Varieties of Mathematical Prose *

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Abstract

This article begins the development of a *taxonomy of mathematical prose*, describing the precise function and meaning of specific types of mathematical exposition. It further discusses the merits and demerits of a style of mathematical writing that labels each passage according to its function as described in the taxonomy.

Key words

Mathematical exposition, writing style, mathematical argument, formal reasoning, symbolic logic, definitions, proofs, terminology, hypertext.

1 Introduction

1.1 Rationale

Many students of mathematics are not experienced in reading mathematics texts. They may not understand the *nature and use of definitions*. Even if they do, they may not easily distinguish between a definition and an informal discussion of a topic. They may not pick up on the use of a word such as "group" that has a meaning in ordinary discourse but that has been given a special *technical* meaning in their text. They may not distinguish a plausibility argument from a careful proof, and in reading a careful proof they may not grasp the significance of the words and phrases the author uses to communicate the logical structure of the proof.

*Any reference to this paper should say "**PRIMUS** vol. 8, pages 116–136 (1998)". PRIMUS stands for "Problems, Resources and Issues in Mathematics Undergraduate Studies" and its home page is at `http://www.dean.usma.edu/math/resource/pubs/primus/index.htm`.

A thorough investigation of the distinctions and variations in usage recorded in a preliminary way in this paper could be useful to the mathematical community in several ways.

1. We expect that authors aware of the distinctions such as those we have made will produce mathematical exposition that is easier to understand because the status of various parts of the text and the shape of the mathematical argument will be clearly indicated.

2. Students who are aware of the possibilities will, we expect, have an easier time identifying the intent of each part of an exposition. This should be helpful to students in the same way in which an explanation of some aspects of the grammar of a foreign language is helpful to the student of a foreign language.

3. We further speculate that an author's detailed awareness of the status of different parts of a mathematical text should make it possible to produce more effective mathematical exposition in hypertext by allowing consistent treatment of pieces of text with similar status. Indeed, it was the consideration of some of the problems involved in turning a text [42] into hypertext that made it apparent that the distinctions we make here are desirable. (Hypertext is described briefly in Section 4.)

1.2 About this paper

In Section 2, we provide a partial taxonomy of mathematical writing at the paragraph or subsection level. This is intended as a first step toward a more complete classification. We also comment on how some of the concepts we have developed relate to teaching.

Section 3 describes two mathematical writing styles that we call the Narrative Style and the Labeled Style. We argue that some version of the Labeled Style, which makes use of the classification in Section 2 to label each section explicitly, is particularly appropriate for undergraduate texts. However, a knowledge of the classification of kinds of text could be useful for the author of any mathematical text.

Section 4 provides a brief discussion of hypertext.

Many ideas of this paper are treated further in [43].

1.2.1 Caveat Although we expect our efforts to be useful to teachers and writers of mathematics, we are not claiming that the lack of understanding of the structure of mathematical prose is the only stumbling block, or even the most important one, for students trying to learn mathematics. Our hope is to enhance the clarity of mathematical prose and to increase students' understanding of the kinds of prose that occur, thereby helping at both ends with the communication of mathematics.

2 A classification of mathematical prose

A classification of the major modes of mathematical exposition is outlined in this section. We must first establish some terminology.

2.1 Types of mathematical discourse

Part of mathematical discourse is written in a special variety of English or some other natural language that we call the **mathematical register**. A piece of text in a natural language, possibly containing embedded symbolic expressions, is in the mathematical register if it communicates mathematical reasoning and facts directly. The presumption is that discourse in the mathematical register could be translated into a sequence of statements in a formal logical system such as first order logic. The mathematical register is the register in which definitions, theorems and proof steps are written.

Other kinds of mathematical writing describe the history of a concept, how to think about the concept, physical examples, and so on. They are in some kind of general scientific or academic register that could be analyzed further, but they are not in the mathematical register even though they are concerned with mathematics. These are discussed in Section 2.3 below.

The distinction between the mathematical register and other kinds of writing that occur in mathematics texts is certainly not precise. We are after all discussing an aspect of natural language and so cannot expect to give the mathematical register a mathematical definition. Nevertheless, we believe that most mathematical writing can be easily seen to contain segments that are clearly intended to communicate mathematical facts and reasoning, and are thus in the mathematical register, and other segments that are clearly not in that register.

Lanham [22, Chapter 6] compares college students taking courses in more than one department to anthropologists who must come to understand three or four cultures simultaneously while having only occasional fifty-minute contacts with each of them. One aspect of mathematical culture is that mathematicians in speaking and writing pass in and out of the mathematical register freely and without comment. Like many aspects of any culture, this practice is commonly quite unconscious. It is a serious challenge to many students to determine whether a statement is or is not in the mathematical register, a challenge made more difficult by the fact that neither teacher nor student normally makes the distinction explicit. A clear articulation of the concept of the mathematical register (whatever name one uses) should help.

2.1.1 On terminology A "register" of a natural language is a special form of the language used for certain purposes. A good brief introduction to the idea is that of Halliday in [13, pages 86ff]. A register may use special words, special grammatical constructions (such as "thou" in a religious register), and words and grammatical constructions with meanings different from those in other registers, such as "definition" and "if. . . then" in the mathematical register (see 2.2.1 below). Many of these special uses are surveyed in [2], with references to the literature.

2.1.2 References Steenrod [33, page 1] called the mathematical register the "formal structure". That is the earliest reference to it that we know of. Note that word "formal" also refers to a non-colloquial *style* of writing. Writing in the mathematical register can be either formal or informal in style.

Other writers who have discussed the mathematical register include DeBruijn [6] (the "mathematical vernacular" – his work contains a partial taxonomy) and F. Schweiger [31, 32]. Laborde [20] discusses mathematics and language in connection with younger students, and some of her comments amount to a partial taxonomy. Halliday and Martin [13] study the more general "scientific register" in depth, but make little mention specifically of mathematics.

2.2 Kinds of discourse in the mathematical register

We describe briefly the main kinds of discourse in the mathematical register. As in any taxonomy, this inevitably involves occasionally stating the obvious (for example in discussing what a definition or a theorem is). However, it is important in a taxonomy to state as exactly as possible what each descriptive term means: we must *define* "definition", "theorem" and so on. The definitions we give are necessarily of the sort a dictionary would give; they do not constitute mathematical definitions. (As the reader may check, the definitions of these words actually given by dictionaries are quite unsatisfactory as descriptions of their use in mathematics.) Giving a precise description of something "everyone knows" is a valuable exercise which often uncovers ambiguities and misunderstandings and provides new insights.

2.2.1 Definitions A **definition** prescribes the meaning of a word or phrase in terms of other words or phrases that have previously been defined or whose meanings are assumed known. Note that definitions are distinct in subtle ways from other kinds of discourse in the mathematical register. For example, "if" in a definition can mean "if and only if". This sort of thing is discussed in [2] and in [19, pp. 71–72].

2.2.2 Specifications A textbook, particularly at the elementary level, sometimes must describe a type of object for which the author might consider it pedagogically unwise to provide an explicit definition. An explicit definition might be pedagogically unwise because of its technical difficulty (for example defining sets as elements of a model of Zermelo-Fränkel set theory) or because it is unrelated to the way in which we usually think of the object (for example defining an ordered pair $\langle a, b\rangle$ as $\{a, b, \{a\}\}$.)

A **specification** is a description of a type of mathematical object that gives salient features of such objects, but which may fall short of giving a list of properties that completely determine the type of object. The word "specification" is not usually used in this context (it was introduced in [42]), but authors of texts for undergraduate courses often give specifications without calling them specifications.

A specification may be accompanied by some choice of notation and some explanation of how the notation serves to denote the relevant properties of the object being specified. For example, one could describe an ordered pair of elements of a set S as follows:

> "An ordered pair of elements of S is a mathematical object that determines a **first coordinate** and a **second coordinate**, each of which is an element of S. If the first coordinate of an ordered pair is a and the second coordinate is b, the ordered pair is denoted by $\langle a, b\rangle$. For any two (not necessarily distinct) elements a and b of S, there is a unique ordered pair with first coordinate a and second coordinate b."

Rosen [29, pages 38–39] refers to an ordered n-tuple as a collection, but then goes on to give salient properties of an ordered triple in terms similar to the specification of ordered pair just presented. Another example is the way in which Herstein [15, page 114] uses what might be called a specification in that he establishes a notation and specifies a *canonical* form for an arbitrary member of $R[x]$ (without proof). The article [41] urges the use of specification. Other examples of specifications of mathematical objects may be found there and in [42].

2.2.3 Theorems A theorem is a mathematical statement that has been proved. Calling it a theorem indicates also that it is considered important. Some authors distinguish between theorems and propositions, the former being regarded as more important. Lemmas and corollaries are called by those names to indicate that they are *logically* subsidiary to (as opposed to being less important than) some theorem. Here usage is varied and many theorems that are called lemmas (for example, Schanuel's Lemma, König's Lemma and Kuratowski's Lemma) are not in fact a step in the proof of one theorem but instead are quite generally useful.

Students *need to be told* that theorems, propositions, lemmas, and corollaries are all assertions that have been proved and that the different names might be intended by the author to suggest the function or the degree of importance of the assertion.

2.2.4 Proofs A **proof** of a mathematical statement is an argument that shows that the statement is correct. It is important to distinguish between a proof of a theorem in this sense and a "proof" as studied in a text on mathematical logic. The latter typically a list or tree of statements in a formal system, each of which can be deduced from preceding ones by some explicit rule of the system. (See for example [23], [37] and [10].) A proof in this sense is a *mathematical object* about which theorems may be proved! Unfortunately the phrase "formal proof" is used both to mean a proof as mathematical object and also to mean a proof in our sense that is written in a particularly careful and systematic manner. We shall use the word "proof" only in the sense we have given.

A fine classification of proofs must await further analysis; we merely mention a preliminary gross classification here:

1. Some proofs in our sense are more or less directly translatable sentence by sentence into a proof in a formal system. Example:

   > "If $S = T$ then, by definition, S and T have precisely the same elements. In particular, this means that $x \in S$ implies that $x \in T$ and also $x \in T$ implies that $x \in S$. That is, $S \subset T$ and $T \subset S$."

   from [18, p. 40].

2. Some proofs might be described as recipes for constructing proofs in the mathematical register, for example:

> "Lagrange's identity can be established by "multiplying out" the right sides of (3.16) and (3.17) and verifying their equality."

(from [1, page 112].) It is not clear that anyone has ever produced a version of formal logic that included a programming language for writing proofs or transforming proofs into other proofs, but the lack of such a system does not invalidate the requirement that the proof "could in principle be translated into some version of formal logic".

3. Some proofs are merely references, such as

> "**Proof**. See Dunford and Schwartz, volume III, page 495."

The individual sentences in a proof can also be classified further. The following is a partial list. Again, these distinctions, if signaled appropriately, could help inexperienced readers.

1. A proof will contain mathematical statements that follow from previous statements. We call these **proof steps**. They are in the mathematical register, like theorems, but unlike theorems one must often deduce from the context the hypotheses that make them true.

2. **Restatements**. These state what must be proved, or, part way through a proof, what is left to be proved or what has just been proved.

3. **Pointers**. These give the location of pieces of the proof that are out of order, either elsewhere in the current proof or elsewhere in the text or in another text. References to another text are commonly called **citations**.

2.3 Other types of mathematical prose

Mathematical discourse not in the mathematical register falls roughly into two categories: presentations of mathematics not in the mathematical register, although they may contain chunks of prose in the mathematical register within them, and miscellaneous discussions that do not directly present mathematical facts. These are considered in this section and the next. Some categories overlap with others; the same piece of text can be described as being of two kinds simultaneously. For instance, fine points (2.3.3) can also be examples.

2.3.1 Examples We outline here a preliminary taxonomy of the types of examples that occur in mathematical writing. Although it is customary to label pieces of mathematical prose as theorems, proofs, corollaries and the like, it is unusual to distinguish examples explicitly by the various purposes that they serve. It appears to us that inexperienced readers would benefit by having the type of example labeled.

1. An **easy example** is an example that can be immediately verified to be an example with the information already provided, or that is already familiar to the reader. Easy examples are often given immediately before or after a definition. Thus an introduction to group theory may introduce the integers on addition or the cyclic group of order two, describing it both in terms of modular arithmetic and as symmetries of an isosceles triangle.

2. If an easy example is given *before* the definition, we call it a **motivating example**. We have noted a sharp difference of opinion among both students and writers as to whether it is appropriate to have any motivating example at all. Those in favor of having them usually argue that having the example in mind gives the reader something to which he or she may relate various facets of the general, abstract definition as it is being presented. Those opposed usually say that one can't tell from the example how it *is* an example (that is, which aspects are salient) until the definition is known. (However, in giving the example, one may point out the salient features and mention explicitly that the definition following is intended to capture those features more precisely and generally.)

3. A **delimiting example** is an example with the smallest possible number of elements or an example with degenerate structure. The trivial group is a delimiting example of a group, and the discrete and indiscrete topological spaces are delimiting topological spaces.

   It is always worthwhile to try out a purported theorem on delimiting examples. Also, beginning readers appear to have trouble applying definitions to the empty set or singletons. Consider such questions as these:

   (a) What is the value of 0^0?

   (b) Does the 0-dimensional linear space have a basis?

(c) Can the empty set carry a group structure? A semigroup structure?

(d) What is the determinant of the empty matrix?

Consideration of such questions assist students in developing a facility with logical manipulation and need not be time-consuming.

4. A **deceptive non-example** is a structure that the reader might mistakenly believe to be an example that actually is not. In the case of groups the set of nonnegative integers with addition as operation is worth mentioning because it reinforces the idea that one has to pay close attention to details when verifying whether some object satisfies a definition. Many students need a lot of time to get used to the idea that the degree of precision in mathematics far exceeds that in day-to-day discourse.

   Some books list conclusions that readers might inadvertently jump to as pitfalls. Bourbaki used the dangerous bend sign Z to indicate such pitfalls.

5. An **elucidating example** is one that serves either as a counterexample to unconscious assumptions that a student is likely to make or as an example that is discussed in detail to clarify or elucidate the various facets of the definition. One may repeat the motivating examples here to tie them in. For instance, the group of symmetries of a square is not commutative (counterexample to unconscious assumption). The group of invertible 2×2 matrices illustrates that verifying associativity need not be trivial.

6. A concept that overlaps with the kinds of examples listed above is the notion of a set of **inventory examples**, a short list of examples, not too general and not too special, to keep in mind when considering the assertions made in the development of the theory. In the early stages of group theory, for example, one might use $\mathbb{Z}$, the nonzero rationals with multiplication as operation, the Klein four-group, the symmetric group on three letters, and perhaps the group of all bijections from some infinite set to itself.

2.3.2 Applications An application of mathematics involves the recognition that some aspects of a phenomenon in some field of study (not necessarily mathematics) can be seen to be an instance of some mathematical structure, and that known mathematical facts about the structure yield insights into that phenomenon. For example, crystallographic groups can be used to describe certain physical properties of crystals. The classification of such groups put a severe restriction on the sorts of crystals that one may encounter in nature.

A paragraph that *mentions* applications is a distinct kind of discourse discussed in Section 2.4.4. A presentation of an application of a concept, on the other hand, is a complete piece of exposition in its own right. Such an exposition may have definitions, perhaps some theorems, and an explicit discussion of the application.

2.3.3 Fine points and warnings A text may contain small sections labeled "fine points" or "warnings" that mention exceptional cases, warn against possible unwarranted assumptions, or describe other usage that may cause confusion. These fall under other kinds of mathematical discourse discussed here, but in a hypertext document such things would perhaps be better labeled as warnings or fine points to give guidance to the user.

2.4 Prose that does not directly present mathematics

The kinds of mathematical discourse discussed in this section do not present mathematical concepts as such, but rather provide motivation, ways of thinking about concepts, and so on. Steenrod [33, page 9] provides a similar discussion.

2.4.1 Mental representations A textbook often tries to give an explanation of a useful way of thinking about a concept. This might be a rough description, ignoring subtleties, intended as a first approximation. Such a thing is in many cases an *explication of the author's mental representation* of the concept and may use pictures. Authors often begin such explanations with words such as "intuitively", although that word is seriously misleading since the statements following the word "intuitively" may not be at all intuitive to a particular reader. It would be better to say something such as: "A way of thinking about this concept that many mathematicians have found useful is. . . ", since different mathematicians, not to mention different students, have very different mental representations of concepts.

Mathematicians exhibit different attitudes toward the use of mental representations in teaching. Some prefer to avoid discussing mental representations, either because they feel students should develop their own mental representations or because they want to emphasize the primacy of explicit and precise definitions, statements of theorems, and proofs. Such things are public and checkable, whereas some forms of mental imagery may have private features that are difficult to communicate.

Many mathematicians hold, however, that the more distinct mental representations one has and the more facility one has in going back and forth among these, the better one's understanding. One can hardly deny that certain facts become particularly transparent in certain representations. Also, different representations have different associations and this can help one's grasp of a concept.

Many articles in the book [34] discuss mental representation (under various names, often "concept image") in depth, particularly [35], [9] and [14]. See also [8, page 163], [16], [36], [41].

Mental imagery is discussed from a philosophical point of view, with many references to the literature, by Dennett [7, Chapter 10], Chapter 10. Lakoff [21] is concerned with concepts in general, with more of a linguistic emphasis. One of Lakoff's points is that human categories are not generally defined as the set of all things that have certain properties (mathematical concepts are *all* like that), but (to vastly oversimplify) typically have *prototypical members*, and one reasons about the category primarily by considering the prototypes. (See also [28].) Halliday and Martin [13, Chapter 8] is concerned with scientific concepts. It is notable that the only mention in that text of a concept that is like a mathematical concept in being an intersection of sets with defining predicates is in one paragraph on page 152. All the other types of concepts discussed do not fit the mathematical model at all.

2.4.2 Usage notes Usage notes describe variations in terminology and symbols in the literature. Students are often confused by such variations, particularly since often students don't realize that there *are* variations.

This phenomonenon often involves some very common notation. Examples include the following:

1. For some authors the natural numbers start at 0 and for others they start at 1.

2. For some authors the statement "r is positive" (for a real number r) means $r > 0$; for others it means $r \geq 0$. (The former is apparently nearly universal in American secondary schools, but the latter is used in many European systems. It dates back at least to Bourbaki [5, pp. 4–5].)

3. For some authors the expression $A \subset B$ means A is a subset of B (specifically allowing $A = B$); for others it means A is included in *but not equal* to B (strictly included in B). It appears that the former usage is more common in upper-level and graduate texts and the latter is more common in undergraduate texts. This is discussed further in [2].

Recommendations We urge that authors tell their readers about all the variations in terminology that they know of. Authors may harbour strong prejudice about what constitutes correct terminology, and discuss pros and cons of other extant usage, but, to suppress alternative usage appears arrogant to us. Moreover, to us it seems particularly irresponsible for a *textbook* not to mention common variations in usage.

We recommend that an author *not use an old word or symbol with a new meaning.* It is too late to suppress the variation in the meaning of "positive" or the inclusion symbol, but we hope that in the future authors who wish to revise the terminology or symbolism would introduce entirely new words or symbols instead of changing the meaning of an old one.

2.4.3 Historical discussions Historical discussions occur principally in two ways. Many authors of undergraduate texts include short biographies of mathematicians who invented the concepts and results being discussed. Such ideas fascinate some but bore others. At any rate, this is a mode of writing that appears in mathematical texts; we will call it **history**.

2.4.4 Pointers Pointers are statements that tell the reader where to go for further information, either elsewhere in the same document or in some other document. One of the great advantages of hypertext is that it allows the reader efficient access to the text being pointed to. In fact, text that points to more information in the same body of writing will essentially disappear in hypertext, to be replaced by buttons or words to click on.

3 Mathematical writing styles

The second author has taught discrete mathematics from the set [42] of classnotes for about eight years. These notes are written using TeX, which has made it possible to revise them each time the course was taught. The original desire was to make the notes easy to read, and so an effort was made to write in a flowing, persuasive manner that carried the reader along. Over the years, the author came to realize that students had difficulty disentangling the purposes of the different parts of the text. As a result, the text has been gradually broken down into finer and finer bits, each labeled according to its nature: definition, theorem, discussion, fine points, warnings about possible misunderstanding, discussion of usage of words, example, applications, and so on.

The result is that the notes have evolved from flowing mathematical prose (an extreme style that we will call the **Narrative Style**) into another extreme, a sort of engineering manual style with nearly every paragraph labeled and numbered — that we will call the **Labeled Style**.

Let us agree to say that a book is in the Narrative Style if it contains sections that typically are a page or more in length, with little in the way of set-off statements other than displayed equations and perhaps theorems. In contrast, we consider a book in which nearly every paragraph is numbered and perhaps carries a caption in a way that indicates its purpose to be in Labeled Style. The word "labeled" here refers to the labeling of the paragraph not (or not only) by its subject matter, but by its *intent* — example, remark, definition, and so on.

Two well-written books that illustrate the two styles are those by Graham, Knuth, and Patashnik [12] (Narrative Style) and by Barendregt [3] (Labeled Style). The book by Barendregt is formal in style and divided into sections with clear labels such as "Examples", "Theorem", "Remark" and so on that give the intent of the section. The Graham-Knuth-Patashnik book is written in clear, flowing prose with very few subdivisions. Some definitions and theorems are so labeled but others are stated without special indication.

3.1 Examples of the styles

In this section, we illustrate the two styles by two presentations of integer division, written specifically for this article, although they are based on [42]. An attempt has been made to keep the styles of the two examples at the same (colloquial/informal) register.

3.1.1 An example of the Narrative Style

1. Division

An integer n is said to **divide** an integer m if there is an integer q for which $m = qn$. For example, $3 \mid 6$, since $6 = 2 \times 3$. The integer n is called a **factor** of m if n divides m. The symbol for "divides" is a vertical line: $n \mid m$ means n divides m, or n is a factor of m.

Don't confuse the vertical line "|", a *verb* meaning "divides", with the slanting line "/" used in fractions. The expression "$3 \mid 6$" is a sentence, but the expression "6/3" is the name of a number, and does not form a complete sentence by itself.

Because $6 = (-2) \times (-3)$, it follows that $-3 \mid 6$. On the other hand, it is not true that $4 \mid 14$ since there is no *integer* q for which $14 = 4q$. There is of course a *fraction* $q = 14/4$ for which $14 = 4q$, but $14/4$ is not an integer.

Because $0 = 0 \times 0$, 0 divides itself. However, 0 divides no other integer, since if $m \neq 0$, then there is no integer q for which $m = q \times 0$.

Other familiar properties of integers are defined in terms of division. For example, an integer n is **even** if $2 \mid n$.

2. Properties of Division

Some properties of division are listed in the following theorem. Part (b) implies that 0 is even.

Theorem *a. Every integer is a factor of itself.*

b. Every integer is a factor of 0.

c. 1 *is a factor of every integer.*

To prove (a), note that if m is any integer, then $m = 1 \times m$, so by Definition 1.1, m divides itself. Similarly the fact that $m = m \times 1$ means that (c) is true.

Part (b) may be surprising. Here is why that statement is true: for any integer m, $0 = m \times 0$, so by Definition 1.1, m is a factor of 0. □

3.1.2 An example of the Labeled Style In this presentation of the Labeled Style, we have followed the suggestion by Steenrod [33] that the section numbers be outdented. He said that doing that was typographically difficult, but these days it is a trivial change to the TEX header file.

1. Division

1.1 Definition An integer n **divides** an integer m if there is an integer q for which $m = qn$.

Related terminology The integer n is called a **factor** of the integer m if n divides m.

Notation The symbol for "divides" is a vertical line: $n \mid m$ means n divides m, or n is a factor of m.

1.2 Example $3 \mid 6$, since $6 = 2 \times 3$.

Warning Don't confuse the vertical line "|", a *verb* meaning "divides", with the slanting line "/" used in fractions. The expression "$3 \mid 6$" is a sentence, but the expression "$6/3$" is the name of a number, and does not form a complete sentence by itself.

1.3 Example Because $6 = (-2) \times (-3)$, it follows that $-3 \mid 6$. On the other hand, it is not true that $4 \mid 14$ since there is no *integer* q for which $14 = 4q$. There is of course a *fraction* $q = 14/4$ for which $14 = 4q$, but $14/4$ is not an integer.

1.4 Example Because $0 = 0 \times 0$, 0 divides itself. However, 0 divides no other integer, since if $m \neq 0$, then there is no integer q for which $m = q \times 0$.

1.5 Definition An integer n is **even** if $2 \mid n$.

2. Properties of division

Some properties of division are listed in the following theorem. Part (b) implies that 0 is even.

2.1 Theorem *a. Every integer is a factor of itself.*

b. Every integer is a factor of 0*.*

c. 1 *is a factor of every integer.*

Proof If m is any integer, then $m = 1 \times m$, so by Definition 1.1, m divides itself. Hence (a) is true. Similarly the fact that $m = m \times 1$ means that (c) is true.

Part (b) may be surprising. Here is why that statement is true: for any integer m, $0 = m \times 0$, so by Definition 1.1, m is a factor of 0. □

3.1.3 Paragraph numbering The Labeled Style could incorporate an even more extreme form of numbering in which every paragraph has a separate number. We observe a decided split of opinion among expositors concerning the question: Should everything be numbered or only the important things? The argument for numbering everything in undergraduate texts is that it allows ease of reference. The argument against is that numbering serves to indicate importance, and this device fails, if everything is numbered.

3.2 Discussion of the two styles

Perhaps most mathematicians prefer the Narrative Style. Steenrod [33, page 2] advocated labeling the informal parts of the text, but it is not clear whether he meant to use formal labels as in the example of Labeled Style above, or that the lead sentence should indicate clearly the point of the paragraph or the subsection.

However, the Labeled Style, we think, has the advantages of uniformity and clarity of structure that makes it much more accessible to the reader. This is one of the statements in this article that lend themselves to being investigated by methods used in educational research, for instance by running pilot programs with control groups.

Let us consider definitions in more detail. As mentioned before, novices often do not readily appreciate abstraction. Thus the idea that, in simple cases, one can prove a theorem by writing out the definitions of the terms involved is new to many students.

Clear awareness of the significance of definitions and their use, we suggest, helps avert several problems that surface only later. The instructor could therefore engage in a two-part process with the students to help them understand the idea of definition:

1. The instructor can explain its constitution-like nature: everything to be said about the concept being defined must ultimately refer back to the definition. The definition is *the source of all truth* about the object defined.

2. *Then* the instructor must spend some time training the students to pick definitions out of texts such as Graham-Knuth-Patashnik [12].

In the example of the Narrative Style in section 3.1.1, the first definition is signaled by the phrase "is said to". Such usage is, however, not uniform and although skilled readers have no trouble at all with this sort of thing, we have encountered bright students who are quite inept at catching the significance of various statements.

If a text has definitions that are clearly, uniformly and explicitly labeled, the instructor has one fewer task. Once a concept such as "definition" has been explained, the students can take advantage of the fact that they are displayed as such. The Labeled Style thus saves one from having to teach the students the many ways in which that authors writing in the Narrative Style indicate their definitions.

The argument on the other side is that if mathematicians do not teach students the ways in which definitions and other types of discourse in the mathematical register are signaled in the Narrative Style, who will? And if we are going to teach it, why not use texts written in the Narrative Style as examples?

Some students, undoubtedly, absorb this information without being taught, and in fact may often not be conscious of this knowledge, and have much more skill than others at picking up clues to the logical structure of mathematical prose than others do. The labeled style can help the ones who are not so skilled and can hardly hurt the skilled ones. Of course, this variation in students' reading skills is a big factor in their performance in courses other than mathematics, as well.

3.2.1 Caveat We have argued that using the labeled style will make it easier for students to pick out the intent of each passage in a mathematical text and as such will be useful for students beginning to study abstract mathematics. We emphasize, however, that persuading the students that the definitions must be taken literally and must be used in proofs is *much more difficult* than teaching them to recognize a definition when they see it. However, it is not necessary that every writer on the problems of teaching write only about the most difficult problems, any more than it is necessary for every mathematician to work on the Riemann Hypothesis!

Discussion of the problems with definitions in teaching may be found in [27] and [39].

4 Hypertext

Hypertext was invented to facilitate navigation through a collection of interrelated topics. The rough idea is that the text is read on a computer screen, and at any point in the exposition, the reader can perform some action (for example, clicking on a word) to obtain more information about some topic, to see illustrations, or to explore related topics.

The form of a hypertext interface is a subject of current research. Consider the difficulties that occur when, for example, the reader follows a trail of concepts and wants to return to some specific spot in the trail [25]. Consider what happens if a student wants to ask a question about a specific topic: one needs an unambiguous way to refer to each piece of the text. A survey of this topic is the book by Nielsen [26] (if you were reading this in a hypertext system, you could click on this to get the full reference) which contains an excellent annotated bibliography.

4.1 Modes of discourse in hypertext

A suitable hypertext document should allow a reader to select further text or illustrations not only with control over the *concept* being presented, but also with control over the *kind of presentation* of the topic. We imagine the reader clicking on the word "group", for example, and being given a choice of more examples, discussion of the motivation of the concept, references for further study, and so on. We hope that the analysis of the conceptual differences between different kinds of mathematical prose given in Section 2 will provide authors of hypertext documents an understanding of the sorts of choices it is reasonable for them to offer the reader, thus contributing to a perspicacious navigation system for the document.

The classification given here is probably too fine to be used in hypertext: the reader can be confused by too many distinctions concerning what can be read next. Icons or words describing the modes of discourse must be carefully chosen. Ultimately, the best organization of hypertext will have to be determined by observing its use.

Finally, we mention that many mathematicians object to a style of writing such as the Labeled Style in which the text is subdivided finely into three or four levels of numbered sections; often in discussions the objection is in part to the numbers themselves. However, in hypertext, the numbers (and even subtitles, although that might be unwise for other reasons) can be omitted. A cross reference may involve merely an underlined word (although it *could* be a number) one clicks on to jump to the section being referred to.

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