple product property). If H is a group and X, Y, Z are three subsets, we say X, Y, Z satisfy the *triple product property* if it is the case that for all $q_x \in Q(X)$, $q_y \in Q(Y)$, $q_z \in Q(Z)$, if $q_x q_y q_z = 1$ then $q_x = q_y = q_z = 1$. Here $Q(X) = Q(X, X)$ is the set of quotients; $Q(S, T) := \{st^{-1} \mid s \in S,\ t \in T\} \subseteq H$.

If $\{(X_i, Y_i, Z_i) \mid i \in I\}$ is a collection of ordered triples of subsets of H, we say that this collection satisfies the *simultaneous triple product property (STPP)* if it is the case that for all $i, j, k \in I$ and all $q_x \in Q(X_i, X_j)$, $q_y \in Q(Y_j, Y_k)$, $q_z \in Q(Z_k, Z_i)$, if $q_x q_y q_z = 1$ then $q_x = q_y = q_z = 1$ and $i = j = k$.

Definition (Abelian STP family). An *Abelian STP family* with growth parameters (α, β) is a collection of ordered triples (H_N, Υ_N, k_N), defined for all $N > 0$, satisfying

(1) H_N is an Abelian group,

(2) $\Upsilon_N = \{(X_i, Y_i, Z_i) \mid i = 1, 2, \ldots, N\}$ is a collection of N ordered triples of subsets of H_N satisfying the simultaneous triple product property,

(3) $|H_N| = N^{\alpha + o(1)}$,

(4) $k_N = \prod_{i=1}^N |X_i| = \prod_{i=1}^N |Y_i| = \prod_{i=1}^N |Z_i| = N^{\beta N + o(N)}$.

Recall from Section 1.3 that in [CKSU05] matrix-multiplication algorithms are constructed based on families of wreath products of Abelian groups.

To get into more details, we must recall basic facts about the discrete Fourier transform of an Abelian group. For an Abelian group H, let $\widehat{H}$ denote the set of all homomorphisms from H to S^1, the multiplicative group of complex numbers with unit modulus. Elements of $\widehat{H}$ are called *characters* and are usually denoted by the letter χ. The sets H, $\widehat{H}$ have the same cardinality. When H_1, H_2 are two Abelian groups, there is a canonical bijection between the sets $\widehat{H}_1 \times \widehat{H}_2$ and $(H_1 \times H_2)^\wedge$; this bijection maps an ordered pair (χ_1, χ_2) to the character χ given by the formula $\chi(h_1, h_2) = \chi_1(h_1)\chi_2(h_2)$. Just as the symmetric group Sym_n acts on H^n via the formula $\sigma \cdot (h_1, h_2, \ldots, h_n) = (h_{\sigma^{-1}(1)}, h_{\sigma^{-1}(2)}, \ldots, h_{\sigma^{-1}(n)})$, there is a left action of Sym_n on the set $\widehat{H}^n$ defined by the formula $\sigma \cdot (\chi_1, \chi_2, \ldots, \chi_n) = (\chi_{\sigma^{-1}(1)}, \chi_{\sigma^{-1}(2)}, \ldots, \chi_{\sigma^{-1}(n)})$.

Notation. The notation $\Xi(H^n)$ will be used to denote a subset of $\widehat{H}^n$ containing exactly one representative of each orbit of the Sym_n action on $\widehat{H}^n$. An orbit of this action is uniquely determined by a multiset consisting of n characters of H, so the cardinality of $\Xi(H^n)$ is equal to the number of such multisets, i.e. $\binom{|H|+N-1}{N}$.

Given an Abelian STP family, the corresponding recursive matrix multiplication algorithm is defined as follows. Given a pair of n-by-n matrices A, B, find the minimum N such that $k_N \cdot N! \geq n$, and denote the group H_N by H. If $N! \geq n$, multiply the matrices using an arbitrary algorithm. (This is the base of the recursion.) Otherwise reduce the problem of computing the matrix product AB to $\binom{|H|+N-1}{N}$ instances of $N! \times N!$ matrix multiplication, using a reduction based on the discrete Fourier transform of the Abelian group H^N.

Padding the matrices with additional rows and columns of 0's if necessary, one may assume that $k_N \cdot N! = n$. Define subsets $X, Y, Z \subseteq H \wr \mathrm{Sym}_N$ as

$$X = \Big(\prod_{i=1}^{N} X_i\Big) \times \mathrm{Sym}_N, \quad Y = \Big(\prod_{i=1}^{N} Y_i\Big) \times \mathrm{Sym}_N, \quad Z = \Big(\prod_{i=1}^{N} Z_i\Big) \times \mathrm{Sym}_N\,.$$

These subsets satisfy the triple product property [CKSU05]. Note that $|X| = |Y| = |Z| = n$. Now treat the rows and columns of A as being indexed by the sets X, Y, respectively; treat the rows and columns of B as being indexed by the sets Y, Z, respectively.

The algorithm uses two auxiliary vector spaces $\mathbb{C}[H \wr \mathrm{Sym}_N]$ and $\mathbb{C}[\widehat{H}^N \rtimes \mathrm{Sym}_N]$, each of dimensionality $|H|^N N!$ and each with a specific basis: the basis for $\mathbb{C}[H \wr \mathrm{Sym}_N]$ is denoted by $\{\, \mathbf{e}_g \mid g \in H \wr \mathrm{Sym}_N \,\}$, and the basis for $\mathbb{C}[\widehat{H}^N \rtimes \mathrm{Sym}_N]$ is denoted by $\{\, \mathbf{e}_{\chi,\sigma} \mid \chi \in \widehat{H}^N,\ \sigma \in \mathrm{Sym}_N \,\}$.

The Abelian STP algorithm from [CKSU05] performs the following steps, which will be labelled according to whether they perform arithmetic or not. (For example, a permutation of the components of a vector does not involve any arithmetic.)

1. **Embedding** (NO ARITHMETIC).
   Compute the following pair of vectors in $\mathbb{C}[H \wr \mathrm{Sym}_N]$:

$$a := \sum_{x \in X} \sum_{y \in Y} A_{xy} \mathbf{e}_{x^{-1}y},$$

$$b := \sum_{y \in Y} \sum_{z \in Z} B_{yz} \mathbf{e}_{y^{-1}z}.$$

2. **Fourier transform** (ARITHMETIC).
   Compute the following pair of vectors in $\mathbb{C}[\widehat{H}^N \rtimes \mathrm{Sym}_N]$:

$$\hat{a} := \sum_{\chi \in \widehat{H}^N} \sum_{\sigma \in \mathrm{Sym}_N} \Big(\sum_{h \in H^N} \chi(h) a_{\sigma h} \Big) \mathbf{e}_{\chi,\sigma},$$

$$\hat{b} := \sum_{\chi \in \widehat{H}^N} \sum_{\sigma \in \mathrm{Sym}_N} \Big(\sum_{h \in H^N} \chi(h) b_{\sigma h} \Big) \mathbf{e}_{\chi,\sigma}.$$

3. **Assemble matrices** (NO ARITHMETIC).
For every $\chi \in \Xi(H^N)$, compute the following pair of matrices A^χ, B^χ, whose rows and columns are indexed by elements of Sym_N:

$$A^\chi_{\rho\sigma} := \hat{a}_{\rho\cdot\chi,\sigma\rho^{-1}},$$
$$B^\chi_{\sigma\tau} := \hat{b}_{\sigma\cdot\chi,\tau\sigma^{-1}}.$$

4. **Multiply matrices** (ARITHMETIC).
For every $\chi \in \Xi(H^N)$, compute the matrix product $C^\chi := A^\chi B^\chi$ by recursively applying the Abelian STP algorithm.

5. **Disassemble matrices** (NO ARITHMETIC).
Compute a vector $\hat{c} := \sum_{\chi,\sigma} \hat{c}_{\chi,\sigma}\mathbf{e}_{\chi,\sigma} \in \mathbb{C}[\widehat{H}^N \rtimes \mathrm{Sym}_N]$ whose components $\hat{c}_{\chi,\sigma}$ are defined as follows. Given χ, σ, let $\chi_0 \in \Xi(H^N)$ and $\tau \in \mathrm{Sym}_N$ be such that $\chi = \tau \cdot \chi_0$. Let
$$\hat{c}_{\chi,\sigma} := C^{\chi_0}_{\tau,\sigma\tau}.$$

6. **Inverse Fourier transform** (ARITHMETIC).
Compute the following vector $c \in \mathbb{C}[H \wr \mathrm{Sym}_N]$.

$$c := \sum_{h\in H^N} \sum_{\sigma\in \mathrm{Sym}_N} \left(\frac{1}{|H|^N} \sum_{\chi\in\widehat{H}^N} \chi(-h)\hat{c}_{\chi,\sigma} \right) \mathbf{e}_{\sigma h}.$$

7. **Output** (NO ARITHMETIC).
Output the matrix $C = (C_{xz})$ whose entries are given by the formula

$$C_{xz} := c_{x^{-1}z}.$$

The main result of [DDHK07] establishes the numerical stability of all Abelian STP algorithms.

Theorem ([DDHK07, Theorem 4.13]). *If $\{(H_N, \Upsilon_N, k_N)\}$ is an Abelian STP family with growth parameters (α, β), then the corresponding Abelian STP algorithm is stable. It satisfies the error bound (6), with the Frobenius norm and the function μ of order*
$$\mu(n) = n^{\frac{\alpha+2}{2\beta} + o(1)}.$$

Remark ([DDHK07, Remark 4.15]). The running time of an Abelian STP algorithm can also be bounded in terms of the growth parameters of the Abelian STP family. Specifically, the running time is [CKSU05] $O\left(n^{(\alpha-1)/\beta + o(1)}\right)$. Note the curious interplay between the two exponents, $(\alpha-1)/\beta$ and $(\alpha+2)/2\beta$: their sum is always bigger than 3, since $\alpha \geq 2\beta+1$ is one of the requirements for an Abelian STP construction:

$$\frac{\alpha-1}{\beta}+\frac{\alpha+2}{2\beta}=\frac{3\alpha}{2\beta}\geq\frac{6\beta+3}{2\beta}>3.$$

2.3. Matrix decompositions and other linear problems. The results about matrix multiplication from the previous section can be extended to show that essentially *all* linear algebra operations can also be done stably, in time $O(n^{\omega})$ or $O(n^{\omega+\eta})$, for arbitrary $\eta>0$ [DDH07]. For simplicity, whenever an exponent contains "$+\eta$", it will henceforth mean "for any $\eta>0$". Below we summarize the main results of [DDH07].

The first result in [DDH07] can be roughly summarized by saying that n-by-n matrices can be multiplied in $O(n^{\omega+\eta})$ operations *if and only if* n-by-n matrices can be inverted stably in $O(n^{\omega+\eta})$ operations. Some extra precision is necessary to make this claim; the cost of extra precision is included in the $O(n^{\eta})$ factor.

Other results in [DDH07] may be summarized by saying that if n-by-n matrices can be multiplied in $O(n^{\omega+\eta})$ *arithmetic* operations, then the QR decomposition can be computed stably (moreover, linear systems and least squares problems can be solved stably) in $O(n^{\omega+\eta})$ *arithmetic* operations. These results do not require extra precision, which is why one needs to count arithmetic operations rather than bit operations.

The QR decomposition can be used to stably compute a rank-revealing decomposition, the (generalized) Schur form, and the singular value decomposition, all in $O(n^{\omega+\eta})$ *arithmetic* operations. Computing (generalized) eigenvectors from the Schur form, can be done by solving the (generalized) Sylvester equation, all of which can be done stably in $O(n^{\omega+\eta})$ *bit* operations.

Here are a few more details about the work in [DDH07]. The paper starts off by reviewing conventional block algorithms used in libraries like LAPACK [ABB+99] and ScaLAPACK [BCC+97]. The normwise backward stability of these algorithms was shown earlier [Hig90], [DHS95], [Hig02] using (6) as an assumption. This means that these algorithms are guaranteed to produce the exact answer (e.g., solution of a linear system) for a matrix $\hat{C}$ close to the actual input matrix C, where close means close in norm:

$$\|\hat{C}-C\|=O(\varepsilon)\|C\|.$$

Here the $O(\varepsilon)$ is interpreted to include a factor n^c for a modest constant c.

The running-time analysis of these block algorithms in [DDH07] shows that these block algorithms run only as fast as $O(n^{\frac{9-2\gamma}{4-\gamma}})$ operations, where $O(n^{\gamma})$ is the operation count of matrix multiplication, with γ used instead of $\omega+\eta$ to simplify notation. Even if γ were to drop from 3 to 2, the exponent $\frac{9-2\gamma}{4-\gamma}$ would only drop from 3 to 2.5, providing only a partial improvement. However, further results in [DDH07] demonstrate that one can do better.

The next step in [DDH07] is the application of known divide-and-conquer algorithms for reducing the complexity of matrix inversion to the complexity of matrix multiplication. These algorithms are not backward stable in the conventional sense. However, they can be shown to achieve the same forward error bound (bound on the norm of the error in the output) as a conventional backward stable algorithm, provided that they use just $O(\log^p n)$ times as many bits of precision in each arithmetic operation (for some $p > 0$) as a conventional algorithm. Such algorithms are called *logarithmically stable.*

Incorporating the cost of this extra precise arithmetic in the analysis only increases the total cost by a factor at most $\log^{2p} n$. Therefore, if there are matrix multiplication algorithms running in $O(n^{\omega+\eta})$ operations for any $\eta > 0$, then these logarithmically stable algorithms for operations like matrix inversion also run in $O(n^{\omega+\eta})$ operations for any $\eta > 0$, and satisfy the same error bound as a conventional algorithm.

A divide-and-conquer algorithm for QR decomposition from [EG00] is simultaneously backward stable in the conventional normwise sense (i.e., without extra precision), and runs in $O(n^{\omega+\eta})$ operations for any $\eta > 0$. This algorithm may be in turn used to solve linear systems, least-squares problems, and compute determinants equally stably and fast. The same idea applies to LU decomposition but stability depends on a particular pivoting assumption [DDH07].

The QR decomposition can then be used to compute a rank-revealing URV decomposition of a matrix A. This means that U and V are orthogonal, R is upper triangular, and R reveals the rank of A in the following sense: Suppose $\sigma_1 \geq \cdots \geq \sigma_n$ are the singular values of A. Then for each r, $\sigma_{\min}(R(1 : r, 1 : r))$ is an approximation of σ_r and $\sigma_{\max}(R(r+1 : n, r+1 : n))$ is an approximation of σ_{r+1}. The algorithm in [DDH07] is *randomized,* in the sense that the approximations of σ_r and σ_{r+1} are reasonably accurate with high probability.

Finally, the QR and URV decompositions in algorithms for the (generalized) Schur form of nonsymmetric matrices (or pencils) [BDG97] lower their complexity to $O(n^{\omega+\eta})$ arithmetic operations while maintaining normwise backward stability. The singular-value decomposition may in turn be reduced to solving an eigenvalue problem with the same complexity. Computing (generalized) eigenvectors can only be done in a logarithmically stable way from the (generalized) Schur form. This is done by providing a logarithmically stable algorithm for solving the (generalized) Sylvester equation, and using this to compute eigenvectors.

This covers nearly all standard dense linear algebra operations (LU decomposition, QR decomposition, matrix inversion, linear equation solving, solving least squares problems, computing the (generalized) Schur form, computing the SVD, and solving (generalized) Sylvester equations) and shows that all those problems can be solved stably and asymptotically as fast as the fastest matrix multiplication algorithm that may ever exist (whether the fastest matrix multiplication algorithm is stable or not). For all but matrix inversion and solving (generalized) Sylvester equations, stability means backward stability in a normwise sense, and the complexity is measured by the usual count of arithmetic operations.

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5ECM Amsterdam 2008

High-dimensional distributions with convexity properties

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Abstract. We review recent advances in the understanding of probability measures with geometric characteristics on $\mathbb{R}^n$, for large n. These advances include the central limit theorem for convex sets, according to which the uniform measure on a high-dimensional convex body has marginals that are approximately gaussian.

Mathematics Subject Classification (2010). 46B07, 60F05, 52A20.

Keywords. Convex sets, high dimension, central limit theorem.

1. Introduction

This article is concerned with probability measures in high dimension that satisfy certain geometric convexity assumptions. Probability distributions on high dimensional spaces appear in quite a few branches of mathematics and mathematical physics. From probability theory to quantum physics, from analysis and combinatorics to statistical mechanics, it is not uncommon to study a distribution, or a family of distributions, on a space of many "equally important" parameters. These high-dimensional measures are usually, but not always, quite concrete. A general study of probability distributions in high dimension is likely hopeless, as such distributions may exhibit a wide range of entirely unrelated phenomena (see [36] for a possible slight exception).

There seem to exist, nevertheless, some large classes of distributions which obey some interesting, non-trivial principles. One of the earliest such examples is provided by the classical *Central Limit Theorem*. Suppose we are given a probability density $f : \mathbb{R}^n \to [0, \infty)$ which is a product density, i.e.,

$$f(x_1, \dots, x_n) = \prod_{i=1}^{n} f_i(x_i)$$

for some functions $f_1, \dots, f_n$. Then f is the joint density of n independent random variables $X_1, \dots, X_n$. Assume that the dimension n is large. Under mild integrability

*The author is a Clay Research Fellow, and is also supported by NSF grant #DMS-0456590.

assumptions on the f_i's, it is guaranteed that

$$\mathbb{P}\Big(\sum_{i=1}^{n} \theta_i X_i \leq t\Big) \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\big(-(s-b)^2/2\big)\, ds \quad \text{for all } t \in \mathbb{R}, \tag{1}$$

for appropriate coefficients $b, \theta_1, \dots, \theta_n \in \mathbb{R}$. Stated differently, any product density f has marginals that are approximately gaussian. This fact demonstrates that product densities enjoy strong regularity properties in high dimension. Moreover, when the density f is properly normalized (such that $X_1, \dots, X_n$ have mean zero and variance one), the gaussian approximation (1) actually holds for "most" choices of $\theta_1, \dots, \theta_n \in \mathbb{R}$ with $\sum_i \theta_i^2 = 1$. By "most" we mean that the coefficients $\theta_1, \dots, \theta_n$ may be chosen randomly, uniformly on the unit sphere S^{n-1} in $\mathbb{R}^n$.

The case of independent random variables is perhaps the paradigmatic example for high-dimensional measures with a clear *structure*, distributions that are composed of basic building blocks. Another source for regularity in the study of high-dimensional measures is *symmetry*; Measures that possess symmetries, whether they be apparent or hidden, are usually easier to analyze.

In this article, we revisit the central limit theorem and related principles from a more geometric point of view. Rather than exploiting the structure or symmetries of a given high-dimensional distribution, our plan is to investigate classes of densities with certain geometric characteristics. In particular, we shall see that *convexity* conditions fit very well with high dimensionality. The study of uniform measures on arbitrary high-dimensional convex sets turns out to be quite fruitful, as well as the study of probability densities of the form $\exp(-H)$, for a convex function $H : \mathbb{R}^n \to \mathbb{R}$. The spatial arrangement of volume due to the geometry of $\mathbb{R}^n$, for large n, imposes regularity and order on such convexity-related measures.

This text is based on a talk given by the author at the fifth European Congress of Mathematics. It is not intended as a comprehensive survey of the subject, as we are far from exhausting all of the relevant literature. I would like to thank Emanuel Milman, Vitali Milman and Sasha Sodin for reading a preliminary version of this note.

2. An example: The sphere

Write $|x|$ for the standard Euclidean norm of $x \in \mathbb{R}^n$, and $x \cdot y$ for the usual scalar product of $x, y \in \mathbb{R}^n$. The unit sphere in $\mathbb{R}^n$ is $S^{n-1} = \{x \in \mathbb{R}^n\,;\, |x| = 1\}$. For a set $A \subseteq S^{n-1}$ and $\varepsilon > 0$ denote

$$A_\varepsilon = \{x \in S^{n-1}\,;\, d(x, y) \leq \varepsilon \text{ for some } y \in A\},$$

the ε-neighborhood of A. Here, d is the geodesic distance on the sphere S^{n-1}, i.e., $\cos d(x, y) = x \cdot y$. As a first example of a truly high-dimensional measure of geometric origin, we will discuss the uniform probability measure on S^{n-1}, denoted by σ_{n-1}. The rotational-symmetry of σ_{n-1} yields simple answers to many geometric questions. Consider for instance the isoperimetric inequality on S^{n-1}, going back to Lévy [38] and to Schmidt [51] (see the appendix in Figiel, Lindenstrauss and Milman [22] or Benyamini [5] for simple proofs). This inequality states that for any Borel set $A \subset S^{n-1}$ and $\varepsilon > 0$,

$$\sigma_{n-1}(A) = 1/2 \implies \sigma_{n-1}(A_\varepsilon) \geq \sigma_{n-1}(H_\varepsilon), \tag{2}$$

where $H = \{x \in S^{n-1}; x_1 \leq 0\}$ is a hemisphere. There are only a handful of scenarios, in addition to the sphere, where the isoperimetric problem is completely solved (see the recent survey by Ros [50]). In order to appreciate the quantitative consequences of the isoperimetric inequality (2), we need to estimate $\sigma_{n-1}(H_\varepsilon) = \mathbb{P}(Y_1 \leq \sin\varepsilon)$, where $Y = (Y_1, \ldots, Y_n)$ is a random vector in S^{n-1}, distributed according to σ_{n-1}. When the dimension n is large, according to Maxwell's principle,

$$\mathbb{P}(Y_1 \leq t) = C_n^{-1} \int_{-1}^{t} \left(1 - s^2\right)^{\frac{n-3}{2}} ds = \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{t} e^{-\frac{s^2 n}{2}} ds + O\left(\frac{1}{n}\right), \tag{3}$$

for $C_n = \int_{-1}^{1} (1-s^2)^{(n-3)/2} ds$. Hence Y_1 is distributed approximately like a gaussian random variable of mean zero and variance $1/n$. The variance of Y_1 is very small; even though Y_1 attains values in the entire range $[-1, 1]$, it is very rare for $|Y_1|$ to reach values as high as $1/10$. We thus arrive at the following surprising conclusion, which seems to contradict our low-dimensional intuition: Most of the mass of the sphere S^{n-1} in high dimension, is concentrated on a very narrow strip near the equator $\{x \in S^{n-1}; x_1 = 0\}$. The same is true, of course, for all other equators in S^{n-1}. This peculiar high-dimensional effect is called the "concentration of measure" phenomenon. See Milman [42], [43] for a thorough review of this phenomenon and its applications.

Returning to the isoperimetric inequality (2), standard computations (see, e.g., Section 2 in [44]) show that

$$\sigma_{n-1}(H_\varepsilon) \geq 1 - \exp(-\varepsilon^2 n/2). \tag{4}$$

The strong quantitative information (4), when plugged into the isoperimetric inequality (2) shows that whenever $A \subset S^{n-1}$ has measure $1/2$, its ε-neighborhood covers almost the entire sphere, in sense of measure. Another useful consequence is the following corollary (see [44], Section 2 and Appendix V).

Corollary 2.1 (Lévy's lemma). *Let $f : S^{n-1} \to \mathbb{R}$ be a 1-Lipschitz function (i.e., $f(x) - f(y) \leq d(x, y)$). Denote*

$$M = \int_{S^{n-1}} f(x) d\sigma_{n-1}(x).$$

Then for any $\varepsilon > 0$,

$$\sigma_{n-1}\left(\left\{x \in S^{n-1}\,;\,|f(x) - M| \geq \varepsilon\right\}\right) \leq C\exp(-c\varepsilon^2 n),$$

where $c, C > 0$ *are universal constants.*

Corollary 2.1 roughly states that Lipschitz functions on the high-dimensional sphere are effectively constant. When one evaluates such a function at, say, five randomly selected points, the typical answer will be five numbers that are very close to one another.

2.1. Sudakov's theorem. One of the conclusions we mentioned in passing was Maxwell's observation that the marginals of σ_{n-1} are approximately gaussian when n is large. What other distributions in high dimension have approximately gaussian marginals? A fundamental result in this direction is a theorem going back to Sudakov [53] and to Diaconis and Freedman [20], to be described next. Let $X = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$ with $\mathbb{E}|X|^2 < \infty$. We assume that X is normalized as follows:

$$\mathbb{E}X_i = 0, \quad \mathbb{E}X_iX_j = \delta_{i,j} \quad \text{for all } i, j = 1, \ldots, n. \tag{5}$$

Equivalently, all of the one-dimensional marginals of X have mean zero and variance one. A random vector that satisfies the normalization condition (5) will be called "isotropic". It turns out that the crucial property of X in the context of gaussian marginals is a certain *thin spherical shell* bound:

Theorem 2.2 (Sudakov [53], Diaconis and Freedman [20], von Weizsäcker [54], Anttila, Ball and Perissinaki [1], Bobkov [6], ...). *Let* X *be an isotropic random vector in* $\mathbb{R}^n$ *and let* $\varepsilon > 0$. *Assume that*

$$\mathbb{P}\left(\left|\frac{|X|}{\sqrt{n}} - 1\right| \geq \varepsilon\right) \leq \varepsilon. \tag{6}$$

Then, there exists a subset $\Theta \subseteq S^{n-1}$ *with* $\sigma_{n-1}(\Theta) \geq 1 - \exp(-c\sqrt{n})$, *such that for any* $\theta \in \Theta$ *and* $t \in \mathbb{R}$,

$$|\mathbb{P}(X \cdot \theta \leq t) - \Phi(t)| \leq C\left(\varepsilon + \frac{1}{n^\alpha}\right) \tag{7}$$

where $\Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^t \exp\left(-s^2/2\right) ds$ *and* $C, c, \alpha > 0$ *are universal constants.*

The main assumption in Theorem 2.2, the inequality (6), states that most of the mass of the random vector X is contained in a thin spherical shell, whose width is only ε times its radius. This thin shell assumption is in fact necessary for the conclusion of the theorem to hold (the necessity follows from (8) below). The proof of Theorem 2.2

is a beautiful manifestation of the concentration of measure phenomenon. Let us briefly sketch the main ideas. Let Y be a random vector, distributed uniformly on the sphere S^{n-1}, which is independent of X. Fix $t \in \mathbb{R}$. Consider the function $F_t(\theta) = \mathbb{P}(X \cdot \theta \leq t)$, defined on the sphere S^{n-1}. Then,

$$\int_{S^{n-1}} F_t(\theta) d\sigma_{n-1}(\theta) = \mathbb{P}\left(|X|\, Y_1 \leq t\right).$$

Note that according to (6) and (3), the random variable $|X|$ is typically very close to $\sqrt{n}$, hence $|X|Y_1$ is approximately a standard normal random variable. Consequently,

$$\int_{S^{n-1}} F_t(\theta) d\sigma_{n-1}(\theta) = \mathbb{P}\left(|X|\, Y_1 \leq t\right) = \Phi(t) + O\left(\varepsilon + \frac{1}{n}\right). \tag{8}$$

In order to deduce Theorem 2.2, we would like to show that

$$F_t(\theta) = \Phi(t) + O\left(\varepsilon + \frac{1}{n^\alpha}\right) \quad \text{for most } \theta \in S^{n-1},$$

where "most" is interpreted in the sense of σ_{n-1}. We already know from (8) that the average of F_t on the unit sphere is close to $\Phi(t)$. We thus need to show that for most $\theta \in S^{n-1}$, the value $F_t(\theta)$ is close to the average of F_t. To that end, we will employ Corollary 2.1: Recall that Lipschitz functions deviate very little from their mean. The function F_t is not necessarily Lipschitz (nor continuous), yet it is possible to construct Lipschitz approximations for F_t: Take

$$\tilde{F}_t(\theta) = \mathbb{E} I_t(X \cdot \theta) \approx F_t(\theta)$$

where I_t is a Lipschitz approximation of the characteristic function of $(-\infty, t]$, see Bobkov [6] for details. This is roughly the sketch of the proof of (7) for a single, fixed value $t \in \mathbb{R}$. By considering simultaneously the values $t_i = \Phi^{-1}(i\varepsilon)$ for $i = 1, \ldots, \lfloor 1/\varepsilon \rfloor$, one concludes Theorem 2.2.

The above discussion demonstrates that the gaussian approximation property of the marginals is not necessarily associated with independent random variables. The geometry of the high-dimensional sphere is another protagonist related to gaussian approximation principles. As a matter of fact, in comparison with the case of independent random variables, the proof that the sphere's marginals are close to normal seems quite straightforward.

3. Convexity

It is easy to construct natural, isotropic probability distributions that strongly violate the *thin shell estimate* (6), and consequently do not have many approximately gaussian marginals. For instance, write σ_{n-1}^t for the uniform probability measure on the sphere of radius t, centered at the origin in $\mathbb{R}^n$, and consider the measure

$$\frac{1}{2}\left[\sigma_{n-1}^{r_1} + \sigma_{n-1}^{r_2}\right]$$

for $r_1 = \sqrt{n}/2$ and $r_2 = \sqrt{7n}/2$. All marginals of this probability measure are far from normal. Therefore, a geometric condition is needed in order to avoid this kind of examples and deduce the existence of approximately gaussian marginals. Here we follow the approach suggested by Anttila, Ball and Perissinaki [1] and by Brehm and Voigt [16], and consider the relationship between thin shell bounds and convexity assumptions.

3.1. Basic volumetric properties of convex sets. A convex body in $\mathbb{R}^n$ is a bounded, open convex set. The uniform measure on a convex body has several regularity features that are prominent mostly in high dimension. Some of these features will be reviewed next. For subsets $A, B \subset \mathbb{R}^n$ we write $A + B = \{a + b\,;\, a \in A, b \in B\}$ and also $\lambda A = \{\lambda a\,;\, a \in A\}$ for $\lambda \in \mathbb{R}$. The classical Brunn–Minkowski inequality states that for any non-empty Borel sets $A, B \subset \mathbb{R}^n$,

$$\mathrm{Vol}_n\,(A + B)^{1/n} \geq \mathrm{Vol}_n(A)^{1/n} + \mathrm{Vol}_n(B)^{1/n},$$

where Vol_n is the Lebesgue measure. The Brunn–Minkowski inequality is a fundamental fact regarding the uniform measure on a convex domain (even though its formulation does not mention convexity), see, e.g., Schneider [52]. A function $f : E \to [0, \infty)$ is *log-concave* if for any $x, y \in E$ and $0 < \lambda < 1$,

$$f(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}.$$

That is, a function f is log-concave if it takes the form $\exp(-H)$ for a convex function $H : E \to (-\infty, \infty]$. In particular, the characteristic function of a convex body is log-concave.

Let $K \subset \mathbb{R}^n$ be a convex body, and suppose that X is a random vector distributed uniformly on K. Let $E \subset \mathbb{R}^n$ be a subspace, and denote by Proj_E the orthogonal projection operator onto E in $\mathbb{R}^n$. One of the consequences of the Brunn–Minkowski inequality is that the random vector $\mathrm{Proj}_E(X)$ has a density in the subspace E, and this density is log-concave. Characteristic functions of convex bodies and their marginals are our main source of examples for log-concave densities. All marginals of all dimensions of a log-concave density are again log-concave, see Borell [13]. The latter fact also follows from the Prékopa–Leindler inequality which is a variant of the Brunn–Minkowski inequality, see, e.g., [31] and references therein, or the first pages of Pisier's book [49].

Many questions regarding the uniform measure on a convex body may be reduced to one-dimensional calculus problems, by using the log-concavity of the marginals.

For instance, suppose that K is a convex body of volume one whose barycenter lies at the origin, and let $\theta \in S^{n-1}$. Denote $H = \{x \in \mathbb{R}^n\, ;\, x \cdot \theta = 0\}$. Then, as is proven in Ball [4], Fradelizi [24] and Hensley [29],

$$\frac{1}{\sqrt{12}} \leq \mathrm{Vol}_{n-1}(K \cap H) \cdot \sqrt{\int_K (x \cdot \theta)^2 dx} \leq \frac{1}{\sqrt{2}}, \tag{9}$$

where Vol_{n-1} denotes $(n-1)$-dimensional volume. The inequalities in (9) are reduced, according to the Brunn–Minkowski inequality, to lower and upper bounds for $f^2(0) \int_{\mathbb{R}} t^2 f(t)\, dt$ where f is the log-concave density of a real-valued random variable of mean zero. It follows from (9) that when the uniform probability measure on a convex body K is isotropic, then all hyperplane sections of K through the origin have roughly the same volume.

An additional consequence of the Brunn–Minkowski inequality that may be proven in a similar way (see Borell [12]), goes as follows: For any random vector X that is distributed uniformly in a convex body in $\mathbb{R}^n$, and a linear functional φ,

$$\mathbb{P}(|\varphi(X)| \geq tM) \leq C \exp(-ct) \quad \text{for all } t \geq 0 \tag{10}$$

where $M = \mathbb{E}|\varphi(X)|$ and $c, C > 0$ are universal constants. In low dimension, the inequality (10) is trivial and probably useless (in two or three dimensions, the discussed probability is zero already for $t = 10$). It is the high-dimensional case in which (10) is meaningful. The large deviations estimate (10) may be generalized to the case where φ is a polynomial of degree d on $\mathbb{R}^n$, rather than a linear functional. In this case, the right-hand side of (10) has to be replaced by $C \exp(-ct^{1/d})$, see Bobkov [8], Bourgain [14], Carbery and Wright [17] and Nazarov, Sodin and Volberg [46].

3.2. Spectral gap. Let μ be an isotropic probability measure on $\mathbb{R}^n$ with a log-concave density. We are interested in approximately gaussian marginals of μ and consequently also in spherical thin shell bounds for μ. The thin shell estimate (6) would follow from a variance bound

$$\int_{\mathbb{R}^n} \left(\frac{|x|^2}{n} - 1 \right)^2 d\mu(x) \ll 1, \tag{11}$$

via Chebyshev's inequality. Here is a common line of attack on the thin-shell hypothesis (see, e.g., [9]): Rather than proving (11) directly, try to establish the inequality

$$\alpha \int_{\mathbb{R}^n} \varphi^2 d\mu \leq \int_{\mathbb{R}^n} |\nabla \varphi|^2 d\mu \tag{12}$$

for *all* smooth, μ-square-integrable functions φ with $\int \varphi d\mu = 0$. If (12) indeed holds with $\alpha \gg 1/n$, then (11) follows easily: It is simply the case $\varphi(x) = |x|^2/n - 1$.

Note that (12) is actually a spectral gap problem: Write $\exp(-H)$ for the log-concave density of μ. For a smooth function φ satisfying certain growth conditions, denote

$$\triangle_\mu \varphi = \triangle \varphi - \nabla H \cdot \nabla \varphi$$

(for simplicity, assume that $H : \mathbb{R}^n \to \mathbb{R}$ is smooth). Integrating by parts, we see that

$$\int_{\mathbb{R}^n} |\nabla \varphi|^2 d\mu = -\int_{\mathbb{R}^n} \varphi \triangle_\mu \varphi d\mu.$$

The operator $-\triangle_\mu$ is thus a positive semi-definite, densely defined symmetric operator in $L^2(\mu)$, and hence it admits an extension to a self-adjoint operator (see, e.g., [19]). The minimal eigenvalue of $-\triangle_\mu$ is zero, with a constant eigenfunction. The inequality (12) is equivalent to a lower bound α for the second eigenvalue of $-\triangle_\mu$. A conjecture going back to Kannan, Lovász and Simonovits [30] is that (12) holds with $\alpha = c$, for all isotropic, log-concave probability measures, where $c > 0$ is a universal constant. Part of the appeal of this conjecture is its equivalent formulation in terms of an isoperimetric inequality for the measure μ, see Ledoux [37].

3.3. Strong uniform convexity assumptions. The spectral gap inequality (12) is known to hold, for reasonable values of α, under some strong uniform convexity assumptions. For example, denote by $\nabla^2 H$ the hessian of H. Then $\nabla^2 H \geq 0$, in the sense of symmetric matrices, as H is convex. Suppose that the following strong convexity assumption is fulfilled:

$$\nabla^2 H(x) \geq \delta I \quad \text{for all } x \in \mathbb{R}^n, \tag{13}$$

for some $\delta > 0$, where I is the identity matrix. A Bochner-type integration by parts formula (see, e.g., [3], [18]) then states that

$$\begin{aligned} \int_{\mathbb{R}^n} (\triangle_\mu \varphi)^2 d\mu &= \int_{\mathbb{R}^n} |\nabla^2 \varphi|^2_{HS} d\mu + \int_{\mathbb{R}^n} (\nabla^2 H)(\nabla \varphi) \cdot \nabla \varphi d\mu \\ &\geq \delta \int_{\mathbb{R}^n} |\nabla \varphi|^2 d\mu \end{aligned} \tag{14}$$

under some smoothness and growth conditions for φ, where $|\cdot|_{HS}$ is the Hilbert–Schmidt norm. Consequently,

$$(-\triangle_\mu)^2 \geq -\delta \triangle_\mu$$

in the sense of symmetric operators. Thus (12) holds with $\alpha = \delta$, as was observed by Brascamp and Lieb [15]. The assumption (13) implies, in fact, much stronger conclusions, see Bakry and Émery [3]. An additional strong convexity assumption that is known to imply a variance bound like (11) is related to the modulus of convexity.

Suppose $K \subset \mathbb{R}^n$ is a convex body which is centrally symmetric (i.e., $K = -K$). Consider the norm $\| \cdot \|_K$ on $\mathbb{R}^n$ whose unit ball is K. The modulus of convexity of K is defined as

$$\delta_K(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|_K}{2} \, ; \, \|x\|_K \leq 1, \ \|y\|_K \leq 1, \ \|x-y\|_K \geq \varepsilon \right\}$$

for $\varepsilon > 0$. The modulus of convexity is always non-negative, and it is linearly invariant (unlike the condition (13)). The larger the modulus of convexity of K, the more "strictly-convex" is the boundary of K. Under certain assumptions on the modulus of convexity of K and its diameter, a thin shell bound (11) was proven by Anttila, Ball and Perissinaki [1], following the works of Arias-de-Reyna, Ball and Villa [2] and Gromov and Milman [28]. See also Milman and Sodin [40] for related isoperimetric inequalities.

4. A central limit theorem for convex bodies

Next we formulate a gaussian approximation result for marginals of general log-concave densities.

Theorem 4.1 (see [33], [34]). *Let X be an isotropic random vector in $\mathbb{R}^n$, with a log-concave density. Then there exists a subset $\Theta \subseteq S^{n-1}$, with $\sigma_{n-1}(\Theta) \geq 1 - \exp(-\sqrt{n})$, such that for any $\theta \in \Theta$ and any measurable set $A \subseteq \mathbb{R}$,*

$$\left| \mathbb{P}(X \cdot \theta \in A) - \frac{1}{\sqrt{2\pi}} \int_A \exp(-s^2/2) ds \right| \leq \frac{C}{n^\alpha},$$

where $C, \alpha > 0$ are universal constants.

The isotropic normalization in Theorem 4.1 is used only to infer that *most* marginals are approximately gaussian. Without assuming that X is isotropic, we can still assert the existence of at least one approximately gaussian marginal. In accordance with the discussion above, a central ingredient in the proof of Theorem 4.1 is the following thin shell estimate: For any isotropic random vector X with a log-concave density in $\mathbb{R}^n$,

$$\mathbb{E}\left(\frac{|X|^2}{n} - 1\right)^2 \leq \frac{C}{n^\alpha}, \tag{15}$$

for universal constants $C, \alpha > 0$ (the proof in [34] yields α close to $1/5$). We thus arrive at the following fundamental, non-intuitive conclusion, conjectured by Anttila, Ball and Perissinaki [1]: Most of the volume of a convex body in high dimension, with the isotropic normalization, is concentrated in a very thin spherical shell.

How can we prove the bound (15) for general log-concave densities, without making strong uniform convexity assumptions? Let us consider first the very particular case where the density of X is not only log-concave, but also *radially symmetric* in $\mathbb{R}^n$. Write $f(|x|)$ for the radial density of X, where f is a log-concave function on $[0,\infty)$. Integrating in polar coordinates, we see that the density of the random variable $|X|$ is

$$t \mapsto n\kappa_n t^{n-1} f(t) \quad (t > 0),$$

where $\kappa_n = \pi^{n/2}/\Gamma(1+n/2)$ is the volume of the n-dimensional unit ball. Such densities are necessarily very peaked: Denote by $t_0 > 0$ the point where the maximum of $t \mapsto t^{n-1} f(t)$ is attained. A standard application of Laplace's method (see [33]) yields the bound

$$t^{n-1} f(t) \le t_0^{n-1} f(t_0) \exp\left(-c(t-t_0)^2\right) \quad \text{for } |t-t_0| \le c\sqrt{n}, \tag{16}$$

where $c > 0$ is a universal constant. The log-concavity of f is crucial for the success of Laplace's method, since it implies upper bounds for the second derivative of $\log(t^{n-1} f(t))$ at the point t_0. The bound (16) entails that $|X|$ is very concentrated near its mean: Even though $\mathbb{E}|X|$ has the order of magnitude of $\sqrt{n}$, the standard deviation of $|X|$ is bounded by a universal constant. The inequality (15) follows with $\alpha = 1$, see [33] for details. An elegant argument leading to the same conclusion, using convexity properties of the moment function, is given by Bobkov [7].

We explained how to deduce (15) in the radial case. The general log-concave case may be reduced to the radial one by using concentration of measure techniques. This idea is very much related to a remark by Gromov [26], Section 1.2.F. Denote by $G_{n,\ell}$ the grassmannian of all ℓ-dimensional subspaces in $\mathbb{R}^n$. The grassmannian $G_{n,\ell}$ is a metric space, and it carries a unique rotationally-invariant probability measure, denoted by $\sigma_{n,\ell}$, which we refer to as the uniform measure on $G_{n,\ell}$. When the dimension n is large, the uniform measure on $G_{n,\ell}$ enjoys concentration properties similar to those described in Corollary 2.1 (see Gromov and Milman [27]).

Next, suppose that X is an isotropic random vector with a log-concave density in $\mathbb{R}^n$. For a subspace $E \subset \mathbb{R}^n$, denote by $f_E : E \to [0,\infty)$ the log-concave density of the random vector $\mathrm{Proj}_E(X)$. Let ℓ be a parameter, which will have the order of magnitude of a small, positive power of n. Our main object of study is projections of X to different ℓ-dimensional subspaces of $\mathbb{R}^n$. Fix $r > 0$. Using the log-concavity of f, it is possible to show that the map

$$(E,\theta) \quad \mapsto \quad \log f_E(r\theta) \quad (E \in G_{n,\ell}, \theta \in S^{n-1} \cap E)$$

may be approximated by a Lipschitz function. This is a rather technical part of the argument, see [34] for the details. Then, we use concentration of measure principles on the grassmannian $G_{n,\ell}$, to conclude that the function $f_E(r\theta)$, as a function of E and θ, is "effectively constant": For "most" subspaces $E \in G_{n,\ell}$ and for "most"

$\theta \in S^{n-1} \cap E$, the value $\log f_E(r\theta)$ is approximately the same. With a bit of analysis, we deduce that for "most" subspaces $E \in G_{n,\ell}$ and for *all* $\theta \in S^{n-1} \cap E$, the value $f_E(r\theta)$ is roughly the same.

By considering several values of r simultaneously, we conclude that for most subspaces $E \in G_{n,\ell}$, the function f_E is approximately radial. Recall that f_E is the marginal of the log-concave density f, and consequently f_E is also log-concave. To summarize, for most ℓ-dimensional subspaces $E \subset \mathbb{R}^n$, the function f_E is the log-concave, approximately-radial density of the isotropic random vector $\mathrm{Proj}_E(X)$. According to the already established thin-shell bound for radial, log-concave densities, for most subspaces $E \in G_{n,\ell}$,

$$\mathbb{E}\left(\frac{|\mathrm{Proj}_E(X)|}{\sqrt{\ell}} - 1\right)^2 \leq \frac{C}{\ell}. \tag{17}$$

Introduce a random ℓ-dimensional subspace $E \subset \mathbb{R}^n$, uniformly distributed in $G_{n,\ell}$, independent of X. It is well-known that $|\mathrm{Proj}_E(X)| = \sqrt{\ell/n}|X| + O(|X|/\sqrt{n})$ with large probability of selecting E. The desired bound (15) thus follows from (17), modulo details we skipped, see [33], [34], or [23] for the complete proof.

Theorem 4.1 is concerned with one-dimensional marginals. There are also corresponding principles for multi-dimensional marginals:

Theorem 4.2 (Eldan and Klartag [21], Klartag [33], [34]). *Let X be an isotropic random vector in $\mathbb{R}^n$ with a log-concave density, and let $\ell \leq cn^\alpha$ be an integer. Then there exists a subset $\mathcal{E} \subseteq G_{n,\ell}$, with $\sigma_{n,\ell}(\mathcal{E}) \geq 1 - \exp(-\sqrt{n})$, such that for all $E \in \mathcal{E}$ the following holds:*

(1) *For any measurable set $A \subseteq E$,*

$$\left|\mathbb{P}(\mathrm{Proj}_E(X) \in A) - \int_A \varphi_E(x)dx\right| \leq \frac{C}{n^\alpha},$$

where $\varphi_E(x) = (2\pi)^{-\ell/2}\exp(-|x|^2/2)$.

(2) *Denote by f_E the density of the random vector $\mathrm{Proj}_E(X)$. Then for any $x \in E$ with $|x| \leq cn^\alpha$,*

$$\left|\frac{f_E(x)}{\varphi_E(x)} - 1\right| \leq \frac{C}{n^\alpha}.$$

Here, $C, c, \alpha > 0$ are universal constants.

When X has a log-concave density but is not required to be isotropic, we can still assert that $\mathrm{Proj}_E(X)$ is approximately gaussian for *some* ℓ-dimensional subspace $E \subset \mathbb{R}^n$. Theorem 4.2 should be compared with the classical Dvoretzky's theorem. The relation between measure projections and Dvoretzky's theorem was noted already by Gromov [26], Section 1.2.F. In Milman's form [41], Dvoretzky's theorem states that for any convex body $K \subset \mathbb{R}^n$, there exists a subspace $E \subset \mathbb{R}^n$ of dimension $\lfloor c \log n \rfloor$, such that the geometric projection

$$\mathrm{Proj}_E(K) = \{\mathrm{Proj}_E(x)\,;\, x \in K\}$$

is approximately a Euclidean ball in the subspace E. Here, $c > 0$ is a universal constant. The logarithmic dependence on the dimension is tight. Theorem 4.2 is concerned with the full measure projection, or marginal, of the uniform measure on K. We learn that one can project the uniform measure of the convex body $K \subset \mathbb{R}^n$ to dimensions as large as $\lfloor n^c \rfloor$, and obtain an approximate gaussian. Here, again, $c > 0$ is a universal constant.

Both the geometric projection and the measure projection of a convex body bring regularity of the best kind, either in the form of a Euclidean ball or in the form of a gaussian distribution. One may argue, however, that on a quantitative level, the projection of the uniform measure on convex bodies behaves better, in a sense, then the geometric projection: We observe a power-law dependence on the dimension, rather than a logarithmic dependence.

5. Rate of convergence

We are still lacking optimal rate of convergence results for the central limit theorem for convex bodies. The exponents α that our proofs yield for Theorem 4.1 and Theorem 4.2 are probably inferior. The main problem is with the thin shell estimate (15); it is conceivable that the correct bound should be

$$\mathbb{E}\left(\frac{|X|^2}{n} - 1\right)^2 \leq \frac{C}{n}, \tag{18}$$

for all isotropic random vectors X with a log-concave density in $\mathbb{R}^n$, see Anttila, Ball and Perissinaki [1] and Bobkov and Koldobsky [9]. There are some cases where the sharp thin shell bound (18) is proven. For example, it is common to say that a log-concave density $f : \mathbb{R}^n \to [0, \infty)$ is "unconditional" if

$$f(x_1, \ldots, x_n) = f(\pm x_1, \ldots, \pm x_n) \quad \text{for all } x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

for any choice of n signs. That is, f is unconditional if it is invariant under coordinate reflections. A convex body is called unconditional if its characteristic function is unconditional. Our general philosophy is that convexity is a great source of regularity in the study of high-dimensional distributions, which may substitute for structure and symmetry. The fact that sharp thin shell bounds were proven, at least so far, only under additional symmetry assumptions is certainly a weak point in our approach. Note that nevertheless, an unconditional log-concave density is only "mildly" symmetric, and that convexity plays a significant role in the analysis of these densities.

When X is an isotropic random vector in $\mathbb{R}^n$ with an unconditional, log-concave density, the bound (18) is proven and Theorem 4.1 holds with $\alpha = 1$. The proof in [35] for the unconditional case is based on the integration by parts formula (14) and some L^2-technique. An additional advantage of the unconditional case is that we may precisely describe the subset $\Theta \subseteq S^{n-1}$ from Theorem 4.1. Specifically, for any $\theta = (\theta_1, \ldots, \theta_n) \in S^{n-1}$, it is proven in [35] that

$$\left| \mathbb{P}\Big(\sum_{i=1}^{n} \theta_i X_j \leq t \Big) - \Phi(t) \right| \leq C \sum_{i=1}^{n} \theta_i^4 \quad \text{for all } t \in \mathbb{R},$$

where $C > 0$ is a universal constant. We may thus take Θ in Theorem 4.1 to be the set of all $\theta \in S^{n-1}$ with $\sum_i \theta_i^4 \leq 50/n$. Note that for this choice, $\sigma_{n-1}(\Theta) \geq 1 - \exp(-\sqrt{n})$. Additionally, for $t \in [0, 1]$ let us define

$$Y_t = \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor tn \rfloor} X_j .$$

The analysis in [35] may be used to show that the stochastic process $(Y_t)_{0 \leq t \leq 1}$ converges, in an appropriate sense, to the standard Brownian motion.

In the unconditional case there are also available sharp large-deviations results, proven by Bobkov and Nazarov [10], [11]. For example, when X is an isotropic random vector in $\mathbb{R}^n$ with an unconditional, log-concave density, it is shown that

$$\mathbb{P}\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \geq t \right) \leq C \exp\left(-ct^2\right) \quad \text{for all } 0 \leq t \leq \sqrt{n}, \tag{19}$$

where $c, C > 0$ are universal constants. When the random vector X is uniform in an unconditional convex body (a slightly stronger assumption than log-concavity), the inequality in (19) holds for all $t \geq 0$. The sub-gaussian behavior in (19) in the unconditional case should be compared with the general, sub-exponential bound of (10). One may also obtain an almost sub-gaussian bound in the general, non-unconditional case. The following was proven in [32] and in Giannopoulos, Pajor and Paouris [25]: For any random vector X distributed uniformly in a finite-dimensional convex body, there exists a non-zero linear functional φ for which the right-hand side of (10) may be improved upon to $C_\varepsilon \exp(-c_\varepsilon t^{2-\varepsilon})$, for arbitrarily small $\varepsilon > 0$. The positive coefficients $C_\varepsilon, c_\varepsilon$ depend solely on ε.

To prove (19), Bobkov and Nazarov use the Prékopa–Leindler inequality in order to reduce the problem from a general unconditional log-concave density to the "worst possible" unconditional one, which is $\exp(-\sum_i |x_i|)$ in this case. The latter density is then analyzed directly. A similar approach leads to the sharp large-deviations bound

$$\mathbb{P}\left(|X| \geq t\right) \leq C \exp\left(-ct\right) \quad \text{for } t \geq C\sqrt{n}, \tag{20}$$

valid for all isotropic random vectors X with an unconditional, log-concave density (see [10], [11]). Here, $c, C > 0$ are universal constants.

One of the most significant and influential developments in recent years in the study of high-dimensional convex bodies is the Paouris theorem [47], [48]. It is one of the very few *sharp* quantitative results that are valid for all high-dimensional log-concave distributions. Paouris proved that the bound (20) actually holds for all isotropic random vectors X with a log-concave density, without the assumption that the density is unconditional. Paouris observed that when E is a random ℓ-dimensional subspace in $\mathbb{R}^n$, for any $\ell \leq c\sqrt{n}$, then the density f_E of $\mathrm{Proj}_E(X)$ is typically approximately radial, in the following sense: The level set

$$\left\{x \in E\,;\; f_E(x) \geq e^{-\ell} f_E(0)\right\} \tag{21}$$

is roughly a Euclidean ball. The dependence of ℓ on the dimension n is optimal. The proof that (21) is indeed approximately Euclidean is based on a clever use of the quantitative theory of Dvoretzky's theorem, developed mostly by Milman (see, e.g., [44]), with contributions by Litvak and Schechtman [45], [39]. Once it is known that the "effective support" of $\mathrm{Proj}_E(X)$ (i.e., the set in (21)) is approximately a Euclidean ball, some analysis of log-concave densities – mostly one-dimensional analysis – leads to the bound (20), see [47], [48] for details.

There are currently no sharp inequalities that complement (20) for smaller values of t, in the general case. The best available thin shell bound is that for any isotropic random vector X with a log-concave density in $\mathbb{R}^n$,

$$\mathbb{P}\left(\left|\frac{|X|^2}{n} - 1\right| \geq t\right) \leq C \exp\left(-cn^{\alpha} t^{\beta}\right) \quad \text{for } 0 < t < 1, \tag{22}$$

with, say, $\alpha = 0.33$ and $\beta = 3.33$, and $c, C > 0$ are universal constants (see [34]).

Some of the arguments we presented, especially those in Section 4, might be robust enough to permit possible generalizations to other notions of convexity. One may consider, for instance, probability measures on the surface of a convex body, rather than the on the body itself, or probability densities of the form $V^{-\beta}$ for a convex function V and $\beta > 0$, as long as the tail is not "too heavy" (see Bobkov [8] for the terminology and for a review of such densities). We expect that convexity-related properties will play a role in the study of some high-dimensional distributions in the future.

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5ECM Amsterdam 2008

Some recent results about the sixth problem of Hilbert: hydrodynamic limits of the Boltzmann equation

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Abstract. The behaviour of a gas can be described by different models depending on the time and space scales to be considered. A natural question is therefore to know if there exist continuous transitions between these models: this is precisely the matter of the sixth problem asked by Hilbert on the occasion of the International Congress of Mathematicians held in Paris in 1900.

Recent works allow to understand rigorously the passage from Boltzmann's equation (which provides a statistical description of the microscopic state of the gas) to continuous models of fluid mechanics.

The goal of the present paper is to present briefly the mathematical framework of the study, and to exhibit some difficulties of the asymptotic analysis related to the physics of the problem.

Mathematics Subject Classification (2000). 35B40, 35B65, 76P05, 76D05, 76B99, 82B40.

Keywords. Kinetic theory, Boltzmann equation, fluid mechanics, incompressible hydrodynamic limits, asymptotic analysis, weak compactness method, relative entropy method.

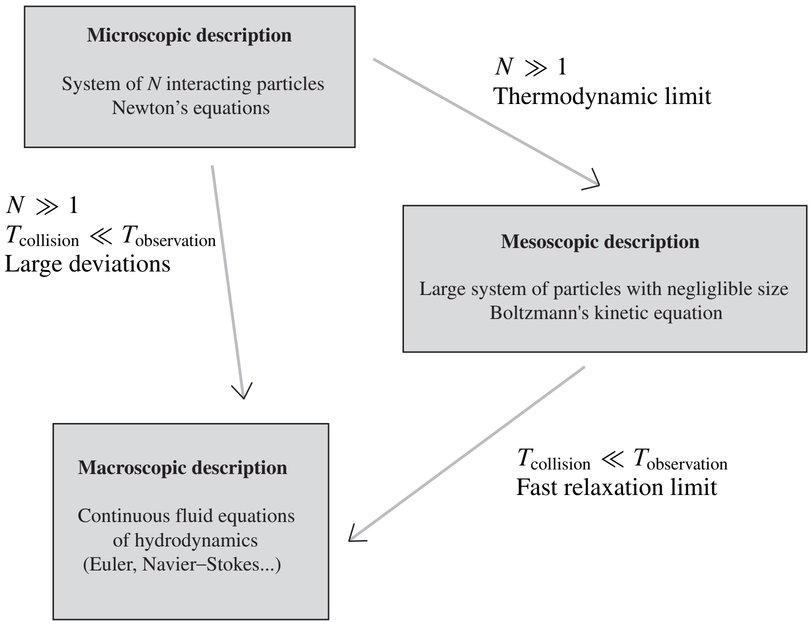
This paper is concerned with hydrodynamic limits of the Boltzmann equation, which is part of the sixth problem posed by Hilbert in 1900, on the occasion of the International Congress of Mathematicians.

Our goal here is to give the reader a description of the problem both from the physical and mathematical points of view, and to point out the main difficulties. In order that a broad audience can understand as far as possible, we will state the results and give some elements of proof only at the very end.

1. The sixth problem of Hilbert

Let us first introduce the general framework of our study. The sixth problem of Hilbert, motivated by Boltzmann's work on the principles of mechanics, consists in *"developing mathematically the limiting processes* [...] *which lead from the atomistic point of view to the laws of motion of continua"*.

In other words, we are interested in getting a unified theory for gas dynamics including all levels of description, namely the atomistic point of view, kinetic theory and hydrodynamics.



1.1. A unified theory of gases. Each transition gives rise to challenging mathematical questions, and therefore to a huge literature. Here we will just mention some significant breakthroughs.

1.1.1. From microscopic to mesoscopic and macroscopic models. For the connection between microscopic and macroscopic models, as far as we know, formal asymptotics go back to Morrey [21].

The probabilistic approach has given nice results as regards the mathematical justification of the limiting process. Let us mention in particular a result by Olla, Varadhan and Yau [23] for the compressible Euler limit, and a result by Quastel and Yau [24] for the incompressible Navier–Stokes limit, starting from lattice systems of particles with white noise.

Concerning the transition from microscopic to mesoscopic models, Lanford [15] has proved that the Boltzmann equation can be derived in the thermodynamic limit for a gas of hard spheres under a strong chaos assumption. Note however that the convergence holds only for very short times, say during one half of the expected time of first collision for a given particle.

1.1.2. From mesoscopic to macroscopic models. From now on, we will actually focus on the connection between the Boltzmann equation and hydrodynamics, which has been investigated at formal level by Hilbert [14], and by Chapman and Enskog [5].

Various mathematical theories have been developed to obtain rigorous convergence results, most of which are based on some approximation methods and therefore require regularity to build the successive correctors. We refer for instance to the works by Caflisch [4], De Masi, Esposito and Lebowitz [6] and Liu, Yang and Yu [20] for convergence results based on asymptotic expansions. We also mention that, using slightly different arguments, Nishida [22], and Bardos and Ukai [2] have investigated hydrodynamic limits of the Boltzmann equation in the framework of smooth solutions (either local in time, or for small data).

Another approach has been initiated by Bardos, Golse and Levermore [1] following more or less the moment method of Grad [13]. The idea is to consider all the solutions to the Boltzmann equation that satisfy the fundamental physical estimates. Later works by Lions and Masmoudi [19] or Golse and the author [10], [11], [25] have shown that this strategy provides global convergence results.

1.2. From the Boltzmann equation to fluid models. Let us then explain a bit more the physical meaning of the Boltzmann equation as well as its fundamental properties.

1.2.1. The Boltzmann equation. As usual in kinetic theory, the unknown is the distribution function f depending on t, x and v, which gives the density of particles having position x and velocity v at time t.

The evolution equation for f expresses a balance between free transport and binary elastic collisions that are further assumed to be localized both in t and x. In nondimensional variables, it states

$$\underbrace{\mathrm{Ma}\,\partial_t f + v\cdot\nabla_x f}_{\text{free transport}} = \underbrace{(\mathrm{Kn})^{-1}Q(f,f)}_{\text{localized binary collisions}},$$

$$Q(f,f) = \iint [f(v')f(v_1') - f(v)f(v_1)]b(v-v_1,\omega)dv_1 d\omega,$$

denoting by v' and v_1' the pre-collisional velocities given by

$$v_1 = v - (v-v_1)\cdot\omega\omega, \quad v_1' = v' + (v-v_1)\cdot\omega\omega.$$

The Mach number Ma is the ratio between the bulk velocity and the thermal speed of the gas, and the Knudsen number Kn is the ratio between the mean free path and the observation length scale.

Because elementary collisions are elastic, meaning that they conserve momentum and energy, the collision operator has invariance properties

$$\int Q(f,f)dv = \int Q(f,f)v_i dv = \int Q(f,f)|v|^2 dv = 0,$$

which guarantee that the Boltzmann equation satisfies the first principle of thermodynamics, and more precisely that one has some local conservation laws for mass, momentum and energy.

The same symmetries of the collision operator also imply that

$$\int Q(f,f)\log f dv \leq 0.$$

We then deduce that $\iint f \log f dv dx$ is a Lyapunov functional for the Boltzmann equation, usually called the entropy. This property, known as Boltzmann's H-theorem gives the irreversibility associated to the second principle of thermodynamics. The collisions actually induce some relaxation process, the equilibria of which are the Gaussian distributions predicted by Maxwell:

$$\int Q(f,f)\log f dv \leq 0 \iff Q(f,f) = 0 \iff f \text{ Maxwellian.}$$

1.2.2. The compressible Euler limit. Because of this last property, in the fast relaxation limit, i.e., when the Knudsen number is very small, $\mathrm{Kn} \ll 1$, we have

$$Q(f,f) = \mathrm{Kn}\,(\mathrm{Ma}\,\partial_t f + v \cdot \nabla_x f)$$

so that the distribution f should be well approximated by the corresponding local thermodynamic equilibrium.

More precisely, we expect f to be equal to some Maxwellian distribution

$$f(t,x,v) \sim \frac{R(t,x)}{(2\pi\Theta(t,x))^{3/2}} \exp\left(-\frac{|v-U(t,x)|^2}{2\Theta(t,x)}\right)$$

up to some remainder of order Kn.

Plugging that Ansatz in the local conservations of mass, momentum and energy, we get a closed system of conservation laws, which is nothing else than the compressible Euler equations up to small corrections:

$$\begin{aligned}
\mathrm{Ma}\,\partial_t R + \nabla_x \cdot (RU) &= 0,\\
\mathrm{Ma}\,\partial_t RU + \nabla_x \cdot (RU \otimes U + RT) &= O(\mathrm{Kn}),\\
\mathrm{Ma}\,\partial_t\left(\tfrac{1}{2}R|U|^2 + \tfrac{3}{2}RT\right) + \nabla_x \cdot \left(\tfrac{1}{2}R|U|^2 U + \tfrac{5}{2}RTU\right) &= O(\mathrm{Kn}).
\end{aligned}$$

1.2.3. The weakly dissipative Navier–Stokes asymptotics. Iterating the same kind of asymptotic analysis, we can characterize the next terms of the expansion. The idea introduced by Chapman and Enskog is to decompose the distribution f as the sum of a hydrodynamic part M_f, and a purely kinetic part $f - M_f$, and to express that purely kinetic part in terms of M_f and of the linearized collision operator at M_f:

$$\begin{aligned} -2Q(M_f, f - M_f) &= -\mathrm{Kn}\,(\mathrm{Ma}\,\partial_t f + v \cdot \nabla_x f) + Q(f - M_f, f - M_f) \\ &= -\mathrm{Kn}\left(\mathrm{Ma}\,\partial_t M_f + v \cdot \nabla_x M_f\right) + O(\mathrm{Kn}^2), \end{aligned}$$

or, equivalently,

$$f - M_f = \mathrm{Kn}\mathcal{L}_{M_f}^{-1}\left(\mathrm{Ma}\,\partial_t M_f + v \cdot \nabla_x M_f\right) + O(\mathrm{Kn}^2).$$

Plugging that refined Ansatz in the local conservations of mass, momentum and energy, we get additional small dissipative terms, corresponding to the viscosity and the heat conductivity in the Navier–Stokes–Fourier system:

$$\begin{gathered} \mathrm{Ma}\,\partial_t R + \nabla_x \cdot (RU) = 0, \\ \mathrm{Ma}\,\partial_t RU + \nabla_x \cdot (RU \otimes U + RT) = \mathrm{Kn}\nabla_x \cdot (\mu(R,T)DU) + O(\mathrm{Kn}^2), \\ \mathrm{Ma}\,\partial_t\left(\tfrac{1}{2}R|U|^2 + \tfrac{3}{2}RT\right) + \nabla_x \cdot \left(\tfrac{1}{2}R|U|^2U + \tfrac{5}{2}RTU\right) \\ = \mathrm{Kn}\nabla_x \cdot (\kappa(R,T)\nabla_x T) + \mathrm{Kn}\nabla_x \cdot (\mu(R,T)DU \cdot U) + O(\mathrm{Kn}^2). \end{gathered}$$

Note that in principle we should obtain asymptotic expansions at any order with respect to Kn, but most of them are ill-posed.

1.2.4. Incompressible hydrodynamic limits. In the sequel, we will actually investigate a restricted class of asymptotic regimes corresponding to fluctuations around some global equilibrium and leading to incompressible fluid models. These models could be obtained formally from the previous ones by considering fluctuations with small Mach number $\mathrm{Ma} \ll 1$ around some fixed (R, U, T), but we will derive them directly from the Boltzmann equation.

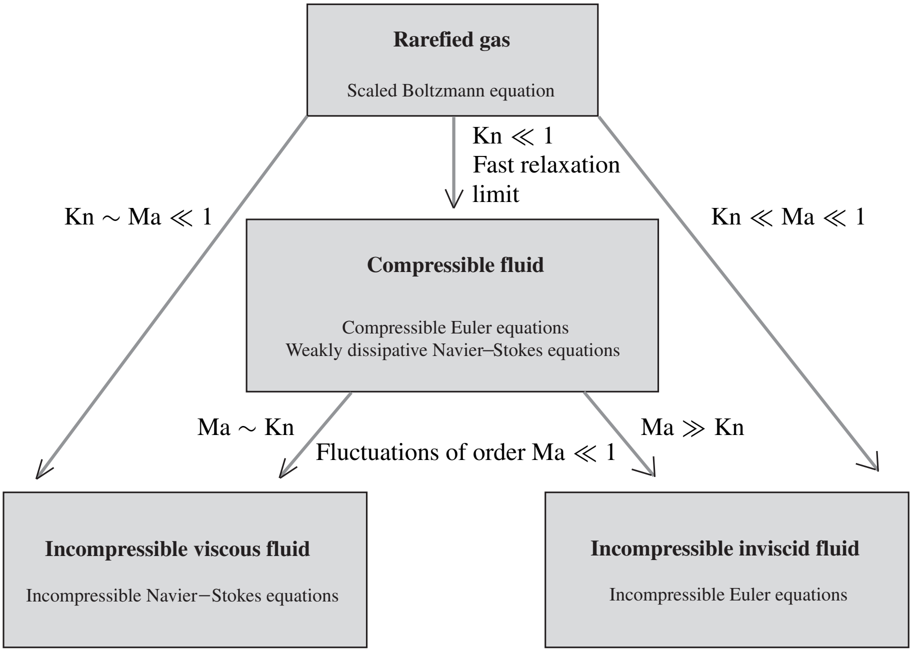
2. Incompressible hydrodynamic limits

Let us therefore sketch the formal derivation of such incompressible limits and then explain the main points where the formal arguments are not correct.

2.1. Mathematical setting. First of all, considering a fluctuation regime around a global equilibrium, for instance

$$M(v) = \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{|v|^2}{2}\right),$$

requires that the size of the fluctuation can be controlled for any time.



2.1.1. The entropy inequality. A good candidate to do that is the relative entropy

$$H(f|M) = \iint \left(f \log \frac{f}{M} - f + M \right) dv dx = \iint M h\left(\frac{f}{M} - 1 \right) dv dx,$$

which is a nonnegative quantity because of the convexity of the entropy functional ($h(z) = (1 + z)\log(1 + z) - z \geq 0$), and which is expected to measure in some sense the distance between f and M.

Because M does not depend neither on t nor on x, and that $\log M$ is a linear combination of 1, v and $|v|^2$, one can use the local conservations of mass, momentum and energy as well as the local entropy inequality to prove that the relative entropy is a Lyapunov functional for the Boltzmann equation

$$H(f|M)(t) + \frac{1}{\mathrm{Kn}\,\mathrm{Ma}} \int_0^t \left(-\int Q(f, f) \log f dv \right) (s, x) ds dx \leq H(f_{\mathrm{in}}|M)$$

meaning that it is controlled for any time by its initial value.

2.1.2. Fluctuations around a global equilibrium. Now, because the function h that arises in the definition of the relative entropy behaves as $h(z) \sim \frac{1}{2}z^2$ in the vicinity of zero, we expect the scaled relative entropy

$$\frac{1}{\mathrm{Ma}^2} H(f|M) \leq \frac{1}{\mathrm{Ma}^2} H(f_{\mathrm{in}}|M) \leq C_{\mathrm{in}}$$

to control essentially the L^2-norm of the fluctuation g defined by

$$f = M(1 + \mathrm{Ma}\, g),$$

at least if the distribution f has no big tail.

What can be proved actually is that the fluctuation is uniformly bounded in some weighted L^1-norm:

$$g \in L^\infty_t(L^1_{\mathrm{loc}}(dx, L^1(M(1+|v|^2)dv)))$$

using Young's inequality

$$pz \leq h^*(p) + h(z) \quad \text{for all } p, z \geq 0,$$

together with the relative entropy bound and the superquadraticity of h^*.

2.2. Formal derivation

2.2.1. Local thermodynamic equilibrium. It is natural to rewrite the Boltzmann equation in terms of the fluctuation g:

$$\mathrm{Ma}\, \partial_t g + v \cdot \nabla_x g = -\frac{1}{\mathrm{Kn}} \mathcal{L}_M g + \frac{\mathrm{Ma}}{\mathrm{Kn}} \frac{1}{M} Q(Mg, Mg),$$

where $\mathcal{L}_M$ is the linearized collision operator defined by

$$\mathcal{L}_M g = -\frac{2}{M} Q(M, Mg).$$

A careful study of that linearized collision operator going back to Grad [13] shows that the kernel of $\mathcal{L}_M$ is constituted of linear combinations of the collision invariants 1, v and $|v|^2$. We then expect that, in the fast relaxation limit $\mathrm{Kn} \to 0$,

$$\mathcal{L}_M g = 0,$$

meaning that the fluctuation behaves as an element of this kernel, referred to as an infinitesimal Maxwellian or as a fluctuation of Maxwellian:

$$g(t,x,v) = \rho(t,x) + u(t,x) \cdot v + \theta(t,x) \frac{|v|^2 - 3}{2}.$$

2.2.2. Macroscopic constraints. Rewriting also the conservation laws in terms of the fluctuation g,

$$\mathrm{Ma}\,\partial_t \int Mg\,dv + \nabla_x \cdot \int vMg\,dv = 0,$$
$$\mathrm{Ma}\,\partial_t \int vMg\,dv + \nabla_x \cdot \int v \otimes vMg\,dv = 0,$$

we obtain that, in the low Mach limit $\mathrm{Ma} \to 0$, the mass and momentum fluxes tend to zero.

Therefore, if the Knudsen and Mach numbers Kn and Ma tend both to zero, whatever their respective sizes, the moments ρ, u and θ satisfy the incompressibility and Boussinesq relations:

$$\nabla_x \cdot \int vMg\,dv = \nabla_x \cdot u = 0,$$
$$\nabla_x \cdot \int v \otimes vMg\,dv = \nabla_x(\rho + \theta) = 0.$$

2.2.3. Motion and heat equations. In order to characterize the limiting fluctuation, it remains then to describe the evolution of the (solenoidal part of the) bulk velocity and of the temperature.

Our starting point is once again the system of scaled conservation laws, and more precisely the following equations:

$$\partial_t \mathbf{P} \int vMg\,dv + \frac{1}{\mathrm{Ma}} \mathbf{P} \nabla_x \cdot \int \left(v \otimes v - \frac{1}{3}|v|^2\,\mathrm{Id}\right) Mg\,dv = 0,$$
$$\partial_t \int (|v|^2 - 5) Mg\,dv + \frac{1}{\mathrm{Ma}} \nabla_x \cdot \int v(|v|^2 - 5) Mg\,dv = 0,$$

denoting by $\mathbf{P}$ the Leray projection onto divergence free vector fields. Note that the conserved quantities have been suitably chosen in order to discard the unbounded parts of the fluxes.

The important point is indeed the fact that the quantities ϕ and ψ defined by

$$\phi(v) = v \otimes v - \frac{1}{3}|v|^2\,\mathrm{Id}, \quad \psi(v) = v(|v|^2 - 5)$$

arising in the momentum and heat fluxes, are orthogonal to the set of infinitesimal Maxwellians.

$$\phi \in (\mathrm{Ker}\,\mathcal{L}_M)^\perp, \quad \psi \in (\mathrm{Ker}\,\mathcal{L}_M)^\perp.$$

Then, as we expect the purely kinetic part of the fluctuation g, that is its projection $\Pi_\perp g$ on $(\mathrm{Ker}\,\mathcal{L}_M)^\perp$, to be small (of same order as the Mach number Ma), the fluxes will be bounded.

Let us indeed recall that the Boltzmann equation for the fluctuation shows that $\mathrm{Ma}^{-1}\mathcal{L}_M g$ can be decomposed as the sum of a nonlinear term of order 1, a stress term the order of which depends on the Reynolds number $\mathrm{Re} = \mathrm{Ma}/\mathrm{Kn}$, and a small remainder term coming from the time derivative:

$$\frac{1}{\mathrm{Ma}}\mathcal{L}_M g = \frac{1}{M}Q(Mg, Mg) - \frac{\mathrm{Kn}}{\mathrm{Ma}}v \cdot \nabla_x g + O(\mathrm{Kn})$$

Using this Ansatz together with the skew-symmetry of $\mathcal{L}_M$ to compute the momentum and heat fluxes, we identify two types of terms, namely convection terms that contain the nonlinearity and diffusion terms that contain the additional spatial derivative:

$$\begin{aligned}\frac{1}{\mathrm{Ma}}\int \zeta M g\, dv &= \frac{1}{\mathrm{Ma}}\int \tilde{\zeta} M \mathcal{L}_M g\, dv \\ &= \underbrace{\int \tilde{\zeta} Q(Mg, Mg)\, dv}_{\text{convection}} - \underbrace{\frac{\mathrm{Kn}}{\mathrm{Ma}}\int \tilde{\zeta}(v\cdot\nabla_x) M g\, dv}_{\text{diffusion}} + O(\mathrm{Kn})\end{aligned}$$

where $\tilde{\phi}$ and $\tilde{\psi}$ are the elements of $(\mathrm{Ker}\,\mathcal{L}_M)^\perp$ defined by

$$\phi = \mathcal{L}_M\tilde{\phi}, \quad \psi = \mathcal{L}_M\tilde{\psi}.$$

All these terms can be computed explicitly if the fluctuation g is an infinitesimal Maxwellian, which is asymptotically satisfied.

We therefore expect the limiting bulk velocity and temperature to satisfy either the incompressible Euler equations, or the incompressible Navier–Stokes–Fourier equations, depending on the respective sizes of the Knudsen and Mach number Kn and Ma,

$$\begin{aligned}\partial_t \mathbf{P}u + \mathbf{P}\nabla_x \cdot (u \otimes u) &= \left(\lim \frac{\mathrm{Kn}}{\mathrm{Ma}}\right)\mu\Delta_x u, \\ \partial_t(3\theta - 2\rho) + 5\nabla_x \cdot (\theta u) &= 5\left(\lim \frac{\mathrm{Kn}}{\mathrm{Ma}}\right)\kappa\Delta_x\theta.\end{aligned}$$

Note however that this formal argument is not correct since the weak compactness inherited from the relative entropy bound is not enough to take limits in the nonlinear convection terms.

2.3. Mathematical difficulties. The mathematical contribution to the study of hydrodynamic limits consists then essentially in obtaining a precise description of the convergences so as to understand the asymptotics of nonlinear terms.

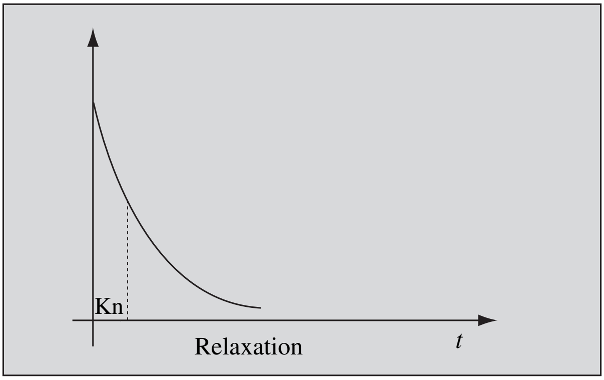
Let us therefore summarize the different approximations we have introduced to describe the asymptotics, and estimate their accuracy.

2.3.1. The relaxation estimate. We have first replaced the fluctuation by the corresponding local thermodynamic equilibrium:

$$g = \Pi g + (g - \Pi g)$$

where Π denotes the projection on Ker $\mathcal{L}_M$.

As this approximation is related to some relaxation process, we expect it to be very good outside from some initial and boundary layers the size of which is characterized by the Knudsen number Kn.



This should be proved using the kinetic equation

$$\mathrm{Ma}\,\partial_t g + v \cdot \nabla_x g = -\frac{1}{\mathrm{Kn}}\mathcal{L}_M g + \frac{\mathrm{Ma}}{\mathrm{Kn}}\frac{1}{M}Q(Mg, Mg)$$

together with the coercivity estimate for $\mathcal{L}_M$ established by Grad [13]

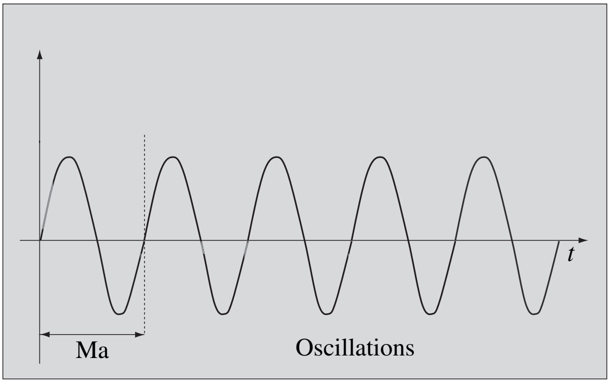
$$\|g - \Pi g\|^2_{L^2(M\,dv)} \leq C_0 \int g\mathcal{L}_M g M\,dv.$$

The point is that the functional framework used by Grad is hilbertian, while the relative entropy bound gives only some L^1-control on the fluctuation.

2.3.2. Time regularity. We have further considered the low Mach limit, especially to obtain the incompressibility and Boussinesq relations. Weak convergence is here really a "convergence in average" since we expect the scaled transport operator $(\mathrm{Ma}\,\partial_t + v \cdot \nabla_x)$ to create a fast time dependence.

More precisely, if we use the local thermodynamic approximation together with the conservation laws, we obtain a precise description of acoustic waves, the period of which is related to the Mach number Ma:

$$\begin{aligned} \mathrm{Ma}\,\partial_t \rho + \nabla_x \cdot u &= o(1), \\ \mathrm{Ma}\,\partial_t u + \nabla_x(\rho + \theta) &= o(1), \\ \mathrm{Ma}\,\partial_t \theta + 2/3\nabla_x \cdot u &= o(1). \end{aligned}$$



One difficulty is to prove that these (fast oscillating) acoustic waves will not disturb the mean motion.

2.3.3. Spatial regularity. The last point we would like to discuss a little bit is the dependence with respect to spatial variables. Because the transport operator is hyperbolic and we have no assumption on the initial regularity, we do not expect the fluctuation g to be smooth in x. On the other hand, taking limits in the (nonlinear) convection terms requires some strong compactness, at least on the moments of g.

Averaging lemma [8] which give some regularity on the moments are actually the useful tool to bypass this last difficulty if we have some suitable control on the transport.

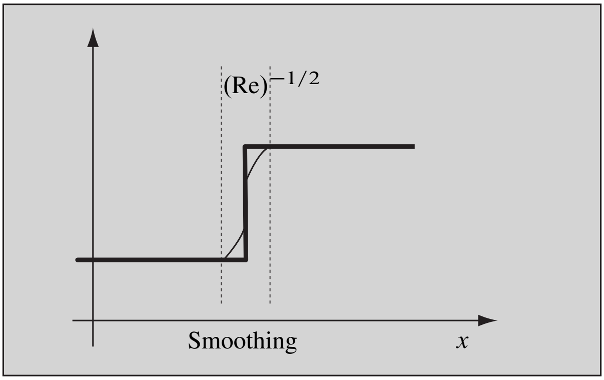
Using the bound on the entropy dissipation

$$D(f) = -\int Q(f, f) \log f \, dv = O(\mathrm{Ma}^3\, \mathrm{Kn})$$

we are able to control the free-transport as follows:

$$(\mathrm{Ma}\,\partial_t + v \cdot \nabla_x) M g = \frac{1}{\mathrm{Ma}\,\mathrm{Kn}} Q(f, f) = O\left(\sqrt{\frac{\mathrm{Ma}}{\mathrm{Kn}}}\right).$$

We therefore expect the situation to be very different depending on the respective sizes of the Mach and Knudsen numbers Ma and Kn.



3. Convergence results and elements of proofs

3.1. The mathematical framework. Our results are therefore not so optimal in inviscid regime as in viscous regime. Let us now give the main statements as well as some elements of proof. For the sake of simplicity, we consider a spatial domain Ω that has no boundary (for instance the whole space $\mathbb{R}^3$ or the three-dimensional torus $\mathbb{T}^3$).

3.1.1. Renormalized solutions to the Boltzmann equation. We start from very weak solutions of the Boltzmann equation, called renormalized solutions because they are only known to satisfy a family of formally equivalent kinetic equations obtained from the Boltzmann equation by truncation of large tails. Such renormalized solutions have been built by DiPerna and Lions [7], [17] twenty years ago. They exist globally in time without restriction on the size of the initial data, but they are known neither to be unique, nor to satisfy some of the fundamental physical properties we would expect, namely the local conservations of momentum and energy, even nor the global conservation of energy.

Theorem 3.1 (DiPerna & Lions). *Assume that the cross-section b satisfies Grad's cutoff assumption for some $\beta \in [0,1]$:*

$$0 < b(z,\omega) \le C_b(1+|z|)^\beta |\cos(\widehat{z,\omega})| \quad \text{a.e. on } \mathbb{R}^3 \times \mathbb{S}^2,$$
$$\int_{\mathbb{S}^2} b(z,\omega)\, d\omega \ge \frac{1}{C_b} \frac{|z|}{1+|z|} \qquad \text{a.e. on } \mathbb{R}^3.$$

Let $f_{\rm in} \in L^1_{\rm loc}(\Omega \times \mathbb{R}^3)$ be such that

$$H(f_{\rm in}|M) = \int_\Omega \int \left(f_{\rm in} \log \frac{f_{\rm in}}{M} - f_{\rm in} + M \right)(x,v)\, dv\, dx < +\infty.$$

Then there exists (*at least*) *one renormalized solution* $f \in C(\mathbb{R}^+, L^1_{\mathrm{loc}}(\Omega \times \mathbb{R}^3))$ *to the Boltzmann equation, meaning that for any* $\Gamma \in C_c^\infty(\mathbb{R}^+)$,

$$\mathrm{Ma}\,\partial_t \Gamma\left(\frac{f}{n}\right) + v \cdot \nabla_x \Gamma\left(\frac{f}{n}\right) = \frac{1}{\mathrm{Kn}} \frac{1}{M} \Gamma'\left(\frac{f}{M}\right) Q(f,f) \quad \text{on } \mathbb{R}^+ \times \Omega \times \mathbb{R}^3,$$
$$f(0,x,v) = f_{\mathrm{in}}(x,v) \quad \text{on } \Omega \times \mathbb{R}^3.$$

Moreover, f *satisfies*

- *the continuity equation*
$$\mathrm{Ma}\,\partial_t \int f\,dv + \nabla_x \cdot \int f v\,dv = 0;$$
- *the momentum equation with some positive definite matrix-valued defect measure* m
$$\mathrm{Ma}\,\partial_t \int f v\,dv + \nabla_x \cdot \int f v \otimes v\,dv + \nabla_x \cdot m = 0;$$
- *the entropy inequality with defect measure*
$$H(f|M)(t) + \int \mathrm{trace}\, m(t) + \frac{1}{\mathrm{Ma}\,\mathrm{Kn}} \int_0^t \int_\Omega D(f)(s,x)\,ds\,dx \le H(f_{\mathrm{in}}|M).$$

This lack of understanding concerning the physical properties of renormalized solutions generates of course additional technical difficulties when considering hydrodynamic limits.

Note however that the same kind of difficulty appears for Leray solutions to the incompressible Navier–Stokes equations, which are not known to satisfy the global conservation of energy.

3.1.2. General strategy. Our strategy is therefore to take advantage of these similarities and to proceed by analogy. More precisely, the main idea is to recognize in the scaled Boltzmann equation the same mathematical structure as in the asymptotic hydrodynamic equations, especially

- weak stability in viscous regime (controlled by the dissipation terms in the energy/entropy inequalities),
- strong-weak stability in inviscid regime (controlled by some energy/entropy functionals).

3.2. The incompressible Navier–Stokes limit. We first focus on the viscous regime, and recall that the incompressible Navier–Stokes equations have global weak solutions satisfying Leray energy inequality, and which are stable by weak convergence [16].

3.2.1. Leray solutions to the Navier–Stokes equations

Theorem 3.2 (Leray). *Let $u_{\mathrm{in}} \in L^2(\Omega)$ be a divergence free vector field.*

Then there exists (at least) one global weak solution $u \in L^2_{\mathrm{loc}}(\mathbb{R}^+, H^1(\Omega)) \cap C(\mathbb{R}^+, w - L^2(\Omega))$ to the incompressible Navier–Stokes equations

$$\begin{gathered} \partial_t u + (u \cdot \nabla_x)u + \nabla_x p = \mu \Delta_x u, \quad \nabla_x \cdot u = 0 \quad \text{on } \mathbb{R}^+ \times \Omega, \\ u(0,x) = u_{\mathrm{in}}(x) \quad \text{on } \Omega. \end{gathered} \tag{3.1}$$

It further satisfies the energy inequality

$$\|u(t)\|^2_{L^2(\Omega)} + 2\mu \int_0^t \|\nabla_x u(s)\|^2_{L^2(\Omega)}\, ds \le \|u_{\mathrm{in}}\|^2_{L^2(\Omega)}. \tag{3.2}$$

The dissipation term in (3.2) provides indeed some spatial regularity, which combined with the time regularity coming from the evolution equation in (3.1), gives some strong compactness and therefore stability of the (nonlinear) convection term.

A similar mechanism will give the weak convergence of the thermodynamic fields associated to the scaled Boltzmann equation as the Knudsen and Mach number goes to 0 at the same rate. In such a viscous regime, the Leray energy inequality and the DiPerna–Lions entropy inequality are indeed very similar objects.

3.2.2. From Boltzmann to Navier–Stokes. We have actually proved in collaboration with François Golse [10], [11] that the limiting fluctuation is an infinitesimal Maxwellian, the moments of which are weak solutions to the Navier–Stokes–Fourier system.

Theorem 3.3 (Golse & Saint-Raymond). *Let $(f_{\epsilon,\mathrm{in}})$ be a family of initial fluctuations around a global equilibrium M, i.e., such that*

$$\frac{1}{\epsilon^2} H(f_{\epsilon,\mathrm{in}}|M) \le C_{\mathrm{in}}.$$

Let (f_ϵ) be a family of renormalized solutions to

$$\begin{gathered} \epsilon \partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon = \frac{1}{\epsilon} Q(f_\epsilon, f_\epsilon) \quad \text{on } \mathbb{R}^+ \times \Omega \times \mathbb{R}^3, \\ f_\epsilon(0,x,v) = f_{\epsilon,\mathrm{in}}(x,v) \quad \text{on } \Omega \times \mathbb{R}^3. \end{gathered}$$

Then the family of fluctuations (g_ϵ) defined by $f_\epsilon = M(1+\epsilon g_\epsilon)$ is relatively weakly compact in $L^1_{\mathrm{loc}}(dt\,dx, L^1(M dv))$; and for any limit point g of (g_ϵ),

$$g(t,x,v) = u(t,x) \cdot v + \theta(t,x) \frac{|v|^2 - 5}{2},$$

where u is a weak solution to the Navier–Stokes equations (3.1) *and θ satisfies the convection-diffusion equation*

$$\partial_t \theta + \nabla_x \cdot (u\theta) = \kappa \Delta_x \theta.$$

The only assumption we need is to know that the initial fluctuation has the "good" size, or in other words that the initial relative entropy is scaled as Ma^2.

3.2.3. Strategy of the proof: the moment method. The strategy of the proof is very similar to that used in the formal derivation.

In order to make the arguments rigorous, we work with the renormalized fluctuation

$$\hat{g}_\epsilon = \frac{2}{\epsilon}\left(\sqrt{\frac{f_\epsilon}{M}} - 1\right)$$

instead of the fluctuation

$$g_\epsilon = \frac{1}{\epsilon}\left(\frac{f_\epsilon}{M} - 1\right)$$

since both quantities are asymptotically equivalent, but the first one is in some weighted L^2-space

$$\iint M \hat{g}_\epsilon^2 \, dx dv \le \frac{2}{\epsilon^2} H(f_\epsilon | M).$$

We then start from some approximate conservation laws obtained by integrating the renormalized kinetic equation against truncated collision invariants.

The scheme of the proof is therefore

- to prove that the conservation defects go to zero,
- to find a suitable decomposition of the flux terms,
- to take limits in the diffusion and convection terms.

We will not enter the details of the proof here since it relies on very careful (and sometimes technical) estimates of the remainder terms. We will just give some highlights concerning the main difficulties that have been mentioned in Section 2.3.

• *The relaxation estimate* is obtained from the identity

$$\frac{1}{\epsilon}\mathcal{L}_M \hat{g}_\epsilon = \frac{1}{2M} Q(M\hat{g}_\epsilon, M\hat{g}_\epsilon) - \frac{2}{\epsilon^2}\frac{1}{M} Q(\sqrt{M f_\epsilon}, \sqrt{M f_\epsilon})$$

coming from the bilinearity of Q, together with the coercivity estimate on $\mathcal{L}_M$.

For bounded cross-sections b, the first term in the right-hand side is indeed estimated by the scaled relative entropy

$$\left\| \frac{1}{2M} Q(M\hat{g}_\epsilon, M\hat{g}_\epsilon) \right\|^2_{L^1(dx, L^2(M dv))} \le C \|\hat{g}_\epsilon\|^2_{L^2(dx M dv)} \le \frac{2C}{\epsilon^2} H(f_\epsilon | M)$$

while the scaled entropy dissipation controls the second term

$$\left\| \frac{2}{\epsilon^2} \frac{1}{M} Q(\sqrt{M f_\epsilon}, \sqrt{M f_\epsilon}) \right\|_{L^2(M dv)} \leq \frac{C}{\epsilon^4} D(f_\epsilon).$$

For general cross-sections b, we introduce some modified collision operator and then use essentially the same arguments.

• *The spatial regularity of the moments* is obtained by the averaging lemma, and more precisely by some L^1 version of averaging lemma [12] that requires a control on the transport

$$\begin{aligned}(\epsilon\partial_t + v \cdot \nabla_x) &\frac{2}{\epsilon}\left(\sqrt{\frac{f_\epsilon}{M} + \epsilon^\alpha} - 1\right) \\ &= O(1)_{L^2(dtdxv^{-1}Mdv)} + O(\epsilon)_{L^1_{\mathrm{loc}}(dtdx, L^2(v^{-1}Mdv))} + O(\epsilon^{2-\alpha/2})_{L^1(dtdxMdv)}\end{aligned}$$

and some equiintegrability with respect to v-variables which is inherited from the relaxation estimate

$$\hat{g}_\epsilon = \Pi \hat{g}_\epsilon + (\hat{g}_\epsilon - \Pi \hat{g}_\epsilon) = \hat{\rho}_\epsilon + \hat{u}_\epsilon \cdot v + \frac{1}{2}\hat{\theta}_\epsilon(|v|^2 - 3) + O(\epsilon).$$

• *The acoustic waves* are dealt with finally, using a “compensated compactness” or “transparency” argument. Because of the algebraic structures of both the wave equations

$$\begin{aligned}\partial_t \frac{3}{5}(\rho_\epsilon + \theta_\epsilon) + \frac{1}{\epsilon}\nabla_x \cdot (\mathrm{Id} - \mathbf{P}) u_\epsilon &= o\left(\frac{1}{\epsilon}\right), \\ \partial_t (\mathrm{Id} - \mathbf{P}) u_\epsilon + \frac{1}{\epsilon}\nabla_x (\rho_\epsilon + \theta_\epsilon) &= o\left(\frac{1}{\epsilon}\right),\end{aligned}$$

and the nonlinear convection terms

$$\mathbf{P}\nabla_x \cdot (u_\epsilon \otimes u_\epsilon) \quad \text{and} \quad \nabla_x \cdot (u_\epsilon \theta_\epsilon),$$

it turns out that the coupling of fast oscillating components does not produce any contribution to the non-oscillating part of the motion.

3.3. The incompressible Euler limit. Of course the same strategy fails when considering inviscid regimes, that are regimes such that the Knudsen number is negligible compared to the Mach number $\mathrm{Kn} \ll \mathrm{Ma}$. We have indeed no control on the free transport and therefore no a priori spatial regularity on the moments.

Note that because of a similar lack of a priori regularity, the incompressible Euler equations are not known to have global weak solutions, at least in three-dimensional spaces Ω.

3.3.1. Dissipative solutions to the Euler equations. Lions has therefore proposed a very weak notion of solutions, called dissipative solutions since they are typically obtained by considering the inviscid limit of the incompressible Navier–Stokes equations [18]. These solutions exist globally but are not known to satisfy any evolution equation in the sense of distributions. They only satisfy a stability inequality, that ensure some strong-weak uniqueness: as long as a smooth solution exists for the incompressible Euler equations, any dissipative solution having the same initial data, coincides with that smooth solution.

Theorem 3.4 (Lions). *Let $u_{\mathrm{in}} \in L^2(\Omega)$ be a divergence free vector field.*

Then there exists (at least) one global dissipative solution $u \in C([0,T), w-L^2(\Omega))$ to the incompressible Euler equations

$$\nabla_x \cdot u = 0, \quad \partial_t u + (u \cdot \nabla_x) u + \nabla_x p = 0, \tag{3.3}$$

meaning that, for all t and all $\tilde{u} \in C_c^\infty(\mathbb{R}^+ \times \Omega)$,

$$\begin{aligned}
&\|u(t) - \tilde{u}(t)\|^2_{L^2(\Omega)} \\
&\quad \le \|u_{\mathrm{in}} - \tilde{u}_{\mathrm{in}}\|^2_{L^2(\Omega)} \exp\left(\int_0^t \|\nabla_x \tilde{u}(s)\|_{L^\infty(\Omega)}\, ds \right) \\
&\qquad + \int_0^t \!\! \int (\partial_t \tilde{u} + \tilde{u} \cdot \nabla_x \tilde{u}) \cdot (\tilde{u} - u)(s,x)\, dx \exp\left(\int_s^t \|\nabla_x \tilde{u}(\tau)\|_{L^\infty(\Omega)}\, d\tau \right).
\end{aligned}$$

A similar stability inequality will be established for the solutions to the scaled Boltzmann equation.

3.3.2. From Boltzmann to Euler. What we are actually able to prove [25] is the global convergence towards dissipative solutions of the incompressible Euler equations starting from renormalized solutions to the scaled Boltzmann equation, but only for very well-prepared initial data.

Theorem 3.5 (Saint-Raymond). *Let $(f_{\epsilon,\mathrm{in}})$ be a family of initial fluctuations around a global equilibrium M, satisfying*

$$\frac{1}{\epsilon^2} H(f_{\epsilon,\mathrm{in}} | \mathcal{M}_{1,\epsilon u_{\mathrm{in}},1}) \to 0 \quad \text{as } \epsilon \to 0,$$

for some given divergence-free vector field $u_{\mathrm{in}} \in L^2(\Omega)$.

Let (f_ϵ) be a family of renormalized solutions to

$$\begin{aligned}
\epsilon \partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon &= \frac{1}{\epsilon^q} Q(f_\epsilon, f_\epsilon) \quad \text{on } \mathbb{R}^+ \times \Omega \times \mathbb{R}^3 \quad (q > 1), \\
f_\epsilon(0,x,v) &= f_{\epsilon,\mathrm{in}}(x,v) \qquad \text{on } \Omega \times \mathbb{R}^3.
\end{aligned}$$

Then the family (u_ϵ) *defined by* $u_\epsilon = \epsilon^{-1}\int f_\epsilon v\,dv$ *is relatively weakly compact in* $L^1_{\mathrm{loc}}(dt\,dx)$*, and any limit point* u *of* (u_ϵ) *is a dissipative solution to the incompressible Euler equations* (3.3).

We have indeed to assume that the initial profile is close to thermodynamic equilibrium so that there is no relaxation layer. Furthermore we consider initial thermodynamic fields that satisfy both the incompressibility and Boussinesq relations in order that there is no acoustic wave. Finally (for technical reasons that we cannot explain here) we restrict our attention to the case when the initial fluctuation of temperature is zero.

$$\frac{1}{\epsilon^2}H(f_{\epsilon,\mathrm{in}}|\mathcal{M}_{1,\epsilon u_{\mathrm{in}},1}) \to 0 \quad \text{as } \epsilon \to 0,\ \text{with } \nabla_x \cdot u_{\mathrm{in}} = 0.$$

All these restrictions come from the fact that we use here a completely different strategy, which leads to some strong convergence result. For general initial data, we would therefore have to build a more accurate approximation, with correctors taking into account the initial layer and acoustic waves.

3.3.3. Strategy of the proof: the modulated entropy method. Let us first recall that, in inviscid regime, the formal proof fails because of a lack of spatial regularity on the moments (no uniform control on the free-transport). The alternative strategy consists then in establishing a stability inequality similar to that defining dissipative solutions. The quantity we are interested in is the scaled modulated entropy

$$\frac{1}{\epsilon^2}H(f_\epsilon|\mathcal{M}_{1,\epsilon\tilde{u},1})$$

measuring in some sense the distance of f_ϵ to the test Maxwellian $\mathcal{M}_{1,\epsilon\tilde{u},1}$.

Such a method, called relative entropy method, has been developed by Golse [3] in the framework of the Boltzmann equation, following an idea of Yau [27].

The scheme of the proof is as follows. We first use the relative entropy inequality to obtain weak compactness (and convergence in the linear relations) as previously. Then, computing the time derivative of the modulated entropy, we can identify two terms: the first one,

$$\iint f_\epsilon(v-\epsilon\tilde{u})\cdot(\partial_t\tilde{u} + (\tilde{u}\cdot\nabla_x)\tilde{u})\,dv\,dx,$$

converges weakly and its limit is zero if $\tilde{u}$ is a strong solution to the Euler equation. The other one,

$$\iint f_\epsilon\big((v-\epsilon\tilde{u})^{\otimes 2} - \frac{1}{3}|v-\epsilon\tilde{u}|^2\big)\,dv\,dx,$$

can be estimated in terms of the modulated entropy and the entropy dissipation. We then deduce by Gronwall's lemma that

$$\begin{aligned}
&\frac{1}{\epsilon^2} H(f_\epsilon | \mathcal{M}_{1,\epsilon\tilde{u},1})(t) + \frac{1}{2\epsilon^{q+3}} \int_0^t \iint D(f_\epsilon) ds dx \\
&\leq \frac{1}{\epsilon^2} H(f_{\epsilon,\mathrm{in}} | \mathcal{M}_{1,\epsilon\tilde{u}_{\mathrm{in}}1}) \exp\left(C \int_0^t \|D\tilde{u}(s)\|_{L^1 \cap L^\infty(\Omega)} ds \right) \\
&- \frac{1}{\epsilon} \int_0^t \iint f_\epsilon (v - \epsilon\tilde{u}) \cdot (\partial_t \tilde{u} + \tilde{u} \cdot \nabla_x \tilde{u}) \exp\left(C \int_s^t \|D\tilde{u}(s)\|_{L^1 \cap L^\infty(\Omega)} ds \right) dv dx ds
\end{aligned}$$

for any test divergence-free vector field $\tilde{u}$. The convergence statement follows then by convexity arguments.

Note that the argument can be improved a little bit to consider general initial data, but in that case we need some extra integrability assumptions on the solutions to the Boltzmann equation to obtain a Gronwall type estimate [26]. Under these additional conditions on (f_ϵ), one can actually prove that the relative entropy between f_ϵ and a precise approximation f_{app} tends to zero

$$\frac{1}{\epsilon^2} H(f_\epsilon | f_{\mathrm{app}}) \to 0 \text{ as } \epsilon \to 0.$$

This approximation f_{app} has an hydrodynamic part obtained as the sum of the weak limit and of acoustic waves, and a purely kinetic part which is non zero only in a thin time layer.

• *Acoustic waves* are described by

$$\begin{pmatrix}
\partial_t \tilde{\rho} + (\tilde{u} \cdot \nabla_x)\tilde{\rho} + \frac{1}{\epsilon} \nabla_x \cdot \tilde{u} \\
\partial_t \tilde{u} + (\tilde{u} \cdot \nabla_x)\tilde{u} + \tilde{\theta} \nabla_x \left(\tilde{\rho} - \frac{3}{2}\tilde{\theta} \right) + \frac{1}{\epsilon} \nabla_x (\tilde{\rho} + \tilde{\theta}) \\
\partial_t \tilde{\theta} + (\tilde{u} \cdot \nabla_x)\tilde{\theta} + \frac{2}{3\epsilon} \nabla_x \cdot \tilde{u}
\end{pmatrix} = 0.$$

Since they only modify the hydrodynamic part of the approximation, they can be taken into account in the stability inequality with only minor modifications

• *The initial relaxation layer* is described by the homogeneous kinetic equation

$$\partial_t f = \frac{1}{\epsilon^{q+1}} Q(f, f).$$

The entropy dissipation associated to that relaxation process converges towards a finite (non zero) quantity, meaning that one has to also modulate the entropy dissipation in the initial layer to prove the convergence statement.

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5ECM Amsterdam 2008

Graded algebras associated to algebraic algebras need not be algebraic

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Abstract. It is shown that the associated graded algebras of affine nil algebras need not be nil. A similar argument shows that filtered algebras generated in degree one of nil algebras need not be nil. This answers a question of Riley [5].

Mathematics Subject Classification (2010). 16N20, 16N40, 16W50.

Keywords. Nil algebra, algebraic algebra, associated graded algebra.

1. Introduction

Filtered algebras are useful for calculating the Gelfand–Kirillov dimension of associative algebras. In [3] Purcell gives a condition which guarantees that a filtered ring is isomorphic to its associated graded ring $\mathrm{gr}(A)$. Recall that the associated graded algebra of an associative algebra A is defined by

$$\mathrm{gr}(A) = \bigoplus_{i\geq 1} A^i/A^{i+1} = \bigoplus_{i\geq 1} A_i .$$

Riley asked whether associated graded algebras of finitely generated nil algebras are nil ([5], Problem 1). The purpose of this paper is to answer this question in the negative. Namely, the following holds.

Theorem 1.1. *Let K be a countable field. Then there is a nil K-algebra A generated by three elements, such that the associated graded algebra* $\mathrm{gr}(A)$ *is not nil.*

A theorem of Amitsur says that polynomial rings over nil algebras over uncountable fields are nil. It follows that associated graded algebras of finitely generated nil algebras over uncountable fields are nil. Theorem 1.1 shows that graded algebras associated to algebraic algebras over countable fields need not be nil. The following question remains open (Riley, [5], Problem 1): *Are associated graded algebras of*

*This work was supported by Grant No. EPSRC EP/D071674/1.

finitely generated nil algebras over countable fields Jacobson radical? It was shown by Krempa [2] that the Koethe conjecture (1930) is equivalent to the assertion that polynomial rings over nil rings are Jacobson radical. Hence, if the Koethe conjecture is true, then the answer to Riley's question is in the affirmative. For some related results see [7]. In [8] Stephenson and Zhang showed that filtered algebras of Noetherian algebras need not be Noetherian. Artin, Small and Zhang proved that if A is an algebra whose filtered graded ring is locally finite and right Noetherian, then every prime ideal in R is an intersection of primitive ideals [1].

2. Algebras generated by elements a, b, c

Let K be a countable field and let T be the free K-algebra in generators a, b, c over K subject to relations $ac = 0$ and $c^2 = 0$. Assigning gradation 1 to elements a and b and gradation 0 to c gives a gradation on T. Then $T = T_0 + T_1 + \cdots$.

Let E be the subalgebra of T generated by elements a^3, b^2 and c. Assigning gradation 1 to elements a^3, b^2 and c gives gradation on E, and we can write $E = E_0 + E_1 + \cdots$. Let I be an ideal in E. Let $\mathrm{gr}(E/I) = \bigoplus_{i\geq 1}(E/I)_i$ where $(E/I)_i = (E/I)^i/(E/I)^{i+1}$. By $\tilde{r}$ we will denote the image of $r \in E$ in $\mathrm{gr}(E/I)$. Note that if $r \in E_k$ and $\tilde{r} = 0$, then $r \in \sum_{i>k} E_i + I$.

Let $w(n_1, n_2, n_3) = \{\sum w : w$ is a product of n_1 elements a^6, n_2 elements b^6 and n_3 elements $c\}$. Notice that $w(n_1, n_2, n_3) \in E_{2n_1+3n_2+n_3} \cap T_{6n_1+6n_2}$.

Lemma 2.1. *Let E, T be as above and let J be a homogeneous ideal in T. Denote, $I = E \cap J$. Let $p = a^6 + b^6 + c \in E$ and let $\tilde{p}$ be an image of p in $\mathrm{gr}(E/I)$. If $\tilde{p}^n = 0$ for some n in $\mathrm{gr}(E/I)$, then $w(n_1, n_2, n_3) \in I + \sum_{i>2n_1+3n_2+n_3} E_i$ for all $n_1 + n_2 + n_3 = n$.*

Proof. Let $\tilde{p}^n = 0$. Write $p^n = \sum_j g_j$, with $g_j \in E_j$. Then

$$g_j = \sum_{\substack{n_1+n_2+n_3=n,\\ 2n_1+3n_2+n_3=j}} w(n_1, n_2, n_3).$$

By assumption $\tilde{g}_j = 0$ so $g_j \in I + \sum_{i>j} E_i$. Now observe that $w(n_1, n_2, n_3) \in T_{6n_1+6n_2}$. Recall that J is a homogeneous ideal in T, therefore $I = \sum_{j\geq 0} T_j \cap I$. Moreover, for every i, $E_i = \sum_{j\geq 0} T_j \cap E_i$. It follows that for every p we get $\{\sum w(n_1, n_2, n_3) : 6n_1 + 6n_2 = p, 2n_1 + 3n_2 + n_3 = j, n_1 + n_2 + n_3 = n\} \in I + \sum_{i>j} E^i$. Therefore, $w(n_1, n_2, n_3) \in \sum_{i>2n_1+3n_2+n_3} E_i + I$, for all n_1, n_2, n_3, with $n_1 + n_2 + n_3 = n$, as required. $\square$

3. Elements x, y, z and elements a, b, c

Let A be the subalgebra of T generated by elements a, b and bc. Denote $x = a$, $y = b$, $z = bc$. Note that A is the free algebra in generators x, y, z. Assigning gradation 1 to x, y, z gives a gradation on A, $A = H_1 + H_2 + \cdots$ where $T_i = H_i + cH_i$.

Let $z(n_1, n_2, n_3) = \{\sum w : w$ is a product of n_1 elements x^6, n_2 elements y^6 and n_3 elements $y^5 z\}$. If either $n_1 < 0, n_2 < 0$ or $n_3 < 0$ we put $z(n_1, n_2, n_3) = 0$. Assigning gradation $\frac{1}{3}$ to element x, $\frac{1}{2}$ to element y and $\frac{3}{2}$ to element z gives a gradation on A. Let $P = \{i : 6i$ is a natural number$\}$. Let $Z_i = \{w \in A : \frac{\deg_x w}{3} + \frac{\deg_y w}{2} + \frac{3\deg_z w}{2} = i\}$. Note that $A = \sum_{i \in P} Z_i$.

Lemma 3.1. *Let F be a right ideal in A. If $w(n_1, n_2, n_3) \in \sum_{i > 2n_1 + 3n_2 + n_3} E_i + F + cF$, then $z(n_1, n_2 - n_3, n_3) \in F + \sum_{i \in P : i > 2n_1 + 3n_2 + n_3} Z_i$.*

Proof. By the definition $w(n_1, n_2, n_3) = z(n_1, n_2 - n_3, n_3) + cz(n_1, n_2 - n_3 + 1, n_3 - 1)$. Note that $E_i \subseteq Z_i + cZ_{i-1}$ (denote $Z_0 = K$). Comparing elements starting with a and b we get the result. □

4. Algebras generated by elements x, y, z

In this section we will only consider algebras generated by x, y, z. Let M_s be a set of products of exactly n elements from the set $\{x, y, z\}$. Then $H_s = KM_s$. Observe that $z(n_1, n_2, n_3) \in Z_{2n_1 + 3n_2 + 4n_3}$. We introduce the following ordering on triples of natural numbers. Let d_1, d_2, d_3, b_1, b_2, b_3 be natural numbers. Let $(d_1, d_2, d_3) \prec (b_1, b_2, b_3)$ if either $2d_1 + 3d_2 + 4d_3 > 2b_1 + 3b_2 + 4b_3$, or $2d_1 + 3d_2 + 4d_3 = 2b_1 + 3b_2 + 4b_3$ and $d_1 < b_1$ or $2d_1 + 3d_2 + 4d_3 = 2b_1 + 3b_2 + 4b_3$, $d_1 = b_1$ and $d_2 < b_2$.

Lemma 4.1 (Lemma 7, [6]). *For each $n \geq 0$ the set $S = \{z(n_1, n_2, n_3) : n_1 + n_2 + n_3 = n\}$ is a free basis of a right module SA. Moreover, for arbitrary integers n_1, n_2, n_3 and $r < n_1 + n_2 + n_3$,*

$$z(n_1, n_2, n_3) = \sum_{r_1 + r_2 + r_3 = r} z(r_1, r_2, r_3) z(n_1 - r_1, n_2 - r_2, n_3 - r_3).$$

Lemma 4.2. *Let $f : H_{6p} \to H_{6p}$, $g : H_{6q} \to H_{6q}$, and $h : H_{6p+6q} \to H_{6p+6q}$ be K-linear mappings such that for all $u \in M_{6p}$, $v \in M_{6q}$, $h(uv) = f(u)g(v)$. If $h(z(n_1, n_2, n_3)) \in \sum_{i \in P : i > 2n_1 + 3n_2 + 4n_3} h(Z_i)$ for all $n_1 + n_2 + n_3 = p + q$, then either $f(z(p_1, p_2, p_3)) \in \sum_{i \in P : i > 2p_1 + 3p_2 + 4p_3} f(Z_i)$ for all $p_1 + p_2 + p_3 = p$ or $g(z(q_1, q_2, q_3)) \in \sum_{i \in P : i > 2q_1 + 3q_2 + 4q_3} g(Z_i)$ for all $q_1 + q_2 + q_3 = q$.*

Proof. Suppose that the result does not hold. Let (p_1, p_2, p_3) and (q_1, q_2, q_3) be minimal with respect to the ordering $\prec$ such that $p_1 + p_2 + p_3 = p$, $q_1 + q_2 + q_3 = q$ and $f(z(p_1, p_2, p_3)) \notin \sum_{i\in P:i>2p_1+3p_2+4p_3} f(Z_i)$, $g(z(q_1, q_2, q_3)) \notin \sum_{i\in P:i>2q_1+3q_2+4q_3} g(Z_i)$. Further, let

$$D = H_{6p} \cap \sum_{i\in P:i>2p_1+3p_2+4p_3} f(Z_i)$$

and

$$B = H_{6q} \cap \sum_{i\in P:i>2q_1+3q_2+4q_3} g(Z_i).$$

Observe that $z(p_1+q_1, p_2+q_2, p_3+q_3) = \sum_{r_1+r_2+r_3=p} z(r_1, r_2, r_3)z(p_1+q_1-r_1, p_2+q_2-r_2, p_3+q_3-r_3)$. By Lemma 4.1, $h(z(p_1+q_1, p_2+q_2, p_3+q_3)) = \sum_{r_1+r_2+r_3=p} f(z(r_1, r_2, r_3))g(z(p_1 + q_1 - r_1, p_2 + q_2 - r_2, p_3 + q_3 - r_3))$. Note that if $(p_1, p_2, p_3) \prec (r_1, r_2, r_3)$ with respect to the above ordering, then $(p_1+q_1-r_1, p_2+q_2-r_2, p_3+q_3-r_3) \prec (q_1, q_2, q_3)$. By the assumptions about the minimality of (p_1, p_2, p_3), if $(r_1, r_2, r_3) \prec (p_1, p_2, p_3)$, then $f(z(r_1, r_2, r_3)) \in \sum_{i\in P:i>2p_1+3p_2+4p_3} f(Z_i)$. Similarly, if $(v_1, v_2, v_3) \prec (q_1, q_2, q_3)$, then we have $g(z(v_1, v_2, v_3)) \in \sum_{i\in P:i>2q_1+3q_2+4q_3} g(Z_i)$. So $h(z(p_1 + q_1, p_2 + q_2, p_3 + q_3)) \in f(z(p_1, p_2, p_3))g(z(q_1, q_2, q_3))+DH_{6q}+H_{6p}B$. By assumption, $h(z(p_1+q_1, p_2+q_2, p_3+q_3)) \in \sum_{i>2p_1+2q_1+3p_2+3q_2+4p_3+4q_3} h(Z_i)$. Therefore, since A is generated in degree one, $Z_i \cap H_{6p+6q} \subseteq \sum_{j\in P:0\le j\le i}(H_{6p}\cap Z_j)(H_{6q}\cap Z_{i-j})$, so $h(Z_i) \subseteq DH_{6q} + H_{6p}B$. Hence, $h(z(p_1+q_1, p_2+q_2, p_3+q_3)) \in DH_{6q}+H_{6p}B$. It follows that $f(z(p_1, p_2, p_3))g(z(q_1, q_2, q_3)) \in DH_{6q} + H_{6p}B$. Recall that $f(z(p_1, p_2, p_3)) \in H_{6p}$ and $D \in H_{6p}$. Therefore either $f(z(p_1, p_2, p_3)) \in D$ or $g(z(q_1, q_2, q_3)) \in B$, a contradiction. □

Lemma 4.3. *Let p, r be integers such that $p + r > 10^8$, p, r are divisible by 240, $r > 10^5$ and $r > 5p$. Let $f: H_{6p} \to H_{6p}$, $g: H_{6r+6p} \to H_{6r+6p}$ be K-linear mappings such that for $u \in M_{6r}$, $v \in M_{6p}$, $g(uv) = uf(v)$. If for all $n_1+n_2+n_3 = p + r$, $g(z(n_1, n_2, n_3)) \in \sum_{i>2n_1+3n_2+4n_3} g(Z_i) + \sum_{i=1}^{40^{-2}(p+r)^2} Kf_i$ for some $f_i \in A$, then $f(z(p_1, p_2, p_3)) \in \sum_{i>2p_1+3p_2+4p_3} f(Z_i)$ for all $p_1 + p_2 + p_3 = p$. Moreover for all $n_1+n_2+n_3 = p+r$, $g(z(n_1, n_2, n_3)) \in \sum_{i>2n_1+3n_2+4n_3} g(Z_i)$.*

Proof. The result holds if either $p_1 < 0$, $p_2 < 0$, or $p_3 < 0$ since then $z(p_1, p_2, p_3) = 0$ by the definition. Hence, it suffices to show that each $f(z(p_1, p_2, p_3))$, where $p_1+p_2+p_3 = p$, is a linear combination of $f(z(q_1, q_2, q_3))$ with $q_1+q_2+q_3 = p$ and $(q_1, q_2, q_3) \prec (p_1, p_2, p_3)$ and elements from $f(Z_i)$ with $i > 2p_1+3p_2+4p_3$. Let $S = \{n_1, n_2, n_3) : n_1+n_2+n_3 = p+r, \frac{1}{3}(p+r) < n_1 < (p+r)(\frac{1}{3}+\frac{1}{20}), \frac{1}{3}(p+r) < n_2 < (p+r)(\frac{1}{3} + \frac{1}{20})\}$. First we shall prove that $\mathrm{card}(S) \ge (p+r)^2 40^{-2}$. Observe that there are at least $(p+r)20^{-1}-2$ natural numbers laying between $(p+r)\frac{1}{3}$

and $(p+r)(\frac{1}{3}+\frac{1}{20})$. We can choose $((p+r)20^{-1}-2)^2$ distinct pairs (n_1, n_2) such that $\frac{1}{3}(p+r) < n_1 < (p+r)(\frac{1}{3}+\frac{1}{20})$ and $\frac{1}{3}(p+r) < n_2 < (p+r)(\frac{1}{3}+\frac{1}{20})$. For each such pair we can choose a natural number n_3 such that $n_1+n_2+n_3 = p+r$ and $(\frac{1}{3}-\frac{1}{10})(p+r) < n_3 < \frac{1}{3}(p+r)$. Since $p+r > 10^8$, we get that $\mathrm{card}(S) \geq ((p+r)20^{-1}-2)^2 > 40^{-2}(p+r)^2$. Hence the assumption of the theorem implies that $\sum_{(n_1,n_2,n_3)\in S} l_{n_1,n_2,n_3}[g(z(n_1,n_2,n_3)) - t_{2n_1+3n_2+4n_3}] = 0$, for some $t_k \in \sum_{i>k} g(Z_i)$, for some $l_{n_1,n_2,n_3} \in K$, not all of which are zeros. Let (j_1, j_2, j_3) be the maximal element in S, with respect to $\prec$, such that $l_{j_1,j_2,j_3} \neq 0$. Then $g(z(j_1, j_2, j_3)) \in \sum k_{n_1,n_2,n_3} g(z(n_1,n_2,n_3)) + \sum_{i>2j_1+3j_2+4j_3} g(Z_i)$ for some $k_{n_1,n_2,n_3} \in K$, where the sum runs over all $(n_1,n_2,n_3) \in S$ with $(n_1,n_2,n_3) \prec (j_1, j_2, j_3)$.

It follows from Lemma 4.1 that if $n_1+n_2+n_3 = p+r$, then $g(z(n_1,n_2,n_3)) = \sum_{r_1+r_2+r_3=r} z(r_1,r_2,r_3) f(z(n_1-r_1, n_2-r_2, n_3-r_3))$. Moreover, observe that $Z_i \in \sum_{r_1+r_2+r_3=r} z(r_1,r_2,r_3) Z_{i-2r_1-3r_2-4r_3} + wA$ where $w \in M_{6r}$ are monomials which are linearly independent from $z(r_1,r_2,r_3)$ with $r_1+r_2+r_3 = r$. Hence, $g(Z_i) \in \sum_{r_1+r_2+r_3=r} z(r_1,r_2,r_3) f(Z_{i-2r_1-3r_2-4r_3}) + wA$.

By Lemma 4.1, $g(z(j_1, j_2, j_3)) = \sum_{r_1+r_2+r_3=r} z(r_1,r_2,r_3) f(z(j_1-r_1, j_2-r_2, j_3-r_3))$. Therefore, $\sum_{r_1+r_2+r_3=r} z(r_1,r_2,r_3) f(z(j_1-r_1, j_2-r_2, j_3-r_3)) \in \sum_{r_1+r_2+r_3=r} z(r_1,r_2,r_3) c_{r_1,r_2,r_3} + wA$ with $c_{r_1,r_2,r_3} = t_{2j_1-2r_1+3j_2-3r_2+4j_3-4r_3} + \sum_{(n_1,n_2,n_3)\in S:(n_1,n_2,n_3)\prec(j_1,j_2,j_3)} k_{n_1,n_2,n_3} f(z(n_1-r_1, n_2-r_2, n_3-r_3))$ for some $t_i \in \sum_{j>i} f(Z_j)$. This and the first part Lemma 4.1 imply that for each triple (r_1,r_2,r_3) satisfying $r_1+r_2+r_3 = r$, with $z(r_1,r_2,r_3) \neq 0$, we have $f(z(j_1-r_1, j_2-r_2, j_3-r_3)) \in \sum_{(n_1,n_2,n_3)\in S:(n_1,n_2,n_3)\prec(j_1,j_2,j_3)} k_{n_1,n_2,n_3} f(z(n_1-r_1, n_2-r_2, n_3-r_3)) + \sum_{i>2j_1-2r_1+3j_2-3r_2+4j_3-4r_3} f(Z_i)$. The definition of S and the assumption $r > 5p$ imply that $j_i > p$ for $i = 1, 2, 3$. Hence for arbitrary p_1, p_2, p_3, such that $p_1+p_2+p_3 = p$, the integers $r_1 = j_1-p_1$, $r_2 = j_2-p_2$, $r_3 = j_3-p_3$ are positive and $r_1+r_2+r_3 = r$. Thus $f(z(p_1,p_2,p_3)) = f(z(j_1-r_1, j_2-r_2, j_3-r_3)) \in \sum_{(n_1,n_2,n_3)\prec(j_1,j_2,j_3)} k_{n_1,n_2,n_3} f(z(n_1-r_1, n_2-r_2, n_3-r_3)) + \sum_{i>2j_1-2r_1+3j_2-3r_2+4j_3-4r_3} f(Z_i)$. Clearly, $(n_1-r_1, n_2-r_2, n_3-r_3) \prec (j_1-r_1, j_2-r_2, j_3-r_3)$, so the result holds. Now by Lemma 4.1 we get that if $n_1+n_2+n_3 = p+r$, then $g(z(n_1,n_2,n_3)) \in \sum_{i>2n_1+3n_2+4n_3} g(Z_i)$. □

5. Some results from other papers

Let K be a field and Q be a graded K-algebra and $Q = \bigoplus_{i=1}^{\infty} Q_i$. Given a number n and a set $F \subseteq Q$ let $B_n^Q(F)$ denote the right ideal in Q generated by the set $\bigcup_{k=0}^{\infty} Q_{nk}F$, i.e., $B_n^Q(F) = \sum_{k=0}^{\infty} Q_{nk}FQ$, where $Q_0 = K$.

Let P be a K- algebra generated by elements $x_1, \ldots, x_6$ with gradation one. Write $P = P_1 + P_2 + \cdots$ where P_n is the K linear space spanned by monomials of degree n in $x_1,\dots,x_6$. By considering algebras generated by 6 elements instead of 3 elements and using the same proof as the proof of Corollary 3 [6] we get the following theorem.

Theorem 5.1. *Let f_i, $i=1,2,\dots$, be polynomials in P with degrees t_i, and let m_i be an increasing sequence of natural numbers such that $m_i > 6^{6t_i}$. There exist subsets $\overline{F_i}\subseteq P_{m_i}$ with $\operatorname{card}(\overline{F_i}) < m_i 6^{2t_i} t_i^2$ such that the ideal of P generated by $f_i^{10m_{i+1}}$, $i=1,2,\dots$ is contained in the right ideal $B^P_{m_{i+1}}(\overline{F_i})$ for $i=1,2,\dots$.*

We say that an element $r\in T$ (or $p\in P$) has degree d if $r\in T_0+\dots+T_d$ (respectively $p\in P_1+\dots+P_d$) and d is minimal with this property.

Theorem 5.2. *Let h_i, $i=1,2,\dots$, be polynomials in T with $\deg h_i^2=t_i$, and let m_i be an increasing sequence of natural numbers such that $m_i>6^{6t_i}$. Let I be the smallest homogeneous ideal in T containing elements $h_i^{20m_{i+1}}$, $i=1,2,\dots$. Then there exist subsets $F_i\subseteq H_{m_i}\subseteq A$, with $\operatorname{card}(F_i) < 40^{-2}m_i^2$ such that I is contained in $J+cJ$ where J is a right ideal in A and $J=\sum_{i=1}^{\infty} B^A_{m_{i+1}}(F_i)$. Moreover, $I\cap T_{m_n}\subseteq \sum_{i=1}^{n-1}(B^A_{m_{i+1}}(F_i)+cB^A_{m_{i+1}}(F_i))$ for all n.*

Proof. Let $\xi\colon P\to T$ be a ring homomorphism such that $\xi(x_1)=a$, $\xi(x_2)=b$, $\xi(x_3)=ca$, $\xi(x_4)=cb$, $\xi(x_5)=bc$, $\xi(x_6)=cbc$. Observe that if $h_i\in T$, then $h_i^2\in \operatorname{Im}(P)$. Let $f_i\in P$ be such that $\xi(f_i)=h_i^2$ and $\deg(h_i)^2=\deg(f_i)$. By Theorem 5.1 there are subsets $\overline{F_i}\subseteq P_{m_i}$ with $\operatorname{card}(\overline{F_i})<m_i6^{2t_i}t_i^2$ such that the ideal of P generated by $f_i^{10m_{i+1}}$, $i=1,2,\dots$ is contained in the right ideal $\sum_{i=0}^{\infty}B^P_{m_{i+1}}(\overline{F_i})$. By applying mapping ξ we get that the ideal of $\operatorname{Im}(P)$ generated by $\xi(f_i)^{10m_{i+1}}=h_i^{20m_{i+1}}$ is contained in the right ideal $B^T_{m_{i+1}}(\xi(\overline{F_i}))$. Denote $D_i=\xi(\overline{F_i})\bigcup c\xi(\overline{F_i})$. Note that $\operatorname{Im}(P)+cK=T$. Therefore, the ideal D of T generated by $h_i^{20m_{i+1}}$ is contained in $B^T_{m_{i+1}}(D_i)$. Note that $D_i\subseteq F_i+cF_i$, for some set $F_i\subseteq A$ with $Fc\subseteq F$ and $\operatorname{card}(F_i)\le 8\operatorname{card}(D_i)\le 16\operatorname{card}(\overline{F_i})$. Therefore, $\operatorname{card}(F_i)<m_i^2 40^{-2}$. Observe that $F_i\subseteq H_{m_i}$, because $\overline{F_i}\subseteq P_{m_i}$. It follows that the ideal of T generated by $h_i^{20m_{i+1}}$ is contained in $\sum_{i=0}^{\infty}(B^A_{m_{i+1}}(F_i)+cB^A_{m_{i+1}}(F_i))$, as required.

Now we will prove the last part of Theorem 5.2. Fix n. Observe that the homogeneous components of $f_n^{20m_{n+1}}$ have degrees larger than m_{n+1}, hence $I\cap T_{m_n}\subseteq \sum_{i=1}^{n-1}(B^A_{m_{i+1}}(F_i)+cB^A_{m_{i+1}}(F_i))$. This finishes the proof. □

Let mappings $R_i\colon H_{m_i}\to H_{m_i}$ and $c_{R_i(F_i)}$ be defined as in Section 2 in [6] with $F_i=\{f_{i,1},\dots,f_{i,r_i}\}\subseteq H_{m_i}$ be as in Theorem 5.2. Recall that $c_{R_i(F_i)}\colon H_{m_i}\to H_{m_i}$ is a K-linear mapping with $\ker c_{R_i(F_i)}=\{R_i(f_{i,1}),\dots,R_i(f_{i,r_i})\}$. Given

$w = x_1 \dots x_{m_{i+1}} \in M_{m_{i+1}}$, $R_{i+1} \colon H_{m_{i+1}} \to H_{m_{i+1}}$ is a K-linear mapping such that

$$R_{i+1}(w) = c_{R_i(F_i)}(R_i(x_1 \dots x_{m_i})) \prod_{j=2}^{m_{i+1}m_i^{-1}} R_i(x_{(j-1)m_i+1} \dots x_{jm_i}).$$

Moreover, $R_1 = \mathrm{Id}$ and we assume that each m_i is divisible by 6.

Theorem 5.3 (Theorem 4, [6]). *Suppose that* $w \in H_{m_{l+1}} \cap \sum_{i=0}^{l} B^A_{m_{i+1}}(F_i)$. *Then* $R_{l+1}(w) = 0$.

Theorem 5.4 (Theorem 6, [6]). *Let* R_i *be as Theorem* 5.3, $m_{i+1} > m_i 2^{i+100}$, m_{i+1} *divisible by* m_i, m_i *divisible by* 6, *for* $i = 1, 2, \dots$. *For every integer* $i \geq 0$ *there are* s_i *divisible by* 6, $s_i \geq \frac{9}{10} m_i$, $\sigma_i \in S_{m_i}$, $\pi_i \in S_{m_i}$ *and a* K*-linear map* $h_i \colon H_{m_i - s_i} \to H_{m_i - s_i}$, *such that if* $u \in H_{s_i}$ *and* $v \in H_{m_i - s_i}$, *then*

$$R_i((uv)^{\sigma_i}))^{\pi_i} = u h_i(v).$$

Moreover we can assume that permutations σ *and* π *permute segments of length* 6 (*because the* m_i *are divisible by* 6).

6. The main result

Theorem 6.1. *Let* $m_1, m_2, \dots$ *be natural numbers such that for each* i, m_i *divides* m_{i+1}, 240 *divides all* m_i, $m_{i+1} > m_i 2^{i+100}$, *and* $m_1 > 10^8$. *Let* $i > 0$, $F_i = \{f_{i,1}, \dots, f_{i,r_i}\} \subseteq H_{m_i}$ *with* $r_i < 40^{-2} m_i^2$. *For every* $i > 0$ *there are* n_1, n_2, n_3 *such that* $n_1 + n_2 + n_3 = \frac{m_i}{6}$ *and* $R_i(z(n_1, n_2, n_3)) \notin \sum_{j > 2n_1 + 3n_2 + 4n_3} R_i(Z_j)$.

Proof. Suppose the contrary. Let i be the minimal number so that $R_i(z(n_1, n_2, n_3)) \in \sum_{j > 2n_1 + 3n_2 + 4n_3} R_i(Z_j)$ for all $n_1 + n_2 + n_3 = \frac{m_i}{6}$. Clearly $i > 1$, since $R_1 = \mathrm{Id}$. By the definition of R_i and by Lemma 4.2 we get that either

$$R_{i-1}(z(n_1, n_2, n_3)) \in \sum_{j > 2n_1 + 3n_2 + 4n_3} R_{i-1}(Z_j)$$

for all $n_1 + n_2 + n_3 = \frac{m_{i-1}}{6}$, or

$$c_{R_{i-1}(F_{i-1})}(R_{i-1}(z(n_1, n_2, n_3)) \in \sum_{j > 2n_1 + 3n_2 + 4n_3} c_{R_{i-1}(F_{i-1})}(R_{i-1}(Z_j))$$

for all $n_1 + n_2 + n_3 = \frac{m_{i-1}}{6}$. Since i was minimal the former is impossible. Thus suppose the latter holds. It follows that $c_{R_{i-1}(F_{i-1})} R_{i-1}(z(n_1, n_2, n_3) - t_{n_1, n_2, n_3}) =$

0 for some $t_{n_1,n_2,n_3} \in \sum_{j>2n_1+3n_2+4n_4} Z_j$. By the definition of the mapping $c_{R_{i-1}(F_{i-1})}$, we get $R_{i-1}(z(n_1,n_2,n_3) - t_{n_1,n_2,n_3}) \in \sum_{j=1}^{r_{i-1}} KR_{i-1}(f_{i-1,j})$. For $w \in M_{m_{i-1}}$ let $\overline{R}_{i-1}(w) = (R_{i-1}(w^{\sigma_{i-1}}))^{\pi_{i-1}}$.

Clearly $z(n_1,n_2,n_3)^{\sigma} = z(n_1,n_2,n_3)$, because σ permutes segments of length 6, so $\overline{R}_{i-1}(z(n_1,n_2,n_3)) = R_{i-1}(z(n_1,n_2,n_3))^{\pi_{i-1}}$. By Theorem 5.4, $r = s_{i-1}$, $p = m_{i-1} - s_{i-1}$, $f = h_{i-1}$ and $g = \overline{R}_{i-1}$ satisfy the assumptions of Lemma 4.3. It follows that $R_{i-1}(z(n_1,n_2,n_3)) \in \sum_{j>2n_1+3n_2+4n_4} R_{i-1}(Z_j)$ for all $n_1+n_2+n_3 = \frac{m_{i-1}}{6}$. A contradiction. $\square$

Proof of Theorem 1.1. The field K is countable so elements of T can be enumerated, say $h_1, h_2, \ldots$ where degree of h_i^2 is t_i. Let I be the smallest homogeneous ideal of T containing elements $h_i^{20m_{i+1}}$, $i = 1,2,\ldots$ where m_i, $i = 1,2,\ldots$ is an increasing sequence of natural numbers such that $m_i > 50^{18t_i}$, m_i divides m_{i+1}, 240 divides m_i. By Theorem 5.2 there exist subsets $F_i \subseteq H_{m_i} \subseteq A$, with $\mathrm{card}(F_i) < 40^{-2}m_i^2$ such that $I \subseteq F + cF$, where $F = \sum_{i=1}^{\infty} B^A_{m_{i+1}}(F_i)$. Let $B = A/I$ and let $\widetilde{E}$ be a subalgebra of B generated by a^3, b^2 and c. Then $\widetilde{E} = E/(I \cap E)$. Note that $\widetilde{E}$ is nil. We will prove that $\mathrm{gr}(\widetilde{E})$ is not nil. Suppose on the contrary that $\mathrm{gr}(\widetilde{E})$ is nil. Then the image of element $a^6 + b^6 + c$ is nil in $\mathrm{gr}(\widetilde{E})$. By Lemma 2.1 there is n such that $w(n_1,n_2,n_3) \in I + \sum_{i>2n_1+3n_2+n_3} E_i$ for all $n_1+n_2+n_3 > n$. Note that $E_i = \sum_{j\geq 0} T_j \cap E_i$ and that $I = \sum_{j\geq 0} T_j \cap I$. Note that $w(n_1,n_2,n_3) \in T_{6n_1+6n_2}$. Therefore, $w(n_1,n_2,n_3) \in I \cap T_{6n_1+6n_2} + \sum_{i>2n_1+3n_2+n_3} E_i \cap T_{6n_1+6n_2}$. Note that $m_n > 6n$. By Theorem 5.2, $w(n_1,n_2,n_3) \in F + cF + \sum_{i>2n_1+3n_2+n_3} E_i$ for all $n_1 + n_2 = \frac{m_n}{6}$ where $F = \sum_{i=1}^{n-1} B^A_{m_{i+1}}(F_i)$ (because $w(n_1,n_2,n_3) \in T_{m_n}$). Lemma 3.1 yields $z(n_1,n_2-n_3,n_3) \in \sum_{i=1}^{n-1} B^A_{m_{i+1}}(F_i) + \sum_{i>2n_1+3n_2+n_3} Z_i$. By applying the mapping R_n and by Theorem 5.3 we get $R_n(z(n_1,n_2-n_3,n_3)) \in \sum_{i>2n_1+3n_2+n_3} R_n(Z_i)$ for all $n_1+n_2 = \frac{m_n}{6}$. Hence, $R_n(z(q_1,q_2,q_3)) \in \sum_{i>2q_1+3q_2+4q_3} R_n(Z_i)$ for all $q_1+q_2+q_3 = m_n$. By Theorem 6.1 it is impossible. $\square$

7. Conclusions

By changing inequalities $i > 2n_1+3n_2+4n_3$ and similar inequalities to the opposite inequalities $i < 2n_1+3n_2+4n_3$ in the proof of Theorem 1.1 we get an example of a nil algebra $E/(I \cap E)$ with a filtered algebra $\alpha(E/(I\cap E)) = \bigoplus_{i=1}^{\infty} F^i/F^{i-1}$, where $F = Ka^3 + Kb^2 + Kc + (I \cap E)$, $F^0 = K + (I \cap E)$, generated in degree one and not nil (over arbitrary countable field). This contrasts a result of Regev [4] that filtered algebras of algebraic algebras over uncountable fields are algebraic.

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