566 LECTURE NOTES IN ECONOMICS AND MATHEMATICAL SYSTEMS

Michael Genser

A Structural Framework for the Pricing of Corporate Securities

Economic and Empirical Issues

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To my wife Astrid

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Bergen, August 2005 Michael Genser

Contents

List of Symbols

$\mathbf{1}_{\mathcal{A}}$	Indicator function: Assumes 1 if the condition $\mathcal{A}$ is true, else 0
α	Fraction of the value of the bankrupt firm V_B which is lost in bankruptcy
$\alpha(V_B)$	Bankruptcy loss function
α_1, α_1	Parameters of the bankruptcy loss function
A_1, A_2	Constants of the general solution of the ordinary differential equation of claim F if the state variable follows an arithmetic Brownian motion
$\bar{A}_1, \bar{A}_2$	Constants of the general solution of the ordinary differential equation of claim F if the state variable follows a geometric Brownian motion
$\mathcal{A}_i$	All sets of $\mathcal{N}$ with i integers
B_{C_j,T_j}	Market value of a government bond issue with maturity T_j and continuous coupon C_j after taxation
BC	Current value of the bankruptcy costs
BL	Current value of the bankruptcy losses
$\mathcal{B}_n$	Set of all joint events $\mathcal{B}_n$ with $i = \{1, \ldots, n\}$
C_j	Continuous coupon of bond j
C_j^*	Optimal continuous coupon of bond j
CF	Regular payments of a claim F
δ_L	Payout function to all claimants of the levered firm value
δ_U	Payout function to all claimants of the unlevered firm value
d_n	Vector of observed debt prices in period n

D_{C_j,T_j}	Market value of the j-th corporate debt issue with maturity T_j and continuous coupon C_j after taxation
$D^+_{C_j,T_j}$	Market value of the j-th corporate debt issue with maturity T_j and continuous coupon C_j after taxation as long as the firm remains solvent
$D^-_{C_j,T_j}$	Market value of the j-th corporate debt issue with maturity T_j and continuous coupon C_j after taxation in the case of bankruptcy
$D^E_{C_j,T_j}$	Market value of the j-th corporate debt issue with maturity T_j and continuous coupon C_j to equity owners
$DO(T, \eta, \eta_B)$	Down and out option value of EBIT with starting EBIT η, barrier η_B, and maturity T
$dz^{\mathcal{P}}$, $dz^{\mathcal{Q}}$	$\mathcal{P}$-, $\mathcal{Q}$-Brownian motion
ε	Fractional loss of tax recovery in Tax Regime 3
$\varepsilon_{d,n}$	Vector of debt price observation errors in period n
$\varepsilon_{e,n}$	Equity price observation error in period n
ε_n	Vector of observation errors in period n: $\varepsilon_n = (\varepsilon_{e,n}, \varepsilon'_{d,n})'$
$\eta, \bar{\eta}$	Earnings before interest and taxes (EBIT) following an arithmetic or geometric Brownian motion
$\hat{\eta}_n$	A-posteriori update of EBIT in the Kalman filter recursion in period n
$\bar{\eta}_{n\|n-1}$	A-priori update of EBIT in the Kalman filter recursion in period n given the estimate of period $n-1$
$\eta_B(T_j), \bar{\eta}_B(T_j)$	EBIT at which the firm declares bankruptcy in the time interval $]T_{j-1}, T_j]$
$\eta^*_B, \bar{\eta}^*_B$	EBIT at which the equity owners optimally declare bankruptcy
$\eta_{\max,t}$	Maximum EBIT earned from investments
e_n	Observed equity price in period n
E	Market value of equity to investors after taxation
E^+	Market value of equity to investors after taxation as long as the firm remains solvent
E^-	Market value of equity to investors after taxation in the case of bankruptcy

$E_{n\|n-1}$	Expectation operator of period n given information in period $n-1$
$E_t^{\mathcal{Q}}$, $E_t^{\mathcal{P}}$	Expectation operator under the probability measure $\mathcal{Q}$ or $\mathcal{P}$ as of time t
$\mathcal{E}$	Set of all even integers in $\mathcal{N}$
$f(e_n\|e_{n-1})$	Conditional density of e_n given e_{n-1}
$f(\eta_t)$	Payout function of an EBIT claim
$F(\eta_t, T)$	Market value of a claim contingent on the firm's EBIT with maturity T
$g(\eta_t, \eta_B(t))$	Function that transforms the time varying barrier $\eta_B(t)$ with respect to η_t into a constant barrier with respect to $g(\cdot)$
G	Market value of tax payments collected by the government
G^+	Market value of tax payments collected by the government as long as the firm remains solvent
G^-	Market value of tax payments collected by the government in the case of bankruptcy
G_η	Gradient of the model security prices with respect to the state variable η
h	Spacing parameter in the Stirling approximation
$\hat{H}$	Information matrix of the Kalman filter
$\hat{H}_{E,n}$	Asymptotic variance of the equity price estimation error
$\hat{H}_{D,n}$	Asymptotic covariance matrix of the bond price estimation errors
I	Identity matrix
I_t	Invested capital as of time t
k	Proportional debt issuing costs
k_1, k_2	Characteristic roots of the general solution of the ordinary differential equation of security prices in the ABM-Corporate Securities Framework
$\bar{k}_1$, $\bar{k}_2$	Characteristic roots of the general solution to security prices of the ordinary differential equation of security prices in the GBM-Corporate Securities Framework
$K(\cdot)$	Debt issuing cost function
K_n	Kalman gain matrix
λ	EBIT spacing parameter in the trinomial lattice approach

$L(\cdot)$	(Log) likelihood function
μ, $\bar{\mu}$	Drift function of EBIT under the risk-neutral measure $\mathcal{Q}$ if EBIT follows an arithmetic or geometric Brownian motion
μ_η, $\bar{\mu}_\eta$	Drift function of EBIT under the physical measure $\mathcal{P}$ if EBIT follows an arithmetic or geometric Brownian motion
μ_{BL}	Drift function of bankruptcy losses BL
μ_L, μ_U	Drift function of the levered firm value V_L and the unlevered firm value V_U
μ_{TAD}	Drift function of tax advantage to debt claim TAD
M_{t_i}	Process of the running maximum of the process X in the subperiod $]t_{i-1}, t_i]$
ν	Drift of the stochastic process X
$\tilde{\nu}$	Drift of stochastic process X with the bankruptcy claim as numéraire
$n(\mu, \sigma^2)$	Univariate normal density function with expected value μ and standard deviation σ
$n_n(\mu, \Sigma)$	n-variate normal density function with a vector of expected values μ and the covariance matrix Σ
$N(\mu, \sigma^2)$	Univariate cumulative normal distribution function with expected value μ and standard deviation σ
$N_n(\mu, \Sigma)$	n-variate cumulative normal distribution function with a vector of expected values μ and the covariance matrix Σ
$\mathcal{N}$	Set of all integers $\{0, \ldots, n\}$
Ω	Time covariance matrix of a Brownian motion
$\phi(t_0, T, \eta_{t_0}, \eta_T, \eta_B(t))$	Joint probability of the process η starting at η_{t_0} of not hitting the barrier function $\eta_B(T)$ in the time interval $]t_0, T]$ and ending at $d\eta = \eta_T$
$\Phi(t_0, T, \eta_{t_0}, \eta_B(t))$	Probability of the process η starting at η_{t_0} of hitting the barrier function $\eta_B(T)$ in the time interval $]t_0, T]$
$\psi(t_0, s, \eta_{t_0}, \eta_B(t))$	First passage time density of the process η starting at η_{t_0} of hitting the barrier function $\eta_B(T)$ in the time interval $]t_0, T]$
p_j	Reflection counter
$P(\mathcal{A})$	Probability of the event $\mathcal{A}$

P_j	Principal of bond j
$p_B(t_0, T, \eta_{t_0}, \eta_B(t))$	Present value of a security paying one account unit if the process η starting at η_{t_0} reaches $\eta_B(T)$ for the first time in the time interval $]t_0, T]$
$\mathcal{P}$	Physical probability measure
PR	Par coupon of a government bond
$PV(\cdot)$	Present value of some payment function
PY	Par yield of a corporate bond
PYS	Par yield spread as the difference between corporate and government par yields
$\mathcal{Q}$	Risk-neutral martingale measure
r	Continuously compounded risk-free interest rate
R	Covariance matrix of the observation errors ε
$ROI_{\max}$	Maximum return on investment
σ_η, $\bar{\sigma}_\eta$	Volatility function of EBIT if EBIT follows an arithmetic or geometric Brownian motion
$\hat{\sigma}_E$	Estimated equity volatility
$\hat{\sigma}_V$	Estimated firm value volatility
σ_{IV}	Implied Black/Scholes volatility
σ_U	Volatility function of the unlevered firm value
$\bar{\Sigma}_\eta(n\|n-1)$	A-priori update of the variance of the state variable η in period n given the information in period $n-1$
$\hat{\Sigma}_\eta(n)$	A-posteriori update of the variance of the state variable η in period n
$\bar{\Sigma}_Y(n\|n-1)$	A-priori update of the variance of the observation estimates Y in period n given the information in period $n-1$
$\bar{\Sigma}_{Y\eta}(n\|n-1)$	A-priori update of the covariance of the observation estimates Y with the state variable η in period n given the information in period $n-1$
s	Point in time
S_j	Start date of the j-th corporate bond issue
τ	Bankruptcy stopping time
t_0	Point in time (today)
T	Maturity of a security
TAD	Tax advantage to debt
T_O	Option maturity
τ^c	Corporate tax rate
τ^d	Investor's personal tax rate on coupon payments
τ^e	Investor's personal tax rate on dividends

τ^{eff}	Equity investor's effective tax rate on dividends equalling $(1-\tau^e)(1-\tau^c)-1$
θ	Risk premium of the EBIT-process to change the physical measure $\mathcal{P}$ into the risk-neutral measure $\mathcal{Q}$
Θ	Parameter vector
$\hat{\Theta}_{ML}$	Maximum-likelihood estimate of the parameter vector Θ
$\hat{\Theta}_{SAE}$	Estimate of the parameter vector Θ using the sum of absolute errors as an objective function
$\hat{\Theta}_{SSQE}$	Estimate of the parameter vector Θ using the sum of squared errors as an objective function
$\mathcal{U}$	Set of all uneven integers in $\mathcal{N}$
v_n	Vector of security price estimation errors in period n
V	Value of the firm before taxation
V^+	Value of the solvent firm before taxation
V^-	Value of the bankrupt firm at the bankruptcy level η_B before taxation
$V_B(T_j)$	Value of the bankrupt firm before taxation in the subperiod $]T_{j-1}, T_j]$
V_{C_j,T_j}	Value of debt with maturity T_j and a continuous coupon C_j before taxation
$V^+_{C_j,T_j}$	Value of debt with maturity T_j and a continuous coupon C_j before taxation as long as the firm is solvent
$V^-_{C_j,T_j}$	Value of debt with maturity T_j and a continuous coupon C_j before taxation in the case of bankruptcy
V_E	Value of equity before taxation
V_E^+	Value of equity before taxation as long as the firm is solvent
V_E^-	Value of equity before taxation in the case of bankruptcy
w_j	Fraction of the total recovery value $V_B-\alpha(V_B)$ received by investors in debt issue j before taxation
w_E	Fraction of the total recovery value $V_B-\alpha(V_B)$ received by equity investors before taxation
X	Stochastic variable hitting a barrier pattern y_{t_i}

y_n	Vector of security price observations in period n: $y_n = (e_n, d_n')'$
y_s	Continuous corporate bond yield in period s
y_{t_i}	Upper barrier of the process X in the subperiod $]t_{i-1}, t_i]$
Y	Vector of observation functions in the Kalman filter
$\bar{Y}_{n\|n-1}$	A-priori estimate of observed security prices in the Kalman filter in period n
YS_s	Continuous yield spread between corporate and government bonds in period s

1

Introduction

In the last few years, a refined pricing of corporate securities has come into focus of academics and practitioners. As empirical research showed, traditional asset pricing models could not price corporate securities sufficiently well. Time series properties of quoted securities were difficult to replicate.

In the search for more advanced models that capture the empirical findings, researchers followed two approaches. The first stream of research fitted the time series properties of corporate securities directly. We refer to this class of models as being of reduced form. Security prices are assumed to follow more advanced stochastic models, in particular models with e.g. non-constant volatility.[1] All studies of this type do not consider the economics of the issuing companies but simply assume a stochastic behavior of the security or its state variables. In contrast, a second, economic literature developed by studying the firm. We call these kinds of models structural because the limited liability of equity holders is modeled explicitly as a function of firm value.

One problem of the reduced form approach is its difficulty of interpretation in an economic sense. Being technically advanced, reduced form models often lack an intuitive economic model and especially disguise the economic assumptions. If security pricing is the only purpose of the exercise, we might not need an economic model. However, if we want to understand price movements, a serious link with the underlying economics appears important.

The credit risk literature even adopted this particular terminology to categorize its models.[2] Whereas reduced form models take each corpo-

[1] See e.g. Stein and Stein (1991) for a stochastic volatility model and Heston and Nandi (2000) on GARCH option pricing.

[2] See e.g. Ammann (2002).

rate security separately and model a firm's default by a Poisson event[3], structural credit risk models concentrate on a model of the firm value. Bankruptcy occurs when either the firm value falls for the first time to a sufficiently low level so that equity holders are not willing to support the firm for a longer period of time, or when some contractual condition forces the firm into bankruptcy. The setup of structural models allows extensions into refined decision making and the use of game theoretic arguments.

Structural credit risk models were pioneered by the seminal papers of Black and Scholes (1973) and Merton (1974). They assume that the firm value follows a geometric Brownian motion. The firm has one finite maturity zero coupon bond outstanding that the firm will repay if the terminal firm value exceeds the debt notional at maturity. Otherwise the firm defaults on its debt. Black and Cox (1976) extend this setup by allowing bankruptcy before debt maturity when the firm value touches a bankruptcy barrier for the first time.

Further extension of the basic setup introduced optimal future capital structure changes. These dynamic capital structure models were analyzed e.g. in Fischer, Heinkel and Zechner (1989a) and Fischer, Heinkel and Zechner (1989b). In both papers the capital structure of the firm is modeled endogenously in a continuous-time setting assuming equity holders to optimize the value of their claim. They do not concentrate on credit risk and bankruptcy but use an argument from corporate finance in order to explain empirically observed leverage ratios and call premia of callable corporate bond issues. The idea of equity holders maximizing the value of their claim when levering the firm or issuing callable debt is developed further by Leland (1994) and Leland and Toft (1996). They focus on the valuation of corporate debt and the sensitivity of debt value to certain model parameters, extending the Fischer et al. (1989a) framework, and derive a firm value level at which equity owners endogenously trigger bankruptcy, thus linking the dynamic capital structure with credit risk models.

However, dynamic capital structure models of the first generation caused confusion. The model dynamics is driven by a stochastic process of the unlevered firm value which can be interpreted as the value of a fully equity financed firm. All other values of interest such as the levered firm value, debt values, leverage ratios, etc. are derived in an optimal budgeting decision with respect to the process of this unlevered firm value. In such a setup, however, both the levered and unlevered

[3] See e.g. Jarrow and Turnbull (1995), Jarrow, Lando and Turnbull (1997), Duffie and Singleton (1999), Collin-Dufresne, Goldstein and Hugonnier (2004).

value of the firm exist at the same time. The pricing of these securities is only arbitrage-free under certain conditions which are usually not clearly stated because they are not obvious if one models the firm value directly.[4]

One reason for this confusion about levered or unlevered firm values in dynamic capital structure models is due to the lack of a precise definition of firm value. One could think of the market value of assets as a natural candidate. However, the market value of the assets is different to the value generated by these assets. The introduction of corporate and personal taxes, bankruptcy cost, and debt blurs the models further and misleads interpretation. The other reason is that the firm value is modeled directly whereas payments to holders of corporate securities are defined in terms of cash flows. Being unclear about which claimant receives which cash flow, it can happen that the total amount of cash flows paid out to claimants does not sum to the firm's available funds. Taking the investment policy as given and unchanged, the mismatch leads to inconsistencies in the models.[5]

More recent approaches of dynamic capital structure models, e.g. Goldstein et al. (2001), Christensen et al. (2000), and Dangl and Zechner (2004), assume a stochastic process for an income measure that is unaffected by the capital structure decision. Earnings before interest and taxes (EBIT) or free cash flow (FCF) are natural candidates. Both income measures describe the earnings or cash flows of a firm from which the interests of **all** financial claimants, such as stockholders, bondholders, and the government, must be honored. The total firm value – i.e. the value of all claims – can be determined by discounting the income measure by an appropriate discount rate. One of the most important advantages of EBIT-based capital structure models is therefore the thinking in discounted cash flows that generate value. It forces a split of the EBIT into different claims, thus easing the interpretation

[4] Some authors like Goldstein, Ju and Leland (2001, p. 485) try hard to convince the reader that it is reasonable to model unlevered firm values by the argument that unlevered firms exist. They quote Microsoft as an example. However, this argument is void since nobody can prevent firms from issuing debt. So even the price histories of the stocks of these firms already account for the *potential* of a capital budgeting decision optimizing leverage in the future. On the other hand, if Microsoft does not issue debt although there is some tax advantage to do so, there must be a reason if they opt out. Again, none of the models can explain this kind of behavior. See Christensen, Flor, Lando and Miltersen (2000, p. 4f.) for a review of this argument.

[5] A prominent example of inconsistency is the numerical example in Goldstein et al. (2001), where EBIT does not match the sum of coupon, dividend and tax payments. Such a case is not covered in their model.

of derived security values. Consistency is permanently checked. Note that in this framework the bond and equity prices can be derived without confusion although levered and unlevered firm values in the sense of Leland (1994) exist. The necessary connection between the two artificial values becomes obvious.

However, structural credit risk models have mostly been illustrated by simulation studies. The simulation results have then been compared to observed leverage ratios, call premia or other firm specific, financial indicators. Promisingly, most simulations indicate that dynamic capital structure models are able to explain observed phenomena and price behaviors reasonably well.

Ericsson and Reneby (2004) have performed the only direct empirical tests for structural credit risk models to date.[6] One reason for the lack of empirical research stems from the fact that the proposed models offer economic settings which appear too restrictive for the available data. Goldstein et al. (2001) and Leland (1994), for example, analyze only infinite-maturity corporate debt. Perpetual debt is not common in practice. Furthermore, these models cannot incorporate a rich capital structure with multiple debt issues. As a result, time series data of finite maturity corporate bond prices cannot be used to test these models. This is true for the Leland and Toft (1996) model as well. Although Leland and Toft (1996) present a finite-maturity debt formula, their model suffers from the specific refinancing assumption that at each instant the firm issues a portion of a fixed maturity bond. Thus, the firm holds a continuum of bonds maturing at any instant until the fixed maturity. Exactly this assumption makes the model hard to test.

This study deals with some of the deficiencies of the existing literature on structural credit risk models. We develop an economically consistent structural framework to price corporate securities that is open for empirical implementation using several time series of corporate security prices. In particular:

- Structural credit risk models are embedded into an economic model. It is shown how the mathematical model evolves naturally without strong assumptions on the economy. We relate our modeling frame-

[6] We like to stress that this test showed that structural credit risk models are able to price corporate securities quite well although the estimation procedure was probably not the most favorable one. However, they can only handle firms with simple capital structures. Chapter 2 of this study proposes an extension of the model to a general capital structure. Chapter 5 develops a more appropriate estimation procedure.

work to the traditional structural models and resolve the confusion caused by misleading interpretation of the traditional literature.

- We model different debt issues simultaneously to allow for complex capital structures. Multiple debt issues call for a very careful modeling of the bankruptcy event because the bankruptcy value has to be split among claimants. We propose a simple analytical solution.
- Our Corporate Securities Framework does not rely on a specific process assumption for EBIT. Thus, we extend the EBIT-based credit risk models to alternative stochastic processes. For analytical tractability only very few equations need to be solved. We exemplify the claim by deriving solutions under the assumption that EBIT follows an arithmetic or geometric Brownian motion.[7] This opens the discussion of whether geometric Brownian motion is the best assumption for an EBIT-process or whether arithmetic Brownian motion might be more suitable. We favor the arithmetic Brownian motion assumption because it covers more economic situations of firms than geometric Brownian motion.
- We illustrate that our approach is practicable for the pricing of all kinds of corporate securities such as options on equity and that common numerical methods can be used to extend the basic analytical solution to optimal bankruptcy and more complicated tax structures.
- Since the Corporate Securities Framework allows for a complex capital structure, time series of all kinds of corporate securities can be used for estimation. In a simulation study, we show that our model can be implemented directly by using simulated time series of stock and bond prices. The parameters can be identified by our proposed Kalman filter.
- We make a strong case for the structural approach of asset pricing. We show that we can actually replicate empirical findings, such as properties of equity option prices. It is not necessary to assume complicated stochastic processes to get observable structures of implied Black and Scholes (1973) volatilities and to explain implied equity return distributions. Our economic setting is sufficient.

We develop our arguments in this study by a very general exposition of the EBIT-based Corporate Securities Framework in Chapter 2. Starting from the viewpoint of a firm where we take its EBIT as given, we define all claims of financial security holders as cash flows which have to be paid from EBIT. So, a natural definition of firm value is

[7] Other process assumptions are candidates for explicit solutions as well. However, we restrict ourselves to the two examples here.

the discounted value of all future EBITs. Since the splitting of EBIT into payments to different claimants divides the firm value, our model can be extended quite easily to additional claimants. The introduction of bankruptcy and of a tax regime is conceptually easy and makes the model more realistic. As well, each claim can be treated as one element of the whole model. We can show that refining one argument, say the decision of bankruptcy, does not change the structure of the other claims.

Chapter 2 focusses on the economic argument. The economic intuition becomes much clearer if we proceed by topic and discuss different modeling approaches of one economic element in the same section even if they were suggested in the literature at different times. This makes Chapter 2 special with respect to the existing literature: We argue economically and refer to the seminal articles as we proceed. The transparency of assumptions enables us to dissolve interpretational difficulties[8] of which some have been discussed above.

Since we do not strive to make the model analytically solvable in Chapter 2, the restrictive assumptions which are necessary to do so become obvious in Chapter 3. In this chapter, we assume specifically that EBIT follows arithmetic (Section 3.2) or geometric Brownian motion (Section 3.3), which restricts the sample paths of EBIT considerably. We discuss the economic implications of such assumptions and argue that geometric Brownian motion might not be the best assumption for the EBIT-process because many types of firms are excluded from the analysis due to its properties. However, we demonstrate in Section 3.2 that many of the results that are available for geometric Brownian motion still hold although security values are no longer homogeneous with respect to EBIT. This homogeneity of degree one has frequently been exploited when considering dynamic capital structure decisions to find closed form solutions.[9] Therefore, it is useful to think about numerical methods to find bankruptcy probabilities and prices of bankruptcy claims. The same numerical methods can be used to price derivatives on the financial claims within our framework.

One of the main issues of Chapter 3 is to extend the existing literature to complex capital structures with more than one finite maturity debt issue so that time series of stock and bond prices can be used to implement structural models directly. We are able to derive such

[8] In fact, some authors themselves are well aware of these difficulties and even point them out in their articles (see e.g. Leland (1994)). However, they avoid a detailed argument and return to their mathematical specifications.

[9] See Goldstein et al. (2001) and Flor and Lester (2004).

formulas which are then used in Chapter 5 to propose a Kalman filter for parameter estimation that does not need too much computational power.

Chapter 4 is devoted to numerical examples to get a better grip on the behavior of security prices. In particular, we contrast the arithmetic with the geometric Brownian motion assumption since there are crucial differences concerning price behavior. Therefore, we provide extensive comparative statics. For an even better understanding of the models, it is useful to think about equity densities at some future point in time. This type of analysis is usually conducted in option markets, where one common tool of analysis is to estimate equity return densities given a range of option prices with different strikes.[10] Our model has the advantage that we can calculate equity densities at future points in time and analyze their properties directly. In a second step, we then calculate option prices and Black and Scholes (1973)–implied volatility curves. The examples demonstrate that considerably high implied volatilities for low exercise options are not unusual and that implied volatilities are monotonically decreasing if the strike increases. This effect can then directly be linked to the equity densities, i.e. the current parameter values of the firm. A comparison with recent empirical research of implied densities of individual stock options shows that we are able to provide additional insight into the discussion of equity option pricing. It becomes again clear that an economic approach sometimes leads to a much simpler explanation of price behavior than ad-hoc stochastic and econometric analysis.

Chapter 5 closes the gap within the empirical literature. Since direct implementations of structural credit risk models are rare, we propose a Kalman filter for the model developed in Chapters 2 and 3. This has the advantage that both time series of stock and bond prices can be used simultaneously to estimate parameters.[11]

To get a better understanding of the stability of the estimation procedure, we simulate stock and bond prices which are disturbed by a considerably large observation error. We demonstrate that the proposed Kalman filter is able to detect the underlying process parameters, i.e. both EBIT-risk-neutral drift and volatility. Previously proposed estimation procedures could only identify volatility. Convergence is quick

[10] See e.g. Jackwerth (1999) for a literature review and Jackwerth and Rubinstein (1996).

[11] Past empirical studies such as Ericsson and Reneby (2004) usually use only time series of stock prices to estimate model parameters and then use these estimates to price debt issues. So, they need additional assumptions about the mean of the EBIT/firm value process.

if we minimize mean absolute pricing errors in a first step and use the estimated parameters as starting values for a maximum-likelihood function. We can therefore conclude that our approach is promising.

Chapter 6 summarizes the main findings and proposes interesting areas where research should continue. Additionally, we discuss further applications for the developed theory.

2

The Corporate Securities Framework

There exist numerous ways to price corporate securities. Structural firm value models enjoy an intuitive economic interpretation because these models focus on the economics of the issuing firm and deduce the prices of specific securities from the firm's current economic condition.

This chapter establishes the economic framework for the valuation of corporate securities. We show that different approaches from the dynamic capital structure theory can be combined to a more general model. The framework is modular. Different elements can be extended in future research without affecting the other results. In particular, we focus on a class of EBIT-processes and postpone the assumption of a particular EBIT-process and the derivation of closed-form solutions to Chapter 3.

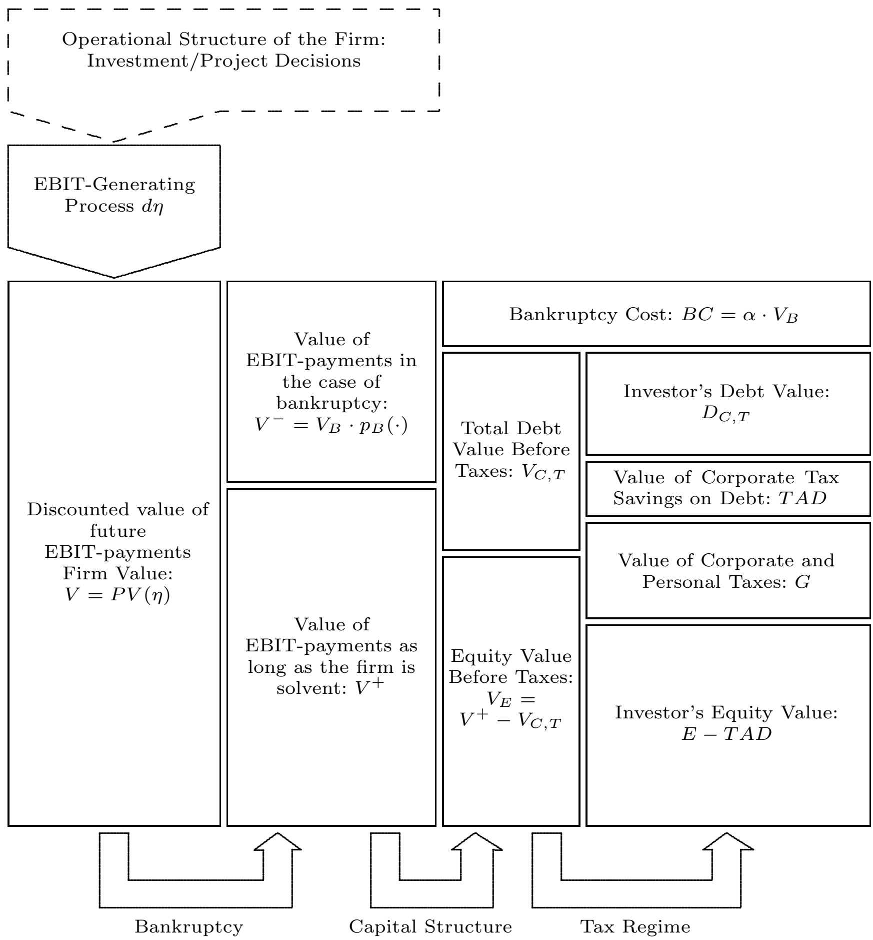
2.1 The Economic Setting

The analysis of business decisions can focus on different aspects. We might on the one hand focus on operational/investment decisions as is done in Dixit and Pindyck (1994). Here we concentrate on financial decisions with a given investment policy. However, to interpret some results it might be useful to remember the economic environment.

2.1.1 EBIT-Generator

Figure 2.1 illustrates the general framework for the pricing of corporate securities. The firm's operational decisions are taken as given. The production technology generates earnings before interest and taxes (EBIT), denoted by η.

Fig. 2.1. Division of claim value within the corporate security framework



The EBIT of the firm has a current value of η_{t_0} and follows a stochastic process[1]

$$d\eta = \mu(\eta, t)dt + \sigma_\eta(\eta, t)dz^{\mathcal{Q}}, \tag{2.1}$$

[1] This study concentrates on the economics of structural firm value models. Therefore, mathematical technicalities are only mentioned if it seems necessary. Throughout the exposition, it is safe to assume that all stochastic integrals exist and are well adapted to the probability space $(\Omega, \mathcal{Q}, \mathcal{F}_t(\eta_t))$ because we are not dealing with information asymmetry and revelation. We refer to Oeksendal (1998) or Duffie (1996) for existence and representation theorems.

where $\mu(\eta, t)$ denotes the instantaneous drift and $\sigma_\eta(\eta, t)$ is the volatility of the EBIT-process. $z^{\mathcal{Q}}$ is a Brownian motion under the risk-neutral martingale measure $\mathcal{Q}$.

Focusing on a flow measure such as EBIT – instead of the firm value directly as suggested in most structural models – introduces a notion of cash flows into the firm value setting, because the use of EBIT lies at the discretion of the firm owners. In particular, the firm owners may decide to issue debt (Subsection 2.1.4) so that not all EBIT can be paid out as a dividend but some must be devoted to honor interest payments to debt holders. An additional claimant to the firm's EBIT is the government, which levies income taxes (Subsection 2.1.5). This special structure of defining all financial claims in terms of payments made from EBIT is advantageous because cash flows are more readily interpreted than values of claims. To ensure that all calculations have been done correctly, the sum of all cash flows paid out has to equal EBIT. As a result, the sum of all values of different payouts must equal the value of all future EBIT-payments, which is indicated by the equal height of all columns in Figure 2.1.

Note that EBIT in equation (2.1) evolves under an equivalent martingale measure $\mathcal{Q}$. This assumption is not restrictive. If a financial claim issued by the firm is traded in an arbitrage-free financial market, a martingale measure $\mathcal{Q}$ exists.[2] Then, the $\mathcal{Q}$-measure of the EBIT-process is implied by this martingale measure of the traded security because the traded security can be interpreted as a derivative of EBIT.[3]

To illustrate this argument in more detail, assume that the EBIT-process under the physical measure $\mathcal{P}$ is[4]

$$d\eta = \mu_\eta(\eta, t)dt + \sigma_\eta(\eta, t)dz^{\mathcal{P}}, \tag{2.2}$$

where $\mu_\eta(\eta, t)$ denotes the respective physical drift. Consider a security $F(\eta, T)$ which depends on EBIT and receives a regular payment of $f(\eta_t)dt$ depending on the prevailing EBIT. To value securities depending on EBIT, it is advantageous to transform the physical measure $\mathcal{P}$ into a risk-neutral measure $\mathcal{Q}$. If the security F is traded, the $\mathcal{Q}$-measure exists and there exists a risk premium θ that adjusts the EBIT-drift μ_η

[2] See Harrison and Kreps (1979) and Harrison and Pliska (1981) for the respective proofs of the connection between arbitrage-free markets and the existence of a martingale measure. For the refined arguments and technicalities in continuous market see e.g. Duffie (1996), chapter 6.

[3] See Ericsson and Reneby (2002b).

[4] See Goldstein et al. (2001), who start their analysis with the physical EBIT-process.

such that the expected $\mathbb{Q}$-return of F is equal to the risk-free interest rate r, i.e.

$$E_t^{\mathbb{Q}}(dF(\eta_t, T)) + f(\eta_t)dt = rF(\eta_t, T)dt. \tag{2.3}$$

If the market is arbitrage-free, the θ-process and an equivalent martingale measure $\mathbb{Q}$ exists. Then, $\mathbb{Q}$ is not only the pricing measure for F but also for all other securities issued by the firm and the EBIT-process of equation (2.1). If none of the firm's securities are traded, the assumption is needed that the θ-process is well defined, which yields the same EBIT-process as of equation (2.1).[5]

Therefore, we can start the analysis with the η-process under the martingale measure $\mathbb{Q}$ defined by equation (2.1) instead of taking the detour via equation (2.2).

The existence of a martingale measure $\mathbb{Q}$ is very useful because it allows the discounting of all $\mathbb{Q}$-expected payments of the firm to investors with the risk-free interest rate.[6]

However, the η-process under the martingale measure $\mathbb{Q}$ does not automatically exhibit a drift equal to the risk-free interest rate $r\eta dt$ – as the securities depending on EBIT-payments do – because EBIT itself is not the value of a traded security but a state variable. Consequently, the risk premium θ will be different to the familiar risk premium $(\mu_\eta - r)/\sigma_\eta$, which is enforced by the arbitrage condition of traded securities.

2.1.2 The Firm's Value and Operations

Continuing the argument of the last subsection, firm value V is defined rather naturally as the discounted value of all future EBIT-payments.[7]

$$V = E_{t_0}^{\mathbb{Q}} \int_{t_0}^{\infty} \eta_s e^{-r(s-t_0)} ds, \tag{2.4}$$

where r denotes a constant risk-free interest rate.[8]

[5] In this case the θ-process needs to be determined by an equilibrium argument. See e.g. Shimko (1992), Chapter 4.

[6] See Musiela and Rutkowski (1997), Chapter 3.

[7] Such a definition of firm value can also be found in Goldstein et al. (2001).

[8] Here and in the rest of the text, the convention will be used that arguments of the value functions are not explicitly stated if the reference is the standard definition. Sometimes it is useful to write out arguments explicitly to emphasize a certain parameter setting. So firm value V denotes the standard t_0 value of the firm depending on the current state of EBIT η_{t_0}, the drift function μ, interest rate r and standard deviation σ_η. $V(\hat{\mu})$ denotes the same value function as V but with a different drift.

Table 2.1. Calculation of cash flows to equity from EBIT

	EBIT
−	Coupon Payments
−	Corporate Taxes
=	Corporate Earnings
	+ *Depreciation*
	− *Gross Investments*
−	Net Investments
	+ *Notional of New Debt Issues*
	− *Repayment of Maturing Debt*
+	Net Debt Financing
=	Cash Flow To Equity

Such a definition of firm value implies a dividend and investment policy of the firm. To illustrate these policies, assume that the company has a level of invested capital I_{t_0}. Depreciation of invested capital is already deducted in EBIT, but reinvested in the company if EBIT is treated as a cash flow to all claimants (including the government) so that net investments equal zero as illustrated in Table 2.1. Therefore, a firm has a constant level of invested capital I. All cash that could remain in the company to increase the invested capital is distributed to equity holders. If bonds have to be repaid, equity holders inject money into the firm to prevent the company from selling assets unless the existing debt issues are replaced by new corporate bonds. As a result, the cash dividend to equity investors (before equity investor's own taxation) is defined as the firm's earnings after taxes adjusted for debt repayments and capital inflows from issuing bonds in the future.

Note that η_t can assume arbitrary values. A negative cash flow to equity holders implies that equity holders need to inject money into the firm.

Although we restrict our analysis to the definition of firm value in equation (2.4), a consideration of potential extensions is illustrative even for the more restrictive setting. Consider that net investments, ΔI_{t_i}, are not unequivocally zero. If the firm disinvests at some points in time t_i with $i = 1, \ldots, n_I$, ΔI_{t_i} becomes negative and affects the available cash flow to financial claimants. Then, equation (2.4) is no longer a correct representation of firm value, which becomes instead

$$V = E^{\mathbb{Q}}_{t_0} \int_{t_0}^{\infty} (\eta_s - \Delta I_s \mathbf{1}_{\{s=t_i\}}) e^{-r(s-t_0)} ds. \tag{2.5}$$

Usually, a changed investment level I_t changes the firm's future growth and risk perspectives by altering $\mu(\cdot)$ and $\sigma_\eta(\cdot)$, respectively. If such a relationship is neglected, equity owners are allowed to sell all physical assets to generate an extra cash flow of I_{t_0} immediately but still receive all future EBIT-payments η_t, $t > t_0$.

As an alternative to changing $\mu(\cdot)$ and $\sigma_\eta(\cdot)$ endogenously, an immediate asset sale can be prevented by defining a maximum return on investments ROI_{max}. Any EBIT that exceeds $\eta_{max,t} = ROI_{max} \cdot I_t$ would be forgone if the firm does not invest to enlarge operations. In contrast, if EBIT falls and the level of assets becomes unreasonably high compared to the value of expected future cash flows, equity owners have an incentive to disinvest because reaching the upper EBIT-barrier is unlikely. Such a setting allows the analysis of a firm's investment/dividend/financing behavior but is beyond the scope of the research question here.

2.1.3 Bankruptcy

The cash flow to equity can eventually become negative. As discussed in the last subsection, the firm's dividend policy implies that equity owners have to infuse money into the firm to support the current obligations. However, equity investors may refuse to pay and the firm goes bankrupt because it cannot pay its contractual commitments. Consider a deterministic function $\eta_B(t)$ at which equity owners declare bankruptcy and define the bankruptcy time τ as

$$\tau = \inf\{s \geq t_0 : \eta_s = \eta_B(s)\}. \tag{2.6}$$

At $t = \tau$ all bankruptcy claims are settled. By liquidating the firm, a residual value denoted by $V_B(\tau)$ is generated. $V_B(\tau)$ is connected to the corresponding EBIT-value $\eta_B(\tau)$ by equation 2.4

$$V_B(\tau) = E_\tau^{\mathbb{Q}} \int_\tau^\infty \eta_s e^{-r(s-\tau)} ds, \tag{2.7}$$

Otherwise bankruptcy can be exploited strategically by equity owners.

Bankruptcy usually occurs not only at predefined dates but at any point in time in the future. To value securities, one might think of each claim as receiving payments in mutually exclusive events: (i) the cash flow received at maturity if the firm is still alive, (ii) the cash flow of contracted regular payments before maturity as long as the firm is solvent, and (iii) a residual cash flow in the case of bankruptcy.[9] Since the three events are disjoint, the three values of the individual claims add to the total claim value. Although one must come up with three different valuation formulas, they are easier to solve in many cases than their partial differential equation counterpart with several boundary conditions.

To see why and to ease the exposition, we introduce the probability $\Phi(\cdot)$ that the firm goes bankrupt before a time $T > t_0$ and the Arrow-Debreu price of the bankruptcy event $p_B(\cdot)$. Define,

$$\phi(t_0, T, \eta_{t_0}, \eta_T, \eta_B(t)) = P(\eta_T \in d\eta | \tau > T) P(\tau > T), \quad (2.8)$$

as the density of η_T at some future point in time T conditional that the firm survives until T. The current EBIT is η_{t_0}. Then, the probability of the firm going bankrupt before T is

$$\Phi(t_0, T, \eta_{t_0}, \eta_B(t)) = 1 - \int_{\eta_B(T)}^{\infty} \phi(t_0, T, \eta_{t_0}, \eta_s, \eta_B(t)) d\eta_s. \quad (2.9)$$

The density of first passage time can be found by taking the derivative with respect to T. Define this density by

$$\psi(t_0, s, \eta_{t_0}, \eta_B(t)) = \frac{\partial \Phi(t_0, T, \eta_{t_0}, \eta_B(t))}{\partial T}. \quad (2.10)$$

Integrating the discounted probability of first passage time over time yields the Arrow-Debreu price of the bankruptcy event

$$p_B(t_0, T, \eta_{t_0}, \eta_B(t)) = \int_{t_0}^{T} e^{-r(s-t_0)} \psi(t_0, s, \eta_{t_0}, \eta_B(t)) ds. \quad (2.11)$$

For derivations to follow, it is convenient to consider default probabilities and Arrow-Debreu bankruptcy prices for future time intervals $]T', T]$, with $t_0 \leq T' < T$. Due to the Markov property, the $]T', T]$-default probability is

$$\Phi(T', T, \eta_{t_0}, \eta_B(t)) = \Phi(t_0, T, \eta_{t_0}, \eta_B(t)) - \Phi(t_0, T', \eta_{t_0}, \eta_B(t)). \quad (2.12)$$

The owner of a claim that pays one currency unit at bankruptcy only if the firm goes bankrupt between T' and T holds a portfolio of two claims of the type $p_B(t_0, \cdot, \eta_{t_0}, \eta_B(t))$. The value of the $]T', T]$-bankruptcy claim is

[9] Black and Cox (1976) were the first to apply this approach when they extended the Merton (1974) model to bankruptcy before debt maturity. The approach is known as the probabilistic approach. See also Shimko (1992).

$$p_B(T', T, \eta_{t_0}, \eta_B(t)) = p_B(t_0, T, \eta_{t_0}, \eta_B(t)) - p_B(t_0, T', \eta_{t_0}, \eta_B(t)). \quad (2.13)$$

Assume for the moment that $\eta_B(s) = \eta_B$ is constant. Then, the t_0-firm value can be split into two periods by the bankruptcy time $\tau(\omega)$.

$$\begin{aligned} V &= E_{t_0}^{\mathbb{Q}} \int_{t_0}^{\tau(\omega)} \eta_s e^{-r(s-t_0)} ds + E_{t_0}^{\mathbb{Q}} \left[e^{-r(\tau(\omega)-t_0)} \int_{\tau(\omega)}^{\infty} \eta_s e^{-r(s-\tau(\omega))} ds \right] \\ &= V^+ + E_{t_0}^{\mathbb{Q}} \left[e^{-r(\tau(\omega)-t_0)} E_{\tau(\omega)}^{\mathbb{Q}} \left[\int_{\tau(\omega)}^{\infty} \eta_s e^{-r(s-\tau(\omega))} ds \right] \right] \\ &= V^+ + \int_{t_0}^{\infty} e^{-r(u-t_0)} E_u^{\mathbb{Q}} \left[\int_u^{\infty} \eta_s e^{-r(s-u)} ds \right] P(\tau(\omega) \in du) \\ &= V^+ + V_B \int_{t_0}^{\infty} e^{-r(u-t_0)} \psi(t_0, u, \eta_{t_0}, \eta_B(u)) du \\ &= V^+ + p_B(t_0, \infty, \eta_{t_0}, \eta_B) V_B \\ &= V^+ + V^-, \end{aligned} \quad (2.14)$$

where we used the law of iterated expectations in line 2 and the definition of the function $\psi(t_0, s, \eta_{t_0}, \eta_B(s))$ as well as the definition of the bankruptcy value of equation (2.7) in line 4. By noting that V^- is readily calculable if the bankruptcy price is known, so is V^+. The splitting of total firm value V into a *solvent* part, V^+, and an *insolvent* part, V^- is very useful when considering a capital structure.

The assumption of a constant bankruptcy barrier is not restrictive because we can always find a transformation $g(\eta_t, \eta_B(t))$ of the state variable η_t relative to the bankruptcy barrier $\eta_B(t)$, so that the bankruptcy barrier is constant with respect to the stochastic process of $g(\eta_t, \eta_B(t))$. So we proceed with a constant bankruptcy barrier V_B and η_B to ease the exposition.

Bankruptcy usually incurs additional cost upon the firm's claimants. The bankruptcy cost BC comprises external claims to bankruptcy proceedings, reorganization costs, lawyer fees, or simply value losses due to the decrease in reputation. Often in the literature, bankruptcy costs are modeled as being proportional to the firm value in bankruptcy V_B. However, alternative specifications appear reasonable if the bankruptcy barrier changes over time. As an example, consider an affine function in V_B

$$\alpha(V_B(t)) = \alpha_1 + \alpha_2(V_B(t) - \alpha_1), \quad (2.15)$$

where α_1 denotes a fixed cost and α_2 a variable cost factor applied to the remaining firm value after direct costs have been deducted from the bankruptcy value. $\alpha_1 = 0$ implies the traditionally used proportional bankruptcy cost structure. The bankruptcy cost can then be valued as

$$BC = \int_{t_0}^{\infty} \alpha(V_B) e^{-r(s-t_0)} \psi_t(t_0, s, \eta_{t_0}, \eta_B) ds. \tag{2.16}$$

If V_B is constant equation (2.16) simplifies to

$$\begin{aligned} BC &= \alpha(V_B) \int_{t_0}^{\infty} e^{-r(s-t_0)} \psi_t(t_0, s, \eta_{t_0}, \eta_B) ds \\ &= \alpha(V_B) p_B(t_0, \infty, \eta_{t_0}, \eta_B). \end{aligned} \tag{2.17}$$

The remaining value $V^- - BC$ represents the present value of the claim distributed among bankruptcy claimants as a recovery value.

2.1.4 Capital Structure

If firms have already stayed in business for some time, they exhibit a capital structure which reflects past financing decisions. So at the current date, a firm has equity and several finite maturity debt issues outstanding. The value of equity before taxes will be denoted by V_E. The jth of the J debt issues, $j = 1, \ldots, J$, which pays a continuous coupon C_j and, matures at T_j has a value of V_{C_j,T_j}. By the same reasoning as above, it is again useful to split each claim into a solvent and an insolvent part, which is represented in the third column of blocks in Figure 2.1.

Pick the insolvent values V_E^- and V_{C_j,T_j}^-, first. If there is more than one debt issue outstanding, the residual firm value has to be split among debt holders according to a reasonable scheme. The residual firm value will not cover all claims of debt holders who usually enjoy preferential treatment in bankruptcy under the debt contracts. In case it does, equity holders receive the excess portion and debt holders recover all of their claims. As many authors noted[10], the absolute priority rule can be violated so that equity owners recover some money although debt holders do not receive their total claim. The framework is flexible enough to cover these cases. Figure 2.1 does not consider equity claims in bankruptcy.

A consistent model of security recovery values needs to be carefully developed. Assume that each of the $j = 1, \ldots, J$ debt securities receives

[10] See e.g. Franks and Tourus (1989), Franks and Tourus (1994), and the references in Longhofer and Carlstrom (1995).

a fraction $w_j(s)$ with $\sum_j w_j(s) \leq 1 \; \forall s > t_0$ of the remaining $V_B(s) - \alpha(V_B)$.[11] Therefore the value of debt holders bankruptcy claim becomes

$$V^-_{C_j,T_j} = \int_{t_0}^{T_j} w_j(s)\alpha(V_B)e^{-r(s-t_0)}\psi(t_0, s, \eta_{t_0}, \eta_B)ds. \qquad (2.18)$$

If $w_j(s)$ is a step function in the subperiod $]T_{i-1}, T_i]$, $i = 1, \ldots, j$, then equation (2.18) simplifies to

$$\begin{aligned} V^-_{C_j,T_j} &= \sum_{i=1}^{j} \alpha(V_B) \int_{T_{i-1}}^{T_i} w_j(T_i)e^{-r(s-t_0)}\psi(t_0, s, \eta_{t_0}, \eta_B)ds \\ &= \sum_{i=1}^{j} \alpha(V_B)w_j(T_i)p_B(T_{i-1}, T_i, \eta_{t_0}, \eta_B). \end{aligned} \qquad (2.19)$$

In case that at some time $s > t_0$, $\sum_j w_j(s) < 1$, there is a residual value for equity owners $w_E(s) = 1 - \sum_j w_j(s)$ if bankruptcy occurs at that point in time.[12] The value of these payments is then

$$\begin{aligned} V^-_E &= \int_{t_0}^{\infty} \max[w_E(s), 0](1 - \alpha(s))V_B e^{-r(s-t)}\psi(t_0, s, \eta_{t_0}, \eta_B)ds \\ &= V^- - BC - \sum_{j=1}^{J} V^-_{C_j,T_j}. \end{aligned} \qquad (2.20)$$

As long as the firm is solvent, debt holders receive coupon payments and the notional at maturity. The debt value of the solvent firm is then

[11] Note that since the capital structure may change in the future, the fraction w_j can also be time dependent. As well, deviations from priority rules might be dependent on η_s.

[12] w_j and w_E can also be thought of being determined by a bankruptcy game in which equity and debt holders split the residual firm value among one another. The framework is capable of incorporating such decision processes. See also Subsections 2.1.3 and 2.2.2.

$$
\begin{aligned}
V^{+}_{C_j,T_j} &= \int_{t_0}^{T_j} C_j e^{-r(s-t_0)} \left[1-\Phi(t_0,s,\eta_{t_0},\eta_B)\right] ds \\
&\quad + P_j e^{-r(T_j-t_0)} \left[1-\Phi(t_0,T_j,\eta_{t_0},\eta_B)\right] \\
&= \int_{t_0}^{T_j} C_j e^{-r(s-t_0)} ds - \int_{t_0}^{T_j} C_j e^{-r(s-t_0)} \Phi(t_0,s,\eta_{t_0},\eta_B) ds \\
&\quad + P_j e^{-r(T-t_0)} \left[1-\Phi(t_0,T_j,\eta_{t_0},\eta_B)\right] \\
&= \left[-\frac{C_j}{r} e^{-r(s-t_0)}\right]_{t_0}^{T_j} - \left[\left[-\frac{C_j}{r} e^{-r(s-t_0)} \Phi(t_0,s,\eta_{t_0},\eta_B)\right]_{t_0}^{T_j} \right. \\
&\qquad \left. - \int_{t_0}^{T_j} -\frac{C_j}{r} e^{-r(s-t_0)} \psi(t_0,s,\eta_{t_0},\eta_B) ds \right] \\
&\quad + P_j e^{-r(T-t)} \left[1-\Phi(t_0,T_j,\eta_{t_0},\eta_B)\right] \\
&= \frac{C_j}{r} + e^{-r(s-t_0)} \left[P_j - \frac{C_j}{r}\right] \left[1-\Phi(t_0,s,\eta_{t_0},\eta_B)\right] \\
&\quad - \frac{C_j}{r} p_B(t_0,T_j,\eta_{t_0},\eta_B),
\end{aligned}
\tag{2.21}
$$

where P_j denotes the principal amount repaid at T_j. Total value of debt consists of two sources, the first one from a regular fulfillment of contracted payments, e.g. coupons and principle repayment, and the second one of the recovery value in the case of bankruptcy.

$$
V_{C_j,T_j} = V^{+}_{C_j,T_j} + V^{-}_{C_j,T_j} \tag{2.22}
$$

Equity investors are bound to the residual EBIT of the solvent firm.

$$
\begin{aligned}
V^{+}_{E} &= E^{\mathbb{Q}}_{t_0} \int_{t}^{\infty} \left[\eta_s - \sum_{j=1}^{J} C_j \mathbf{1}_{\{s \le T_j\}} - P_j \mathbf{1}_{\{s=T_j\}}\right] \\
&\qquad e^{-r(s-t_0)} \left[1-\Phi(t_0,s,\eta_{t_0},\eta_B)\right] ds \\
&= E^{\mathbb{Q}}_{t_0} \int_{t}^{\infty} \eta_s e^{-r(s-t_0)} \left[1-\Phi(t_0,s,\eta_{t_0},\eta_B)\right] ds \\
&\quad - \sum_{j=1}^{J} \left[\int_{t}^{\infty} C_j \mathbf{1}_{\{s \le T_j\}} e^{-r(s-t_0)} \left[1-\Phi(t_0,s,\eta_{t_0},\eta_B)\right] ds \right. \\
&\qquad \left. + P_j e^{-r(T_j-t_0)} \left[1-\Phi(t_0,T_j,\eta_{t_0},\eta_B)\right] \right] \\
&= V^{+} - \sum_{j=1}^{J} V^{+}_{C_j,T_j}
\end{aligned}
\tag{2.23}
$$

In equation (2.23), the indicator functions within the square brackets ensure that no coupons are paid to debt holders of issue j after debt maturity and that the principal is paid at debt maturity if $s = T_j$. Again, the solvent and insolvent equity value sum to total equity value

$$V_E = V_E^+ + V_E^- . \tag{2.24}$$

2.1.5 Tax System

To complete the model of the economic environment of the firm, a tax regime is considered that is imposed upon payments out of EBIT. Introduce three different kinds of taxes. Debt holders' coupon payments are taxed at a tax rate τ^d. A corporate tax rate τ^c is applied to corporate earnings, i.e. EBIT less coupon payments. Corporate earnings after tax are paid out as a dividend, which is then taxed at the personal tax rate of equity owners τ^e.

In the last subsection, valuation equations have been stated for all claims of a firm in a solvent state as well as in bankruptcy before taxation. Each of these values can be split further into a value that belongs to the investors, E^+, E^-, as well as $D^+_{C_j,T_j}$, $D^-_{C_j,T_j}$, and the governments claim G, respectively.

Starting with the bankruptcy case, the standard assumption about bankruptcy proceedings is that the old equity owners hand over the firm to the bankruptcy claimants. The new owners should therefore be treated as equity investors for tax purposes. The recovery value, i.e. the insolvent firm value less bankruptcy cost, can be thought of as a final dividend that is taxed at the corporate and the receiver's level. Denote by $\tau^{eff} = (1-\tau^c)(1-\tau^e) - 1$ the effective tax rate of equity owners in a full double taxation regime. Then

$$D^-_{C_j,T_j} = (1-\tau^{eff})V^-_{C_j,T_j} \tag{2.25}$$

$$E^- = (1-\tau^{eff})V^-_E \tag{2.26}$$

$$G^- = \tau^{eff}(V^- - BC), \tag{2.27}$$

because the before tax security values are all proportional to the bankruptcy tax base $V^- - BC$.

Some authors[13] allow debt holders to relever the firm as a going concern. We do not consider such a possibility here because it is not needed in the current setting where equity owners are not allowed to restructure the firm before debt maturity. Assume debt holders are able

[13] See e.g. Leland (1994) or Fischer et al. (1989a).

to successfully relever the firm and continue operations. They would only bother to do so if there is a benefit to them. Otherwise they will simply sell the production technology and keep the recovery value. If they could make a profit on this strategy, the original equity owners would have considered the sale in the first place and postponed bankruptcy or demand a share of the recovery value, i.e. $w_E > 0$. In fact, the original equity holders would not have allowed the firm to go bankrupt but have changed the capital structure to avoid bankruptcy cost. So if all agents act in their best interest, the debt holders going concern strategy should not add any new insights to the firm's current security values because the insolvent firm value less bankruptcy losses includes already optimization strategies of the new equity owners.[14]

To determine tax payments in the solvent state, tax rules must be defined in more detail. As long as earnings and dividends are positive, the tax rates are applied to the positive tax bases and the government receives tax payments. In most countries the tax treatment of negative earnings is more complicated. Usually, the tax system allows for immediate tax refunds for some portion of negative earnings and an additional loss carry-forward for the rest of the negative earnings.

The taxation of investors is usually split into two parts. First, security income is taxed whenever a payment is received. Second, realized capital gains are taxed accordingly, whereas realized capital losses undergo a more restrictive treatment.

To ease the derivation of closed form solutions and abstract from complications in specific tax codes, some simplifications are made with respect to the tax regimes considered. To investigate the impact of the tax system on security values, we restrict the analysis to three benchmark cases that can be implemented more easily analytically or numerically.

Tax Regime 1: In the base case model, we implement a setting where negative corporate earnings are immediately eligible for a tax refund, so that the company has cash inflows amounting to the negative tax liabilities. By the same token, equity owners who have to balance negative earnings immediately do this with a tax subsidy. Explicitly, we exempt capital repayments of debt at maturity from this taxation on the corporate and equity investor level. This yields the following valuation equations for the debt, equity holders, and the

[14] The bankruptcy strategies here are simple because the only option is liquidation of the firm. In Section 2.2 other models of the bankruptcy event are discussed. However, this literature fits smoothly into our framework if the single liquidation option is replaced by a more complex game between equity and debt holders.

government for the solvent firm after tax.[15]

$$D^{+}_{C_j,T_j} = \int_{t_0}^{T_j} (1-\tau^d) C_j e^{-r(s-t_0)} \left[1 - \Phi(t_0, s, \eta_{t_0}, \eta_B)\right] ds$$
$$+ P_j e^{-r(T_j - t_0)} \left[1 - \Phi(t_0, T_j, \eta_{t_0}, \eta_B)\right] \qquad (2.28)$$

$$E^{+} = E^{\mathbb{Q}}_{t_0} \int_{t_0}^{\infty} \left[(1-\tau^{eff}) \left(\eta_s - \sum_{j=1}^{J} C_j \mathbf{1}_{\{s \le T_j\}} \right) \right.$$
$$\left. - \sum_{j=1}^{J} P_j \mathbf{1}_{\{s = T_j\}} \right]$$
$$e^{-r(s-t_0)} \left[1 - \Phi(t_0, s, \eta_{t_0}, \eta_B)\right] ds \qquad (2.29)$$

$$G^{+} = E^{\mathbb{Q}}_{t_0} \int_{t_0}^{\infty} \left(\tau^{eff} \eta_s + (\tau^d - \tau^{eff}) \sum_{j=1}^{J} C_j \mathbf{1}_{\{s \le T_j\}} \right)$$
$$e^{-r(s-t_0)} \left[1 - \Phi(t_0, s, \eta_{t_0}, \eta_B)\right] ds \qquad (2.30)$$

Note that the value of debt to the firm is different to the one in equation (2.28). Denoting this value by

$$D^{E+}_{C_j,T_j} = \int_{t_0}^{T_j} (1-\tau^{eff}) C_j e^{-r(s-t_0)} \left[1 - \Phi(t_0, s, \eta_{t_0}, \eta_B)\right] ds$$
$$+ P_j e^{-r(T_j - t_0)} \left[1 - \Phi(t_0, T_j, \eta_{t_0}, \eta_B)\right], \qquad (2.31)$$

equation (2.29) can be written more conveniently as

$$E^{+} = (1-\tau^{eff}) V^{+} - \sum_{j=1}^{J} D^{E+}_{C_j,T_j}.$$

Tax Regime 2: This tax regime is similar to Tax Regime 1 but it is allowed to deduct debt repayments at the equity investor level. This changes the advantage of debt issues to equity investors. Equations (2.29) and (2.30) change to

[15] All integrals can be solved along the same lines as equation (2.22)

$$E^{+} = E_{t_0}^{\mathbb{Q}} \int_{t_0}^{\infty} (1-\tau^{e}) \left[(1-\tau^{c}) \left(\eta_s - \sum_{j=1}^{J} C_j \mathbf{1}_{\{s \leq T_j\}} \right) - \sum_{j=1}^{J} P_j \mathbf{1}_{\{s = T_j\}} \right] e^{-r(s-t_0)} \left[1 - \Phi(t_0, T_j, \eta_{t_0}, \eta_B) \right] ds, \tag{2.32}$$

$$G^{+} = E_{t_0}^{\mathbb{Q}} \int_{t_0}^{\infty} \left(\tau^{eff} \eta_s + (\tau^{d} - \tau^{eff}) \sum_{j=1}^{J} C_j \mathbf{1}_{\{s \leq T_j\}} - \tau^{e} \sum_{j=1}^{J} P_j \mathbf{1}_{\{s = T_j\}} \right) e^{-r(s-t_0)} \left[1 - \Phi(t_0, s, \eta_{t_0}, \eta_B) \right] ds. \tag{2.33}$$

Economically, the tax refund of debt repayments is accounted for in the equity investors' tax base. The amount of refunded taxes is approximately the one which the investor would not have paid on dividends if the firm chose to keep the money in its excess cash account. The difference is due to discounting effects of equity owners' tax payments.

Tax Regime 2 can therefore be interpreted as one in which the firm saves money for future debt repayments in a separate account. The money belongs to equity owners but payment of the personal income tax is avoided.

Note that this interpretation of Tax Regime 2 slightly alters the notion of the bankruptcy value of V_B if the repayment of debt is interpreted as coming from a cash account within the firm which is not available to bankruptcy proceedings but locked in by equity investors.

Tax Regime 3: Consider again Tax Regime 1, but assume that the government restricts tax credits. A portion $\varepsilon \in [0,1]$ of the immediate tax credits as of Tax Regimes 1 or 2 is lost for the firm.[16] Then, equations (2.29) and (2.30) change to

[16] Leland (1994) and Goldstein et al. (2001) consider such a tax system. With a simple capital structure and assuming that EBIT follows a geometric Brownian motion, it is possible to derive closed form solutions.

$$E^{+} = E^{\mathbb{Q}}_{t_0} \int_{t_0}^{\infty} \Bigg[\Big(1 - \tau^{eff} + \varepsilon(\tau^{eff} - \tau^{e}) \mathbf{1}_{\{\eta_s \leq \sum_{j=1}^{J} C_j\}}\Big) \left(\eta_s - \sum_{j=1}^{J} C_j \mathbf{1}_{\{s \leq T_j\}}\right) - \sum_{j=1}^{J} P_j \mathbf{1}_{s=T_j} \Bigg] e^{-r(s-t_0)} \left[1 - \Phi(t_0, s, \eta_{t_0}, \eta_B)\right] ds \tag{2.34}$$

$$G^{+} = E^{\mathbb{Q}}_{t_0} \int_{t_0}^{\infty} \Bigg[\Big(\tau^{eff} - \varepsilon(\tau^{eff} - \tau^{e}) \mathbf{1}_{\{\eta_s \leq \sum_{j=1}^{J} C_j\}}\Big) \left(\eta_s - \sum_{j=1}^{J} C_j \mathbf{1}_{\{s \leq T_j\}}\right) + \tau^{d} \sum_{j=1}^{J} C_j \mathbf{1}_{s \leq T_j} \Bigg] e^{-r(s-t_0)} \left[1 - \Phi(t_0, s, \eta_{t_0}, \eta_B)\right] ds \tag{2.35}$$

For $\varepsilon = 0$, Tax Regime 1 is obtained and for $\varepsilon = 1$, the tax system does not allow for any tax loss carry-forwards.
Real tax systems are positioned somewhere between these extreme values for ε. By implementing the two extreme cases as a benchmark, the impact of tax systems on security values can be analyzed.[17]

We might consider a *realistic* tax system as a fourth tax regime where we keep track of a tax loss carry forward that accumulates if corporate earnings are negative and decreases again if corporate earnings turn positive again. An implementation of such a regime in a lattice seems possible but introduces path-dependence because the tax loss carry forwards accumulates differently across each path.

Note that the solution to the values of debt, equity and the government's claim in Tax Regime 1 and 2 reduces to solving for the firm value of equation (2.4), the bankruptcy probability of equation (2.9), and the bankruptcy price of equation (2.11).

2.1.6 Tax Advantage to Debt, and Traditional Firm Value Models

The blocks of Column 4 in Figure 2.1 show the respective value of debt D_{C_j,T_j} and equity E after taxes. These are equal to those prices that

[17] In fact, we demonstrate in Subsection 4.1.2.2 that security values in the two benchmark cases differ substantially. Taxes constitute a substantial portion of firm value overall.

market participants are willing to pay if arbitrage is excluded and the properties of the EBIT-process and its current value are known.

In traditional firm value models, the analysis does not start with the assumption of a stochastic process for EBIT but for an artificially unlevered firm value, i.e. the firm value of a fully equity financed firm. To be able to relate this stream of literature to our setup here, denote the artificially unlevered firm value by V_U which is assumed to follow an Itô process under the risk-neutral martingale measure $\mathbb{Q}$

$$dV_U = \mu_U(V_U, t)dt + \sigma_U(V_U, t)dz^{\mathbb{Q}}, \tag{2.36}$$

where $\mu_U(V_U, t)$ denotes the drift and $\sigma_U(V_U, t)$ the volatility of the process. In an arbitrage-free market and if the artificially unlevered firm value were a traded asset, its gain process, i.e. the payout to investors and the change in firm value, must equal the risk-free return on the investment. Denote by $\delta_U(V_U, t)$ the payout function to all claimants implying $\mu_U(V_U, t) = rV_U - \delta_U(V_U, t)$.

The issuance of debt is motivated by a tax advantage to debt TAD, defined as the corporate tax savings on coupon payments compared to the all equity firm. In Tax Regime 1, TAD is e.g.

$$TAD = \int_{t_0}^{\infty} \tau^{eff} \sum_{j=1}^{J} C_j \mathbf{1}_{\{s \le T_j\}} e^{-r(s-t_0)} \left[1 - \Phi(t_0, s, \eta_{t_0}, \eta_B)\right] ds. \tag{2.37}$$

Additionally, bankruptcy losses[18] BL must be considered so that the levered firm value equals

$$V_L = V_U + TAD - BL. \tag{2.38}$$

Taking differentials of equation (2.38) yields dynamics for the levered firm value to be

$$dV_L = dV_U + dTAD - dBL. \tag{2.39}$$

All dynamics in a risk-neutral setting must have instantaneous gains equal to the risk-free interest rate. The instantaneous drift of the different components of dV_L are then μ_U as defined above, $\mu_{TAD} = rTAD - \tau^{eff} \sum_{j=1}^{J} C_J \mathbf{1}_{\{s \le T_j\}}$, and $\mu_{BL} = rBL$, respectively. V_L itself has a drift of $\mu_L = rV_L - \delta_L(V_L, t)$. Equation 2.39 becomes

[18] Note that the bankruptcy losses BL here do not coincide with the bankruptcy cost BC defined in Subsection 2.1.3 because the reference artificially unlevered firm value V_U does not equal the firm value V as defined above.

$$
\begin{aligned}
rV_L - \delta_L(V_L,t) &= [rV_U - \delta_U(V_U,t)] \\
&\quad + \left[rTAD - \tau^{eff} \sum_{j=1}^{J} C_J \mathbf{1}_{\{s \le T_j\}} \right] - rBL \\
\delta_L(V_L,t) - \delta_U(V_U,t) &= \tau^{eff} \sum_{j=1}^{J} C_J \mathbf{1}_{\{s \le T_j\}}. \qquad (2.40)
\end{aligned}
$$

The difference in the drift rates of the levered and artificially unlevered firm value must be equal to the instantaneous corporate tax savings. Leland (1994), Leland and Toft (1996), Goldstein et al. (2001), and all other previously published traditional firm value models are not explicit on the implications of treating the artificially unlevered firm value as a traded security. Only Leland (1994) mentions in footnote 11 this *delicate issue*. As discussed above, a more rigorous treatment leads to a link between the payout functions of levered and unlevered firm value in equation (2.40). The traditional firm value setup obscures this interdependence. Fischer et al. (1989b) discuss a *no-arbitrage* condition on the drift of the value of unlevered assets A, which acts as their state variable and is comparable to V_U here. Their difference between the drifts is the advantage to leverage as is here. However, they do not interpret it as a cash flow.[19]

Traditional firm value models differ from EBIT-based models. Comparing the notion of firm value in our setting with the unlevered firm value V_U following the stochastic process of equation (2.36), the unlevered firm value V_U would include $\sum_{j=1}^{J} D_{C_j,T_j}$ and the investors' value of equity part 1, in Figure 2.1. The role of the bankruptcy cost BC in the tradition firm value setting is not quite clear. The tax savings of coupon payments are added to get the levered firm value. Assuming a stochastic process for this firm value does not necessarily lead to the same firm value process as implied by assuming a stochastic process for EBIT.[20]

[19] Note, that due to our discussion here, the numerical examples in Leland (1994) and Goldstein et al. (2001) need to be taken with care. If not all EBIT is paid out to security holders and the government, and the firm is able to save on EBIT, the whole model structure changes because one of the crucial assumptions is violated. Again, this point only becomes evident in an EBIT-based framework.

[20] If EBIT follows a geometric Brownian motion all security values become homogenous of degree 1 with respect to EBIT. The dynamics of η, V, V_U, V_L would therefore be identical. All other EBIT-process assumptions entail more complicated relationships.

As has been seen during the discussion of traditional firm value models, the interpretation of security values is much more difficult than in an EBIT-based framework where each security is defined as receiving a portion of the firm's EBIT.

2.1.7 Capital Restructuring and Optimal Bankruptcy

Until so far, the firm has an initial capital structure which reflects past issuing decisions. Due to the dilution of their respective claims, debt holders will not accept an increase of the current debt burden and equity holders will not deliberately buy back debt issues by selling new equity for small changes of the state variable. As a result, the current capital structure will maintain as long as the benefits of changing it are not high enough.

If equity holders issue finite maturity debt, they face refinancing decision whenever a debt issue matures. In traditional firm value models, this issue is often circumvented by assuming that the capital structure is sufficiently simple. Merton (1974) assumes that the firm is financed by a finite maturity zero bond and that the firm is liquidated at the bond's maturity. Most other authors[21] assume that callable perpetual debt is the second source of financing additional to equity. This simplifies the analysis considerably because all valuation formulas become independent of future dates. Unfortunately, only few firms constantly carry one single debt type. Leland and Toft (1996) derive closed form solutions for finite maturity debt issues. However, their refinancing assumption constitutes that a firm instantaneously retires maturing debt by reissuing the original finite maturity debt contract. This prevents equity holders to optimize the capital structure after the initial debt issue.[22]

Flor and Lester (2004) allow for a more flexible treatment of capital restructuring. They let equity holders optimally determine their restructuring dates by allowing the firm to issue finite maturity, eventually callable, debt. At each of the refinancing dates, i.e. when the debt

[21] See, e.g. Leland (1994), Goldstein et al. (2001), Mella-Barral (1999), only to mention a few authors.

[22] Dangl and Zechner (2004) use the same framework to consider voluntary debt reduction. As Leland (1994) already pointed out, without further incentives for equity holders, there is no immediate reason for equity holders to do so. In Dangl and Zechner (2004), equity holders can successfully trade off bankruptcy cost and debt calling cost by issuing short term debt. However, their results concerning the optimality of short-term debt are not convincing in the light of Flor and Lester (2004).

issue is called or matures, equity holders optimize the capital structure by choosing a debt maturity T, a coupon rate C, and a debt notional P, so that their claim is maximized.[23]

Before discussing the equity owners optimization problem, recall from Table 2.1 that if debt is not refinanced by reissuing new debt, equity owners as the residual claimants have to infuse money in case the existing EBIT does not suffice to cover the notional amount. This assumption implies that the firm cannot save for debt repayment and is not allowed to sell assets because the investment volume I is fixed by assumption. This is less stringent as it might appear at first sight. Assuming that residual EBIT of the solvent firm is available to equity owners irrespective of whether it is paid out as a dividend, the money for debt repayments need not come from an equity issue but can stem from internal financing.[24]

The assumption of debt being replaced by equity is one valid refinancing policy. However, a thorough modeling should include (1) an offering procedure for new debt issues to prospective debt holders by equity holders based on agency theory which fixes coupon and maturity and (2) an investment policy that fixes the necessary funds, i.e. the notional. The decisions are taken simultaneously and subject to equity owners discretion of maximizing the value of their claims.[25] If the second requirement is dropped, several conditions can be stated which need to be fulfilled if new debt is issued.[26]

Consider a series of increasing times of capital restructuring dates $\{t_i\}_{i=1}^{\infty}$, $t_i \leq t_{i+1}$. Pick the first date $t = t_1 = T_1$ as the date when the next debt issue matures whereby we denote $t-$ the instance just before the new debt issue and $t+$ the instance thereafter. Equity holders pick a refinancing strategy $\{C, P, T\}_{T_1}$ which replaces the existing debt. Note that, if $P < P_1$, equity holders need to infuse the amount $P_1 - P$ into the firm which depicts the case of voluntary debt reduction as described by Dangl and Zechner (2004).

Usually, the firm has other debt outstanding. In this case the total value of the still outstanding debt sums to

$$D_t(\cdot, V_B(s > t)) = \sum_{j=2}^{J} D_{C_j,T_j}(\cdot, \eta_B(s > t)).$$

[23] The parameter λ which denotes the call premium is dropped here to not deliberately complicate the exposition.

[24] Recall the discussion of Tax Regime 2.

[25] These considerations again introduce path dependence.

[26] See e.g. Flor and Lester (2004).

$D_t(\cdot, \eta_B(s > t))$ explicitly depends on the future bankruptcy strategy $\eta_B(s > t)$ because after the new debt issue the situation of equity changes. However, both equity holders and new debt investors incorporate the issuance decision and the ex-post behavior of equity holders beforehand.

Assume that a new debt issue does not lead to an immediate cash flow to equities but is reduced by $K(D(C,T))$. $K(D(C,T))$ includes issuing cost for the new debt as well as funds that need to stay within the firm. The debt notional cash flows then sum to $P-P_1-K(D(C,T))$. Equity holders try to maximize their equity value

$$\max_{(C,P,T)} E_{t+}$$

with

$$E_{t+} = E_{t-} + P - P_1 - K(D(C,T)) \tag{2.41}$$

conditional that, firstly, the incremental debt issue takes place at par

$$D(C,T) = P. \tag{2.42}$$

Secondly, equity holders optimize their own claim given the new debt package with respect to the bankruptcy level.

If equity holders choose bankruptcy optimally, when would they abandon further operation of the company? Recall that the firm's dividend policy forces equity investors to infuse capital to keep the level of invested capital constant. From the viewpoint of equity investors, they would only honor this call for money if e.g. the expected future value of their claim is worth more than their investment. If the expected net present value of equity is not positive, equity investors will refuse to pay and force the firm into bankruptcy. The firm initiates bankruptcy procedures. As the above argument shows, equity owners will choose a function η_B as to

$$\eta_B^* = \sup_{\eta_B} E(\eta_{t_0}, \cdot) \tag{2.43}$$

It can be shown that the optimal control η_B^* has to satisfy the value matching and smooth pasting conditions[27]

$$E(\eta_B^*, \cdot) = E^- \tag{2.44}$$

$$\left.\frac{\partial E(\eta_t, \cdot)}{\partial \eta}\right|_{\eta=\eta_B^*} = 0, \tag{2.45}$$

[27] See e.g. Dixit (1993).

meaning that equity owners will choose the optimal bankruptcy barrier so that continuing to fund the firm is no longer worthwhile.[28]

So, on the issuance date, the new bankruptcy rule becomes

$$\left.\frac{\partial E_{T_1+}}{\partial \eta}\right|_{\eta=\eta_B^*} = 0. \tag{2.46}$$

Thirdly, anticipating the optimal refinancing strategy at $t-$ yields value matching and smooth pasting conditions on each security. These conditions ensure that financial investors cannot exploit capital structure changes by arbitrage operations. Equity value matching is already achieved by identity (2.41). Debt value matching yields

$$D_{C_j,T_j}(t-) = D_{C_j,T_j}(t+), \tag{2.47}$$

for $j = 2, \ldots, J$ and smooth pasting by

$$\left.\frac{\partial E_{t-}}{\partial \eta}\right|_{\eta=\eta_t} = \left.\frac{\partial E_{t+}}{\partial \eta}\right|_{\eta=\eta_t}, \tag{2.48}$$

$$\left.\frac{\partial D_{C_j,T_j}(t-)}{\partial \eta}\right|_{\eta=\eta_t} = \left.\frac{\partial D_{C_j,T_j}(t+)}{\partial \eta}\right|_{\eta=\eta_t}. \tag{2.49}$$

Analytic solutions to this system of equations can only be obtained in restrictive settings.

In the literature, the above arguments are summarized under the topic of dynamic capital structure theory. Goldstein et al. (2001), Christensen et al. (2000), and Flor and Lester (2004) develop models where equity owners issue one single finite maturity or perpetual debt contract with a call feature. The call feature is the crucial difference to our setting here. If the firm is successful and EBIT increases, it might be profitable for equity owners to call the existing debt before its maturity and relever the firm to better exploit the tax advantage to debt. This kind of behavior can be modeled by introducing an upper EBIT-barrier at which the bond is replaced.[29]

To incorporate optimal calling features some adjustments to the current framework are necessary. In particular, probabilities of hitting the

[28] Note that we can express the same conditions using the bankruptcy firm value V_B, because the firm value in equation (2.4) is an invertible function of η_t. In some cases, this might be useful when discussing debt covenants which ensure that some value of the firm is left to secure debt holders.

[29] If more than one debt issue is outstanding, the bond is replaced whose tax advantage increases most given the cost of calling and issuing.

upper barrier and hitting prices for the upper barrier under the condition that the bankruptcy barrier is not hit before need to be introduced similar to those of the bankruptcy barrier in Subsection 2.1.3.[30] The bankruptcy barrier probabilities and prices need to be conditioned on the event of not hitting the upper barrier before bankruptcy. Optimality conditions for the barrier include the cost of calling the existing and issuing the new bond as well as smooth-pasting conditions on all continuing securities.

2.2 Remarks on Extension of the Framework

Structural credit risk models have been very popular in the past 10 years and the basic models of Black and Scholes (1973) and Merton (1974) have been extended into several directions. Section 2.1 has combined several of the most important developments into one consistent framework. However, some comments on the treatment of refinancing strategies which were made with reference to the structural firm value literature and the model of the bankruptcy event seem warranted.

2.2.1 Flexible Refinancing Policies

In recent years some authors[31] have suggested indirect approaches to refinancing strategies. Instead of modeling debt issues explicitly, a distance to default process is defined which is mean reverting and influenced by the current market condition or interest rate level. Default probabilities are then determined with respect to this process, i.e. bankruptcy occurs if the distance to default is zero. Despite the mathematically interesting setup and its tractability under the assumption of the firm value following a geometric Brownian motion, the concept is difficult to interpret economically. Cash flows to different claimants are disguised.[32] Especially the mean reverting version of cash flows assumes that debt is issued whenever the firm does well, but bought back if the firm's condition deteriorates. In our framework the extra cash flows to equity owners become obvious. The authors would need

[30] See e.g. Lando (2004), Section 3.3, where these models are described.

[31] See, e.g. Collin-Dufresne and Goldstein (2001), and Demchuk and Gibson (2004).

[32] Cash flows are of course not the primary focus of this kind of literature. However, the authors claim that their models are structural and thus economically founded. So, we do not criticize the modeling approach per se which might be well suited for pricing purposes but we draw attention to be careful about the direct economic interpretability of the models.

to make either adjustments to their valuation formulas for equity or they need the assumption that all this is done by changing the firm's assets. However, if the firm can adjust its assets easily, why does this only influence credit spreads and not the payout ratio to equity?

To be more precise with the argument. The firm value is assumed to follow a geometric Brownian motion. This implies that EBIT follows the same stochastic process. However, there is a payout ratio defined. In order to be compliant with the setting developed here, we must require that this payout ratio is equal to $\eta_t dt$. Luckily, EBIT is proportional to the firm value in a geometric Brownian motion setting. Therefore, we can think of $\delta V = \eta$, which implies that $\delta = r - \mu$ where δ is assumed to be constant. If interest rates are stochastic,[33] this relation does not hold any longer and the interpretation is blurred.

Our suggestions in Subsection 2.1.4 seem more appropriate to handle such cases.

2.2.2 Refinement of the Bankruptcy Model

A very rich literature has been developed to refine the model of bankruptcy. Mella-Barral (1999), Anderson and Sundaresan (1996), and Fan and Sundaresan (2000) describe models of strategic debt renegotiations. Their models follow the observation that in the case of bankruptcy, equity owners have an incentive to reduce debt service payments by an amount that debt holders will accept without triggering bankruptcy. Bankruptcy only occurs if the firm value deteriorates further and equity holders reduce coupon payments such that the incurring bankruptcy costs cannot threaten debt holders further. The behavior is the outcome of a game between debt and equity holders whose Nash equilibrium depends on the bargaining power of the two players. The bargaining power determines by how much equity holders can decrease coupon payments without triggering bankruptcy, i.e. how much equity holders can squeeze out of bankruptcy proceedings.

Francois and Morellec (2004) take this reasoning one step further. If a firm files for bankruptcy, it usually urges for creditor protection before actually initiating liquidation. Under this legislation, the firm continues to operate under court supervision over a certain period where the firm has a chance to reorganize and recover from a temporal financial distress. If successful and creditors consent, the firm stays in business. If the value of the firm deteriorates further until the end of the protection period, liquidation is triggered. In contrast to the previously

[33] Collin-Dufresne and Goldstein (2001) e.g. allow interest rates to follow the Vasicek (1977) dynamics.

mentioned literature on strategic debt service, Francois and Morellec (2004) model the protection period explicitly, finding that the existence of the protection period and the chance of debt renegotiations increases the protection triggering firm value and credit spreads.

Refined bankruptcy proceedings fit smoothly into our framework. We only need to reinterpret η_B to be the barrier at which the firm files for creditor protection. V_B is then the result of the bankruptcy game described above. The agents in the game will receive a fraction of the value according to the rules and equilibrium outcome of the game.

2.2.3 Investment Decisions

The connection between operational and purely financial decision are analyzed in Morellec (2001) and Morellec (2004). Morellec (2001) endogenizes the management's decision to sell assets by modeling the firm's profit function as stemming from a Cobb-Douglas production technology. Morellec (2004) allows the firm's manager to earn perquisites from investments. Thus the work scrutinizes on the agency conflict between stock and bondholders first described in Jensen and Meckling (1976).

Introducing such an investment decision in the Corporate Securities Framework implies that the drift and volatility functions of the EBIT-process $\mu(\cdot)$ and $\sigma_\eta(\cdot)$ are changing according to the current amount of the invested capital.[34] The financing decision can then be analyzed with the tools outlined in this chapter.

2.2.4 Unknown Initial EBIT - Incomplete Knowledge

Firm value models have often been criticized because the probability of bankruptcy in the near future is virtually zero if the firm starts above the bankruptcy barrier. This feature leads to almost credit risk-free short term corporate debt which is empirically not observed.[35] Duffie and Lando (2001) propose an economic setting where investors observe the current value of the state variable with an error. They proof that a default intensity and thus a non-zero credit spread for short term bonds exists. The model here can well be adjusted accordingly. One only needs to assume that η_{t_0} obeys a certain distribution with an expected value of $E(\eta_{t_0})$ and an estimation error ε_η. Note that in such

[34] See the discussion in Subsection 2.1.2.

[35] Note that other factors such as liquidity, differential tax treatments between government and corporate bonds, and interest rate dynamics might explain such differences in short-term credit spreads. See e.g. Longstaff, Mithal and Neis (2004) for some evidence in a reduced form framework.

a setting investors' knowledge has to be kept track of in a separate filtration of the probability space because η itself is not part of the investors' information set. Therefore, care has to be taken with respect to the measurability of security prices.

2.3 Summary

In this chapter, the economic foundation of structural credit risk models is reviewed. In contrast to the existing literature, the focus is purely economic. The mathematical model is carefully reasoned and assumptions are clearly stated.

The analysis starts with the firm's earnings before interest and taxes (EBIT) which introduces a notion of cash flow into the valuation of corporate securities. In Subsection 2.1.2 the separation of operating and financial decisions is discussed which leads to a natural definition of firm value as the discounted future EBIT. After introducing a bankruptcy barrier, a complex capital structure with several finite maturity debt issues, and a rich tax regime, market values of each debt issue, the firm's equity, and the government's claim have been derived in Subsection 2.1.5. Subsection 2.1.7 raises the issue of optimal decisions within the framework. However, we note that explicit solutions to the system of equations are difficult to find.

The framework extends the current literature into several directions. Firstly, we consider a complex capital structure with several finite maturity debt issues. In the literature, authors refrain from working with more than one bond issue because it complicates the exposition. Additionally, several authors claim that the extension to several bond issues is an easy exercise. In contrast, we show that a careful model of bankruptcy is needed to keep track of the capital structure in future subperiods. Secondly, the framework is defined quite generally without specifying the functions of the EBIT-drift and volatility explicitly. We are able to demonstrate that an analytical solution of this framework reduces to solving the firm value equation (2.4), the probabilities of bankruptcy of equation (2.9), and the Arrow-Debreu bankruptcy price in equation (2.11). An explicit solution of optimal bankruptcy and capital structure discussed in Subsection 2.1.7 depends crucially on the EBIT-process assumption and the complexity of the capital structure. Thirdly, all claims are easily interpreted because they are all defined as receiving payments from EBIT. Finally, due to its generality extensions can focus on equations (2.4), (2.9), and (2.11). All other valuation formulae remain intact.

3

ABM- and GBM-EBIT-Models

If the economic framework of the last chapter is used for pricing corporate securities, a specific assumption about the EBIT-process has to be made. Commonly, authors assumed that EBIT follows a geometric Brownian motion (GBM) because of its tractability. All claims become homogeneous of degree 1 in the initial EBIT η_{t_0}. Economically, the choice of GBM is debateable because EBIT is severely restricted by this assumption.

We propose to assume the EBIT-process to follow an arithmetic Brownian motion (ABM) instead. Although the mathematical properties of the valuation formulae are not as nice as in the GBM-case and need numerical methods at some points, we get remarkably far.

Therefore, this chapter is structured as follows: Before deriving the explicit and numerical solutions for the ABM-case in Section 3.2 and for the GBM-case in Section 3.3 for comparison, we devote some effort in detailing the arguments for the process assumption in the following Section 3.1. Section 3.4 discusses numerical extensions to the analytical settings to be able to value derivative securities on corporate securities and introduce refined decision making.

3.1 Arithmetic vs. Geometric Brownian Motion

One of the major advantages of starting the analysis of structural credit risk models in a very general way as done in Chapter 2 is that the economic content is easily accessible and therefore economic inconsistencies become apparent. Due to its independence of a specific process assumption for EBIT, we can predicate the choice of EBIT-process on economic arguments.

Assuming that EBIT follows a geometric Brownian motion has the distinct disadvantage, that EBIT cannot become negative. In order to start solvent the initial EBIT must lie above zero and consequently all consecutive EBITs remain positive. This does not imply that equity owners are not asked to infuse money into the firm, since coupons, taxes, and debt repayments reduce the instantaneous dividend to equity owners and can turn it negative. However, two distinct situations are excluded from the analysis: First, newly founded firms which usually start with negative EBIT are inaccessible for the theory. Second, the firm itself will never get into a state where it incurs operational losses.

Whereas the first case only excludes a class of firms over the period until they exit the founding phase, the second case affects all firms. It would not be problematic if it can be observed that firms go bankrupt before they incur operational losses. However, prominent examples of distressed companies around the world violate this assumption. In contrast, debt and equity holders allow a firm to accumulate a quite substantial amount of operating losses even without severe restructuring until they send a firm into bankruptcy. The rationale is that there is a chance of a turnaround and future profits compensate for the current additional funding.

In general, variables following a geometric Brownian motion are expected to grow at an exponential rate equal to the drift function. It might be reasonable to assume that this is the case for quick growing firms – probably newly established firms after they have broken even. However, what about well established firms that generate already a substantial EBIT in highly competitive markets? It appears not appropriate to assume such a growth rate for EBIT.

As a result, from an economical point of view growth firms can be reasonably modeled with geometric Brownian motion EBITs.

The reverse is true for arithmetic Brownian motion. In this case, we do not suffer the non-negativity constraint of GBM. However, the growth perspective is limited to being only additive which seems to be plausible for all firms except quickly expanding growth firms. However, as we will see in the next section, a high firm value can be attributed either to a high EBIT growth or to a high current EBIT. Such a distinction is impossible in firm value models of geometric Brownian motion.[1]

[1] This distinction effectively explains why young firms in the technology sector attained such high values despite the huge losses they incurred. Their growth potential was judged high. However, this opinion was revised before the technology bubble burst.

As a final judgement, arithmetic Brownian motion EBITs appear a more natural assumption as geometric Brownian motion. For this reason, we solve the ABM-case first and state the respective GBM-results for comparison in Section 3.3.

3.2 The Basic ABM-EBIT-Model

This section develops the analytical solution of the economic framework with the assumption that EBIT follows an arithmetic Brownian motion.

3.2.1 EBIT-Process and Firm Value

Define an economy as in Chapter 2 in which the firm generates an EBIT-flow of the form

$$d\eta = \mu dt + \sigma_\eta dz^{\mathbb{Q}}, \tag{3.1}$$

with the risk-neutral drift μ and the standard deviation σ_η as constants. $dz^{\mathbb{Q}}$ denotes a standard Brownian motion under the risk-neutral martingale measure $\mathbb{Q}$. The current EBIT is η_{t_0}.

A claim V that receives $\eta_t dt$, $t > t_0$ forever, must have a law of motion by Itô's lemma of

$$dV = \frac{\partial V}{\partial \eta} d\eta + \frac{1}{2}\frac{\partial^2 V}{\partial \eta^2}\langle d\eta^2 \rangle. \tag{3.2}$$

By arbitrage arguments, the expected capital gain on the claim V must equal a risk free return so that

$$\eta dt + dV = rV dt. \tag{3.3}$$

Guess that the solution of this differential equation is linear in the variable η, i.e. $V = A + B\eta$. Inserting equation (3.1), its square root, and the derivatives of the proposed solution into (3.2), the parameters A and B can be solved for together with equation (3.3) to find a firm value of

$$V = \frac{\mu}{r^2} + \frac{\eta_{t_0}}{r}. \tag{3.4}$$

By Itô's lemma, the process of the firm value is given by

$$dV = \frac{1}{r}\left[\mu dt + \sigma_\eta dz^{\mathbb{Q}}\right], \tag{3.5}$$

and thus follows the same dynamics as EBIT up to the factor $1/r$.

If the detour via the physical measure $\mathcal{P}$ is taken, the $\mathcal{P}$-process of EBIT is assumed to follow

$$d\eta = \mu_\eta dt + \sigma_\eta dz^{\mathcal{P}}, \tag{3.6}$$

where μ_η is the constant physical drift and $dz^{\mathcal{P}}$ is a standard Brownian motion under the measure $\mathcal{P}$. The existence of a martingale measure $\mathcal{Q}$ implies a risk premium θ where $\mu = \mu_\eta - \theta\sigma_\eta$. Together with equation (3.4), this implies a risk premium of

$$\theta = \frac{\mu_\eta - r}{\sigma_\eta} + \frac{r + r \cdot \eta_{t_0} - r^2 \cdot V}{\sigma_\eta}. \tag{3.7}$$

Equation (3.7) has an additional term compared to the usual risk premium of a traded asset. However, substituting equation (3.7) into equation (3.5) yields

$$dV + \eta dt = rVdt + \frac{\sigma_\eta}{r} dz^{\mathcal{Q}}. \tag{3.8}$$

Therefore the drift of the risk-neutral firm value process is equivalent to the risk-free return if adjusted for intermediate cash flows. The risk premium in equation (3.7) is constant and does not depend upon the stochastic factor η. The first derivative of the risk premium with respect to η is zero.

3.2.2 The Case of a Single Perpetual Debt Issue

Consider as a start a capital structure discussed in Goldstein et al. (2001) or Leland (1994) where the firm only issues a single perpetual debt.

3.2.2.1 The Value of Debt, Equity, and the Government's Claim

From standard textbooks on differential equations[2], any claim F with a regular payment flow $f(\eta_t)$ to investors depending on EBIT η must satisfy the partial differential equation (PDE)

$$\mu \cdot F_\eta + \frac{(\sigma_\eta)^2}{2} \cdot F_{\eta\eta} + F_t + f(\eta_t) = r \cdot F. \tag{3.9}$$

[2] See e.g. Shimko (1992).

For perpetual claims, such as preferred dividends and perpetual debt, equation (3.9) simplifies to a time-independent version. The PDE becomes an ordinary differential equation (ODE)

$$\mu \cdot F_{\eta} + \frac{(\sigma_{\eta})^2}{2} \cdot F_{\eta\eta} + f(\eta_t) = r \cdot F. \tag{3.10}$$

The general solution[3] of the homogenous ordinary differential equation disregarding the regular cash flows $f(\eta_t)$

$$\mu \cdot F_{\eta} + \frac{(\sigma_{\eta})^2}{2} \cdot F_{\eta\eta} - r \cdot F = 0 \tag{3.11}$$

is given by

$$F = A_1 \cdot e^{-k_1 \cdot \eta_t} + A_2 \cdot e^{-k_2 \cdot \eta_t} \tag{3.12}$$

with

$$k_{1/2} = \frac{\mu \mp \sqrt{\mu^2 + 2r\,(\sigma_{\eta})^2}}{(\sigma_{\eta})^2}. \tag{3.13}$$

k_1 is negative, k_2 positive. If η_t becomes large, the exponential involving k_1 in equation (3.12) goes to infinity whereas the second exponential involving k_2 converges to zero.

As noted above, the general solution does not account for intermediate cash flows. To find a solution for a specific security, the present value of the specific security's payments to investors must be accounted for. This particular solution F^* is added to the general solution given by equation (3.12). The parameters A_1 and A_2 are determined by boundary conditions of the security under consideration.

Consider first the Arrow-Debreu bankruptcy claim $p_B(t_0, \infty, \eta_{t_0}, \eta_B)$ from Subsection 2.1.3. The constant firm value at bankruptcy is V_B as defined in equation (3.4) with $\eta_{t_0} = \eta_B$.[4] This claim has no maturity date in the current setting nor intermediate cash flows. Therefore the general solution (3.12) can be used directly.

For the Arrow-Debreu bankruptcy price, there exist two boundary conditions. If EBIT η_t increases, the security's value has to converge to zero, since it becomes less likely that the firm goes bankrupt. If EBIT approaches η_B, the security price must reach one. Both conditions imply that $A_1 = 0$, $A_2 = \exp(k_2 \cdot \eta_B)$, and that

[3] See e.g. Shimko (1992, p. 34 ff.).

[4] It is safe to assume that η_B is constant in an infinite maturity setting. Optimality arguments proof later that η_B^* is in effect a constant control.

$$p_B(t_0, \infty, \eta_{t_0}, \eta_B) = e^{-k_2 \cdot (\eta_{t_0} - \eta_B)}. \tag{3.14}$$

Therefore, the insolvent part of the firm as defined in equation (2.14) is valued by

$$V^- = V_B \cdot p_B(t_0, \infty, \eta_{t_0}, \eta_B) \tag{3.15}$$

and the solvent part as

$$V^+ = V - V_B \cdot p_B(t_0, \infty, \eta_{t_0}, \eta_B). \tag{3.16}$$

The solvent value of a perpetual debt issue before taxes $V^+_{C,\infty}$ can be determined similarly. The particular solution $F^*_{C,\infty} = C/r$ is the value of the coupon payments to the debt holders until infinity. The solution of the non-homogenous differential equation is $F + F^*$ or

$$V^+_{C,\infty} = \frac{C}{r} + A_1 \cdot e^{-k_1 \cdot \eta_{t_0}} + A_2 \cdot e^{-k_2 \cdot \eta_{t_0}}. \tag{3.17}$$

Limit conditions $\lim_{\eta_t \to \infty} V^+_{C,\infty} = C/r$ and $\lim_{\eta \to \eta_B} V^+_{C,\infty} = 0$ lead to $A_1 = 0$ and $A_2 = -\frac{C}{r} \cdot \exp(k_2 \cdot \eta_B)$ and

$$\begin{aligned} V^+_{C,\infty} &= \frac{C}{r} - \frac{C}{r} \cdot e^{k_2 \cdot \eta_B} \cdot e^{-k_2 \cdot \eta_{t_0}} \\ &= \frac{C}{r} \cdot \left[1 - e^{-k_2 (\eta_{t_0} - \eta_B)}\right] \\ &= \frac{C}{r} \cdot \left[1 - p_B(t_0, \infty, \eta_{t_0}, \eta_B)\right]. \end{aligned} \tag{3.18}$$

The equity value before taxes then becomes

$$V^+_E = V^+ - V^+_{C,\infty}.$$

Extending the analysis to Tax System 1 as defined in Subsection 2.1.5, we find that the investor values the solvent part of the corporate perpetual $D^+_{C,\infty}$ as

$$D^+_{C,\infty} = (1 - \tau^d) \cdot V^+_{C,\infty} \tag{3.19}$$

because $V^+_{C,\infty}$ is proportional to the coupon level C and thus to taxation itself. A principal repayment, which is usually treated differently for tax purposes, does not appear in the debt value. The solvent part of equity E^+ becomes[5]

[5] Note that $V^{E+}_{C,\infty} = (1 - \tau^c) V^+_{C,\infty}$.

$$E^{+} = (1 - \tau^{eff}) \cdot (V^{+} - V^{+}_{C,\infty}). \tag{3.20}$$

The government's share of the solvent firm's value amounts to

$$G^{+} = \tau^{eff} \cdot (V^{+} - V^{+}_{C,\infty}) + \tau^{d} \cdot V^{+}_{C,\infty}. \tag{3.21}$$

The first term can be interpreted as the tax payment due to corporate earnings and dividends, the second term as the tax payments on coupons. The sum of all three claims in equations (3.19) to (3.21) is equal to the firm's solvent value as defined in equation (3.16). Thus

$$V^{+} = D^{+}_{C,\infty} + G^{+} + E^{+}.$$

If the bankruptcy level is set to η_B, the firm value in the case of bankruptcy is given by equation (3.15) which adds up to total firm value given by equation (3.4) when summed with the solvent firm value given by equation (3.16):

$$V = V^{+} + V^{-}.$$

Usually, it is assumed that there are costs associated with bankruptcy. Set in equation (2.15), $\alpha_1 = 0$ and $\alpha_2 = \alpha$, so that a portion α of V_B is lost when the firm abandons operations. Bankruptcy costs are valued at

$$BC = \alpha \cdot V^{-}. \tag{3.22}$$

By assumption, debt holders become the new owners of the firm. Pick a V_B sufficiently low such that not all of the claims senior to equity can be honored and equity holders receive nothing ($E^{-} = 0$). Because the new owners are subject to the corporate tax rate, they value their claim as

$$D^{-}_{C,\infty} = (1 - \alpha) \cdot (1 - \tau^{eff}) \cdot V^{-}. \tag{3.23}$$

The government again receives the tax payments, namely

$$G^{-} = (1 - \alpha) \cdot \tau^{eff} \cdot V^{-}. \tag{3.24}$$

The three claims described in equations (3.22), (3.23), and (3.24) again sum to the value of the insolvent firm, as specified in equation (3.15)

$$V^{-} = D^{-}_{C,\infty} + G^{-} + BC.$$

3.2.2.2 The Optimal Bankruptcy Level and Coupon

Under the current restrictive capital structure we can continue the exposition by considering the optimal choice of the bankruptcy level η_B and coupon C. As suggested in Subsection 2.1.7, take the debt characteristics as given and maximize equity value with respect to the bankruptcy barrier η_B. The first derivative of equity with respect to EBIT evaluated at the bankruptcy barrier as in equation (2.45) is given by

$$\begin{aligned}\left.\frac{\partial E}{\partial \eta}\right|_{\eta=\eta_B} &= 0 \\ &= (1-\tau^{eff})\left[\frac{1}{r} + k_2\left(V_B - \frac{C}{r}\right)e^{-k_2(\eta_{t_0}-\eta_B)}\right]\Bigg|_{\eta=\eta_B} \\ &= \frac{1}{r} + k_2\left(V_B - \frac{C}{r}\right). \qquad (3.25)\end{aligned}$$

By noting that V_B is defined as

$$V_B = \frac{\eta_B}{r} + \frac{\mu}{r^2},$$

the optimal bankruptcy barrier results as

$$\eta_B^* = C - \frac{\mu}{r} - \frac{1}{k_2}. \qquad (3.26)$$

Consider next the capital structure decision of equity owners. The capital structure is simple so that only one perpetual debt issue is admissible. Before issuing debt, the firm is fully equity financed. Equity owners optimize their equity value at the issuing date. They will receive E as a promise of future residual payments and a cash inflow from the debt issue $P = D_{C,\infty}$ because debt is issued at par. Furthermore, no maturity decision is needed in the perpetual debt case. So the optimization problem is equivalent to optimizing $(1-k)D_{C,\infty} + E$ with respect to the coupon C, where k denotes the portion of the issuing amount which is not available to equity investors. Using equation (3.26), the debt and equity derivative with respect to the coupon level become

$$\begin{aligned}\frac{\partial D_{C,\infty}}{\partial C} = \frac{1-\tau^d}{r} + e^{-k_2(\eta_{t_0}-\eta_B^*)}\Bigg[&k_2\frac{C}{r}\left[(1-\alpha)(1-\tau^{eff}) - (1-\tau^d)\right] \\ &- \frac{1-\tau^d}{r}\Bigg]. \qquad (3.27)\end{aligned}$$

and

$$\frac{\partial E}{\partial C} = \frac{1-\tau^{eff}}{r}\left(-1+e^{-k_2(\eta_{t_0}-\eta_B^*)}\right). \tag{3.28}$$

The optimal coupon C^* solves the equation $(1-k)\partial D_{C,\infty}/\partial C + \partial E/\partial C$. C^* cannot be stated explicitly but is the unique solution to the Lambert W function

$$(1-\tau^{eff})-(1-k)(1-\tau^d) = e^{-k_2(\eta_{t_0}-\eta_B^*)}\Big[(1-\tau^{eff}) \\ +(1-k)\Big(C^*[(1-\alpha)(1-\tau^{eff}) \\ -(1-\tau^d)]-(1-\tau^d)\Big)\Big]. \tag{3.29}$$

Equation (3.29) can be solved for η_{t_0}.[6] Then η_{t_0} is strictly monotonically increasing with the coupon level C^*. Therefore, C^* can easily be found numerically.

3.2.2.3 Minimum Optimal Coupon and Asset Substitution

Two special cases are interesting. Consider first, that the debt issue is needed for investments so that μ and σ_η can be realized in the future. Then $k=1$, and the optimal coupon becomes

$$C^* = k_2\eta_{t_0} + \frac{\mu}{r} + \frac{1}{k_2}. \tag{3.30}$$

The C^* of equation (3.30) is the lowest coupon equity owners pay given the current EBIT. Whenever proceeds from the debt issue can be paid out as a dividend immediately, equity owners are willing to accept a higher coupon.

Second, consider the asset substitution problem raised by Jensen and Meckling (1976) and discussed in Leland (1994) and Leland (1998). Equity owners might reallocate investments after debt is issued into projects that are more risky. Jensen and Meckling (1976) claim that this strategy increases the equity value and harms debt holders due the abandonment option inherent in the equity contract. Leland (1994) shows in a geometric Brownian motion setting that the agency problem only exists if equity holders may choose the bankruptcy level optimally. If a value covenant is enforced, there is no incentive for equity holders to shift risk.

[6] Note that by equation(3.26), C^* appears in the exponent, as well.

Disregarding effects on the risk-neutral drift μ if the risk of the firm σ_η is changed,[7] the effects of an increase in firm risk depend crucially on the parameter k_2. Changing the firm's risk, decreases k_2.

$$\frac{\partial k_2}{\partial \sigma_\eta} = -2\frac{\mu\sqrt{\mu^2 + 2r\sigma_\eta^2} + \mu^2 + r\sigma_\eta^2}{\sigma_\eta^3\sqrt{\mu^2 + 2r\sigma_\eta^2}}$$

has a positive nominator if $\mu > 0$ and if $\mu < 0$ and $|\mu| > \sqrt{r/2}\sigma_\eta$. The second condition is relatively loose. If k_2 has a negative derivative, the optimal bankruptcy barrier decreases, as well because

$$\frac{\partial V_B^*}{\partial \sigma_\eta} = \frac{1}{rk_2^2}\frac{\partial k_2}{\partial \sigma_\eta}.$$

Therefore, the effect on the bankruptcy price becomes ambiguous

$$\frac{\partial p_B(t_0, \infty, \eta_{t_0}, \eta_B^*)}{\partial \sigma_\eta} = -\frac{\partial k_2}{\partial \sigma_\eta}\left(\eta - \eta_B^* - \frac{1}{k_2}\right) p_B(t_0, \infty, \eta_{t_0}, \eta_B^*),$$

because it depends on the sign of $(\eta - \eta_B^* - 1/k_2)$. For

$$V \gtreqless \frac{C}{r} \Rightarrow \frac{\partial p_B(t_0, \infty, \eta_{t_0}, \eta_B^*)}{\partial \sigma_\eta} \gtreqless 0.$$

Equity owners only have an incentive to increase risk if the equity derivative with respect to σ_η is positive.

$$\frac{\partial E}{\partial \sigma_\eta} = -(1 - \tau^{eff})\frac{\partial k_2}{\partial \sigma_\eta} p_B(t_0, \infty, \eta_{t_0}, \eta_B^*)\left[\frac{1}{rk_2^2} - \left(\eta_{t_0} - \eta_B^* - \frac{1}{k_2}\right)\right]$$

The positivity condition reduces to

$$\frac{1}{k_2^2} > r^2\left(V - \frac{C}{r}\right)$$

which is naturally fulfilled if the firm value is low $V_B^* < V < C/r$. For $V > C/r$, risk can be optimized because the equity derivative with respect to the EBIT-volatility can become zero, i.e. equity owners choose an optimal risk level which solves

$$k_2 = (r(rV - C))^{-\frac{1}{2}}.$$

[7] The following results depend on μ being independent of σ_η. It is assumed that the risk premium θ changes accordingly to ensure that independence.

For $V > C/r$-firms, risk optimization becomes possible because the firm can trade off 2 effects. By increasing risk, the bankruptcy probability rises thus lowering debt value. However, future tax savings become less valuable and the solvent firm value decreases. The lower the firm value, the less important is the second effect.

If a debt covenant is negotiated, equity owners contract on a bankruptcy value level V_B^C and implicitly on η_B^C. The equity derivative then becomes

$$\frac{\partial E}{\partial \sigma_\eta} = -(1-\tau^{eff})\frac{\partial k_2}{\partial \sigma_\eta} p_B(t_0, \infty, \eta_{t_0}, \eta_B^C)\left(V_B^C - \frac{C}{r}\right)$$

which is negative if the covenant level is $V_B^C < C/r$. So equity holders always have the incentive to reduce the firm's risk because risk harms their claim by bringing the firm closer to bankruptcy.[8]

3.2.3 Finite Maturity Debt and Multiple Financing Sources

The model discussed in Section 3.2.2 cannot describe a term structure of corporate credit spreads because only perpetual debt was considered. To describe the value of debt with maturities less than infinity, we use the setup of Subsection 2.1.4.

3.2.3.1 Bankruptcy Probabilities and Claims with Finite Maturities

For the derivation of time-inhomogeneous security prices in the current setting, the infinite maturity bankruptcy claim of the last subsection is not sufficient. As discussed in Subsection 2.1.3, bankruptcy probabilities and claim's prices for all future points in time are needed when the capital structure changes.

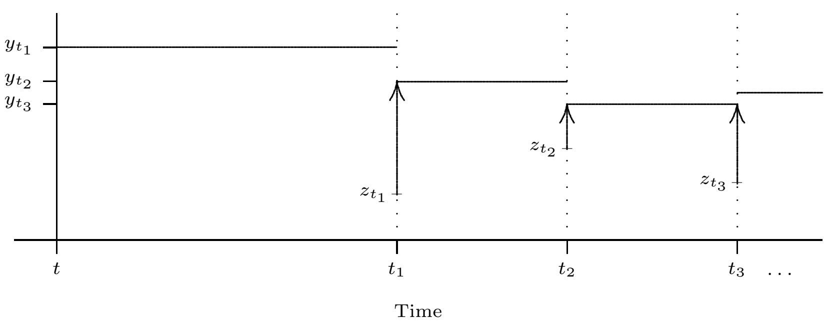
Without loss of generality, the exposition is restricted to the case where an underlying variable X follows an arithmetic Brownian motion with drift ν and standard deviation σ. The current value $X_{t_0} = 0$ and the barrier level is denoted by a sequence of barriers y_{t_j} which are constant in the time interval $]t_{j-1}, t_j]$ for $j = 1 \ldots n$, and $t_n = T$ being the final maturity.[9]

[8] In effect, this result is independent of the process assumption. Leland (1994) shows the same effect for geometric Brownian motion.

[9] Above it is argued that we set $X = \eta$ and take $\nu = -\mu$ to find hitting probabilities with a lower barrier.

Figure 3.1 illustrates such a setting with three subperiods. The barriers might be changing arbitrarily at t_j but remain constant within each subperiod. Hitting probabilities are derived by calculating the complementary probability of not hitting the barrier before or at t_j. For the first subperiod $]t_0, t_1]$, the probability of not hitting the barrier until t_1 is equivalent to the process staying below y_{t_1} and ending at t_1 below y_{t_2}. To derive this probability, $z_{t_1} \leq \min(y_{t_1}, y_{t_2})$ is first chosen arbitrarily so that the next subperiod can be entered without going bankrupt. Letting, $z_{t_1} \rightarrow \min(y_{t_1}, y_{t_2})$ yields the desired result. One can repeat the same kind of analysis by including the next subperiod until we reach the maturity T.

Fig. 3.1. Time structure of barrier levels



To ease the derivation of the probabilities of not hitting the barrier, define the process of the running maximum by M_{t_i} with

$$M_{t_i} = \sup_{t_{i-1} < s < t_i} X_s. \tag{3.31}$$

The joint probability of surviving the first period $]t_0, t_1]$ and ending at t_1 below a certain level z_{t_1} can be written by the law of total probability as

$$\begin{aligned} P_\nu(X_{t_1} \leq z_{t_1}, M_{t_1} < y_{t_1}) = P_\nu(X_{t_1} \leq z_{t_1}) \\ -P_\nu(X_{t_1} \leq z_{t_1}, M_{t_1} \geq y_{t_1}), \end{aligned} \tag{3.32}$$

where the subscript indicates that the probability is taken with respect to the probability measure that has drift ν. The first term is the value of cumulative normal distribution with X_{t_1} $N(\mu(t_1 - t_0), \sigma^2(t_1 - t_0))$.

Harrison (1985) derives the second term by using the reflection principle for a drift-less Brownian motion under the probability measure P_0, then differentiating the probability with respect to z_{t_1}, and lastly applying Girsanov's theorem to change the measure from P_0 to P_ν.

The probability that the process X hits the barrier y_{t_1} and ends up below z_{t_1} at the end of the first period, denoted by $\Phi(t_0, t_1, y_{t_1}, z_{t_1}, \nu)$, can then be found as

$$\begin{aligned}\Phi(t_0, t_1, y_{t_1}, \nu) &= 1 - P_\nu(X_{t_1} \leq z_{t_1}, M_{t_1} < y_{t_1}) \\ &= N(h_1) + e^{\frac{2\nu}{\sigma^2} \cdot y_{t_1}} N(h_2) \end{aligned} \tag{3.33}$$

with

$$\begin{aligned} h_1 &= \frac{-z_{t_1} - \nu \cdot (t_1 - t_0)}{\sigma \sqrt{t_1 - t_0}}, \\ h_2 &= \frac{z_{t_1} - 2y_{t_1} + \nu \cdot (t_1 - t_0)}{\sigma \sqrt{t_1 - t_0}}, \end{aligned}$$

where $N(\cdot)$ denotes the cumulative univariate standard normal distribution function. For $z_{t_1} \to \min(y_{t_1}, y_{t_2})$, we get the desired hitting probability in the subperiod $]t_0, t_1]$.

The above mentioned procedure can be extended to more than one subperiod by iteratively using the law of total probability until we are able to apply the reflection principle. Before stating the general result, it is helpful to illustrate the derivation for $n = 3$.[10] Hereby, we use the convention that the index j denotes events such that $P(\mathcal{B}_j) = P(\bigcup_{i=1}^{n} \mathcal{B}_i)$.[11] Starting again with the joint probability of not hitting the barrier in the three subperiods and ending up at each subperiod below z_{t_i}, one gets

[10] See Carr (1995), Appendix A2, for an illustration of the bivariate case where the barrier is constant over the subperiods.

[11] As an example, consider the event $\mathcal{B}_i = \{X_{t_i} \leq z_{t_i}\}$ of the stochastic variable X_{t_i} being below z_{t_i} at t_i. Then, $\mathcal{B}_j$ denotes the joint event of all $\mathcal{B}_i$ for all $i = 1, \ldots, j$.

$$
\begin{aligned}
&P_0(X_{t_j} \le z_{t_j}, M_{t_j} < y_{t_j}) \\
=&P_0(X_{t_j} \le z_{t_j}, M_{t_i} < y_{t_i}, i = \{1,2\}) \\
&\quad -P_0(X_{t_j} \le z_{t_j}, M_{t_i} < y_{t_i}, i = \{1,2\}, M_{t_3} \ge y_{t_3}) \\
=&P_0(X_{t_j} \le z_{t_j}, M_{t_1} < y_{t_1}) - P_0(X_{t_j} \le z_{t_j}, M_{t_1} < y_{t_1}, M_{t_2} \ge y_{t_2}) \\
&- \big[P_0(X_{t_j} \le z_{t_j}, M_{t_1} < y_{t_1}, M_{t_3} \ge y_{t_3}) \\
&\quad - P_0(X_{t_j} \le z_{t_j}, M_{t_1} < y_{t_1}, M_{t_i} \ge y_{t_i}, i = \{2,3\})\big] \\
=&P_0(X_{t_j} \le z_{t_j}) - P_0(X_{t_j} \le z_{t_j}, M_{t_1} \ge y_{t_1}) \\
&- \big[P_0(X_{t_j} \le z_{t_j}, M_{t_2} \ge y_{t_2}) - P_0(X_{t_j} \le z_{t_j}, M_{t_i} \ge y_{t_i}, i = \{1,2\})\big] \\
&- \big\{P_0(X_{t_j} \le z_{t_j}, M_{t_3} \ge y_{t_3}) - P_0(X_{t_j} \le z_{t_j}, M_{t_i} \ge y_{t_i}, i = \{1,3\}) \\
&\quad - \big[P_0(X_{t_j} \le z_{t_j}, M_{t_i} \ge y_{t_i}, i = \{2,3\}) \\
&\qquad -P_0(X_{t_j} \le z_{t_j}, M_{t_i} \ge y_{t_i}, i = \{1,2,3\})\big]\big\}\,. \qquad (3.34)
\end{aligned}
$$

By reflecting the process at the first and all consecutive barriers, equation (3.34) can be reexpressed by a tri-variate cumulative normal distribution.

$$
\begin{aligned}
&P_0(X_{t_j} \le z_{t_j}, M_{t_j} < y_{t_j}) = P_0(X_{t_j} \le z_{t_j}) - P_0(-X_{t_j} \le z_{t_j} - 2y_{t_1}) \\
&- \left[P_0\begin{pmatrix} X_{t_1} \le z_{t_1} \\ -X_{t_2} \le z_{t_2} - 2y_{t_2} \\ -X_{t_3} \le z_{t_3} - 2y_{t_2} \end{pmatrix} - P_0\begin{pmatrix} -X_{t_1} \le z_{t_1} - 2y_{t_1} \\ X_{t_2} \le z_{t_2} + 2(y_{t_1} - y_{t_2}) \\ X_{t_3} \le z_{t_3} + 2(y_{t_1} - y_{t_2}) \end{pmatrix}\right] \\
&- \left\{P_0\begin{pmatrix} X_{t_i} \le z_{t_i}, i = \{1,2\} \\ -X_{t_3} \le z_{t_3} - 2y_{t_3} \end{pmatrix}\right. \\
&\qquad -P_0\begin{pmatrix} -X_{t_i} \le z_{t_i} - 2y_{t_1}, i = \{1,2\} \\ X_{t_3} \le z_{t_3} + 2(y_{t_1} - y_{t_3}) \end{pmatrix} \\
&- \left[P_0\begin{pmatrix} -X_{t_1} \le z_{t_1} \\ -X_{t_2} \le z_{t_2} - 2y_{t_2} \\ X_{t_3} \le z_{t_3} + 2(y_{t_2} - y_{t_3}) \end{pmatrix}\right. \\
&\qquad \left.\left. -P_0\begin{pmatrix} -X_{t_1} \le z_{t_1} - 2y_{t_1} \\ X_{t_2} \le z_{t_2} + 2(y_{t_1} - y_{t_2}) \\ -X_{t_3} \le z_{t_3} - 2(y_{t_1} - y_{t_2} + y_{t_3}) \end{pmatrix}\right]\right\}. \qquad (3.35)
\end{aligned}
$$

The consecutive reflection of the process X at the barriers can best be illustrated by the triple reflection of the last probability which is depicted in Figure 3.2. Note that the first reflection at y_{t_1} turns the initial upper barrier y_{t_2} into a lower barrier $2y_{t_1} - y_{t_2}$ which is also reflected at y_{t_1}. The second probability limit $2y_{t_1} - 2y_{t_2} + z_{t_2}$ can be derived by the same argument as the first, i.e. reflection of the already once reflected process at the lower barrier $2y_{t_1} - y_{t_2}$. The second reflection turns the third barrier y_{t_3} into an upper barrier located at $2y_{t_1} - 2y_{t_2} + y_{t_3}$ with a reflected probability limit of $2y_{t_1} - 2y_{t_2} + 2y_{t_3} - z_{t_3}$. Taking the correct inequalities into account leads to the above tri-variate probability.

The general structure of equations (3.34) and (3.35) can be summarized by

$$P_0(X_{t_j} \leq z_{t_j}, M_{t_j} < y_{t_j}) = \sum_{i \in \mathcal{E}} \sum_{\mathcal{A}_i} P_0(X_{t_j} \leq z_{t_j}, M_{t_i} \geq y_{t_i}, i \in \mathcal{A}_i)$$
$$- \sum_{i \in \mathcal{U}} \sum_{\mathcal{A}_i} P_0(X_{t_j} \leq z_{t_j}, M_{t_i} \geq y_{t_i}, i \in \mathcal{A}_i), \tag{3.36}$$

where the set $\mathcal{A}_i$ is defined as

$$\mathcal{A}_i = \{\text{All sets of } \mathcal{N} \text{ with i elements}\}, \tag{3.37}$$

and

$$\mathcal{N} = \{0, 1, 2, \ldots, n\}. \tag{3.38}$$

The index sets $\mathcal{E}$ and $\mathcal{U}$ are defined respectively by

$$\mathcal{E} = \left\{ i | i = 2k \leq n, k = 0, \ldots, \frac{n}{2} \right\}$$
$$\mathcal{U} = \left\{ i | i = 2k - 1 \leq n, k = 1, \ldots, \frac{n+1}{2} \right\}.$$

Define a counter p_j for the number of reflections up to time period t_j for an arbitrary index set $\mathcal{B}$ by

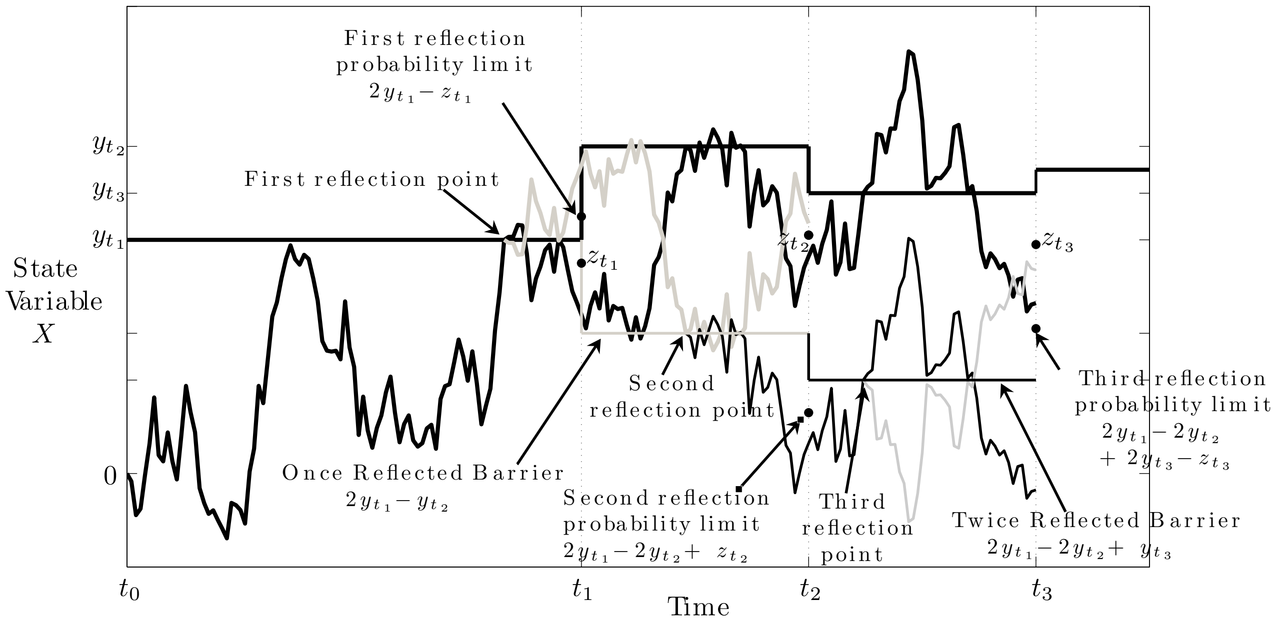
$$p_j(\mathcal{B}) = \sum_{i=1}^{j} \mathbf{1}_{\{i \in \mathcal{B}\}}. \tag{3.39}$$

After consecutively reflecting the probabilities of equation (3.36), one yields

$$P_0(X_{t_j} \leq z_{t_j}, M_{t_j} < y_{t_j})$$
$$= \sum_{i \in \mathcal{E}} \sum_{\mathcal{A}_i} P_0\left((-1)^{p_j(\mathcal{A}_i)} X_{t_j} \leq z_{t_j} + z_j^* \right)$$
$$- \sum_{i \in \mathcal{U}} \sum_{\mathcal{A}_i} P_0\left((-1)^{p_j(\mathcal{A}_i)} X_{t_j} \leq z_{t_j} + z_j^* \right). \tag{3.40}$$

with

Fig. 3.2. Illustration of a triple reflection: The thick black line is the original path of the stochastic process X_t, which faces three absorbing barriers y_{t_1}, y_{t_2}, and y_{t_3}. In order to calculate the probabilities of hitting all three barriers and ending below z_{t_1}, z_{t_2}, and z_{t_3}, the stochastic process is reflected first at the first barrier (thick grey line), second at the reflected second barrier (thin black line), and third at the twice reflected barrier (thin grey line). The limits of the tri-variate normal distribution are indicated within the figure.



$$z_j^* = (-1)^{p_j(\mathcal{A}_i)} \; 2\sum_{k=1}^{j} (-1)^{p_k(\mathcal{A}_i)-1} y_{t_k} \mathbf{1}_{\{i \in \mathcal{A}_i\}}.$$

The variables X_{t_j}, $j = 1, \ldots, n$, in equation (3.40) are n-variate normally distributed with

$$P_0\left((-1)^{p_j(\mathcal{A}_i)} X_{t_j} \leq z_{t_j} + z_j^*\right) \\ = N_n\left(\frac{z_{t_j} + z_j^*}{\sqrt{t_j}}, \Omega(\mathcal{A}_i)\right), \qquad (3.41)$$

where Ω denotes the n x n correlation matrix with entries

$$\Omega_{i,k}(\mathcal{B}) = (-1)^{p_i(\mathcal{B})+p_k(\mathcal{B})} \sqrt{\frac{\min(t_i, t_k)}{\max(t_i, t_k)}}, \quad i, k = 1, \ldots, n.$$

To change the probability measure from P_0 to P_ν with the Radon-Nikodym derivative

$$\frac{dP_\nu}{dP_0} = e^{\nu z_n - \frac{1}{2}\nu^2 t_n},$$

it is necessary to differentiate the multivariate cumulative normal distribution of equation (3.41) with respect to all initial upper limits z_j, $j = 1, \ldots, n$. After the measure transformation[12] and integrating, one gets

$$P_\nu\left((-1)^{p_j(\mathcal{A}_i)} X_{t_j} \leq z_{t_j} + z_j^*\right) \\ = e^{z_n^* \frac{\nu}{\sigma^2}} N_n\left(\frac{z_{t_j} + z_j^* - (-1)^{p_j(\mathcal{B})+p_n(\mathcal{B})} t_j \nu}{\sigma\sqrt{t_j}}, \Omega(\mathcal{A}_n)\right). \qquad (3.42)$$

As in the one-dimensional case the hitting probability for the interval $]t_0, t_j]$ is again

$$\Phi(t_0, t_j, y_{t_j}, \nu) = 1 - P_\nu(X_{t_j} \leq z_{t_j}, M_{t_j} < y_{t_j}), \qquad (3.43)$$

where $z_{t_i} = \min(y_{t_{i-1}}, y_{t_i})$.

In our case, the barrier η_B is below the starting value η_{t_0} of the EBIT-process η, all equations are still valid due to the symmetry of

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[12] See e.g. Carr (1995), Appendix A4. Carr (1995)'s vectors w and μ become $w = [0 \cdots 0, \nu]'$ and $\mu = [z_j^*]$ in out setting.

Brownian motion. The only adjustment is that the drift $\nu = -\mu$ must be used[13] and $y_{t_i} = \eta_{t_i} - \eta_B(t_i)$.[14]

For the bankruptcy prices, the exposition is again restricted to the case of approaching a barrier from below. The changes in variable as discussed above apply here as well.

As shown in Carr (1995), Appendix A1, changing the numéraire from the bank account to the bankruptcy claim itself yields the value of the bankruptcy claim. Define a new drift

$$\tilde{\nu} = \sqrt{\nu^2 + 2\sigma^2 r}.$$

The price of a finite maturity hitting claim for the first subperiod $]t_0, t_1]$ is given by

$$p_B(t_0, t_1, y_{t_1}) = e^{-\frac{\nu - \tilde{\nu}}{\sigma^2} y_{t_1}} \Phi(t_0, t_1, y_{t_1}, \tilde{\nu}). \tag{3.44}$$

All consecutive claims can be defined recursively by

$$\begin{aligned} p_B(t_0, t_j, y_{t_j}) &= p_B(t_0, t_{j-1}, y_{t_{j-1}}) \\ &\quad + e^{-\frac{\nu - \tilde{\nu}}{\sigma^2} y_{t_j}} (\Phi(t_0, t_j, y_{t_j}, \tilde{\nu}) - \Phi(t_0, t_{j-1}, y_{t_j - 1}, \tilde{\nu})). \end{aligned} \tag{3.45}$$

For the first subperiod, the hitting claim's value can be written explicitly as[15]

$$p_B(t_0, t_1, y_{t_1}) = e^{-k_1 \cdot y_{t_1}} N(q_1) + e^{-k_2 \cdot y_{t_1}} N(q_2) \tag{3.46}$$

with

$$\begin{aligned} q_1 &= \frac{-y_{t_1} - \sqrt{\nu^2 + 2r\sigma^2} \cdot (t_1 - t_0)}{\sigma \sqrt{t_1 - t_0}} \\ q_2 &= \frac{z_{t_1} - 2y_{t_1} + \sqrt{\nu^2 + 2r\sigma^2} \cdot (t_1 - t_0)}{\sigma \sqrt{t_1 - t_0}} \end{aligned}$$

and $k_{1/2}$ as defined in equation (3.13), as well as z_{t_1} as defined above.

Equations (3.45) and (3.43) can be interpreted economically. In Subsection 3.2.2 it was argued that $p_B(t_0, \infty, \eta_{t_0}, \eta_B(\infty))$ is the Arrow-Debreu price of a perpetual security that pays one currency unit if bankruptcy occurs. Equivalently, in terms of the option pricing literature, it is the price of a perpetual down-and-in barrier option written

[13] See e.g. the appendix of Duffie and Lando (2001) or Ericsson and Reneby (1998).

[14] Note that $y_{t_i} = g(\eta_{t_i}, \eta_B(t_i))$ mentioned in Subsection 2.1.3 that normalizes the process η with respect to the barrier $\eta_B(t_i)$ so that standard results are applicable.

[15] See e.g. Rubinstein and Reiner (1991). Note that, in contrast to Rubinstein and Reiner (1991), we have here a setting based on arithmetic Brownian motion.

on the EBIT with a single barrier $\eta_B(\infty)$ that pays a unit lump sum upon passing the barrier for the first time. Similarly, equation (3.43) is the price of a finite maturity down-and-in barrier option that pays one currency unit upon passing the barrier $\eta_B(t_0 < t \leq T)$ if bankruptcy occurs before or at maturity T.

$\Phi(t_0, T, \eta_{t_0}, \eta_B(T))$ represents the probability of hitting the barrier $\eta_B(T)$ before or at maturity T. A down-and-out barrier option would have the price $exp(-r(T - t_0)) \cdot (1 - \Phi(t_0, T, \eta_{t_0}, \eta_B(T)))$ and pay one currency unit at maturity T if the option is still alive.

The barrier option framework has merit because we can interpret corporate securities as portfolios of barrier options. Therefore, different bankruptcy models can be addressed by changing the structure of the barrier options which are used to construct the respective corporate security.[16]

In real world applications, the multivariate cumulative normal distribution of equation (3.43) must be evaluated which is computationally demanding. However, the special correlation structure might be exploited here. Tse, Li and Ng (2001) show that the inverse of $\Omega(\mathcal{B})$ is tri-diagonal which reduces the exponential terms of the multivariate normal density considerably. As a result, a direct numerical integration of the simpler density is computationally faster and can therefore be conducted with higher precision. Moreover, Tse et al. (2001) give error bounds for their approximation so that the precision can be controlled with which the multivariate cumulative normal distribution is approximated.

Alternatively, the multivariate normal algorithm proposed in Genz (1992) can be used. It is computationally fast and can be approximated to a desired error level, as well. In this study, Genz (1992)'s method is used.

3.2.3.2 Value of the First Maturing Bond

Using results of the last subsection, closed-form solutions for finite maturity debt issues can be found by directly applying equation (2.31). Recall from Subsection 2.1.4 that the firm not only uses one source of debt financing but issues $j = 1, \ldots, J$ bonds where each bond value to investors after taxes, D_{C_j,T_j}, is characterized by a continuously paid

[16] See e.g. Ericsson and Reneby (1998), who use different barriers for coupon and notional payments. Brockman and Turtle (2003) favorably test a model with default barrier and compare the results with a model of one time bankruptcy like Merton (1974).

coupon rate, C_j, a maturity date, T_j, and the principal, P_j. Without loss of generality, set $j = 1$ for the bond issue that matures next, and order the subsequently maturing bonds in increasing order. Consider the issue $j = 1$ first. The value of the bond $j = 1$ can be expressed as

$$D_{C_1,T_1} = \int_{t_0}^{T_1} (1-\tau^d) e^{-r\cdot s} C_1 [1 - \Phi(t_0, T_1, \eta_{t_0}, \eta_B(T_1))] ds \\ + e^{-r\cdot(T_1-t)} P_1 [1 - \Phi(t_0, T_1, \eta_{t_0}, \eta_B(T_1))] + D^-_{C_1,T_1}. \quad (3.47)$$

The first line and the first term in the second line of equation (3.47) represent the value of the cash flows, interest payments and principal repayment. Each of these payments takes into account the probability of the firm going bankrupt and corresponds to $D^+_{C,\infty}$ of equation (3.19) if we considered a perpetual claim. The term $D^-_{C_1,T_1}$ in the second line denotes the value in the case of bankruptcy. This claim is interpreted similarly to equation (3.23) but accounts for the fact that more than one debt issue is outstanding.

With respect to tax payments, we assume Tax Regime 1. Bond investor must pay a tax of τ^d on coupons received and are fully liable to corporate taxation and equity taxation in the case of bankruptcy. In many countries capital gains for long-term investments are tax exempt, so that no tax rate is applied to the redemption of the principal.

Similar to Subsection 2.1.4, $V_B(T_1)$ is split among debt holders, the government, the loss portion, and — if some value can still be distributed — the equity holders. If the firm goes bankrupt before or at T_1, we assume that the value of the firm is split proportionally among the debt claimants. This implies

$$V^-_{C_1,T_1} = \min\left[(1-\alpha)V_B(T_1); \sum_{j=1}^{J} P_j\right] \frac{P_1}{\sum\limits_{j=1}^{J} P_j} \\ p_B(t_0, T_1, \eta_{t_0}, \eta_B(T_1)). \quad (3.48)$$

The minimum function ensures that not more funds are distributed among debt holders than were originally lent to the firm.

The weighting factor $w_1 = P_1 / \sum_{j=1}^{J} P_j$ does not account for the seniority of debt and treats all bond issues similarly. Altering this weighting factor allows for the integration of junior bonds in the analysis.

The other bonds $j = k$ receive a recovery value similar to (3.48) but with a weighting factor $P_k / \sum_{j=1}^{J} P_j$. Denote the recovery value before taxes by $V^-_{C_k,T_1}$.

If all debt claims have been paid, the remaining funds are distributed among stock holders. Therefore,

$$V_{E,j=1}^{-} = \max\left[(1-\alpha)\cdot V_B(T_1) - \sum_{j=1}^{J} P_j; 0\right] \cdot p_B(t_0, T_1, \eta_{t_0}, \eta_B(T_1)). \tag{3.49}$$

$V_{E,j=1}^{-}$ and $\sum_{j=1}^{J} V_{C_j,T_1}^{-}$ add up to $(1-\alpha)V_B(T_1)p_B(t_0, T_1, \eta_{t_0}, \eta_B(T_1))$. Thus, the value of the loss portion in the period $]t_0, T_1]$ is

$$BC_{j=1} = \alpha \cdot V_B(T_1) \cdot p_B(t_0, T_1, \eta_{t_0}, \eta_B(T_1)). \tag{3.50}$$

Introducing taxes as in Subsection 3.2.2, we find the after-tax bankruptcy values of the first period by

$$E_{j=1}^{-} = (1-\tau^{eff})V_{E,j=1}^{-} \tag{3.51}$$

$$D_{C_j,T_1}^{-} = (1-\tau^{eff})V_{C_j,T_1}^{-} \qquad j = 1 \ldots J \tag{3.52}$$

and the value of taxes paid to the government by

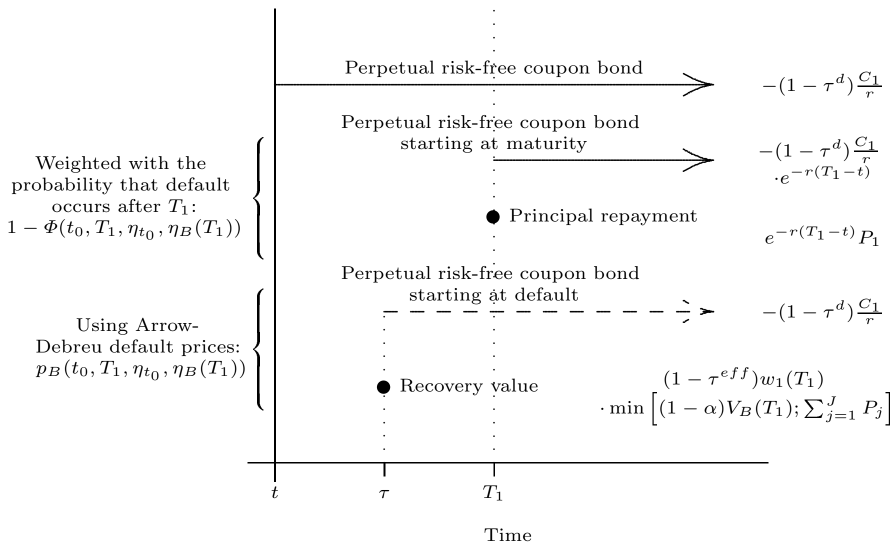
$$G_{j=1}^{-} = \tau^{eff}\left(V_{E,j=1}^{-} + \sum_{j=1}^{J} V_{C_j,T_j}^{-}\right) = \tau^{eff}(1-\alpha)V_B(T_1)p_B(t_0, T_1, \eta_{t_0}, \eta_B(T_1)). \tag{3.53}$$

Note that equations (3.51) through (3.53) only account for the time from t_0 to T_1, i.e., the time when the first debt issue is still outstanding. Since the capital structure changes when the first debt is repaid, more funds may be available for distribution among the remaining debt and equity holders in the case of bankruptcy after this event. Still, we do not make any attempts to find an optimal η_B, which, in principle, can be assumed to be so low such that equity holders never receive anything in the case of bankruptcy.

An explicit solution for the value of the nearest maturing debt can be found. Integrating (3.47) by parts leads to

$$D_{C_1,T_1} = (1-\tau^d)\frac{C_1}{r} + e^{-r(T_1-t)}\left[P_1 - (1-\tau^d)\frac{C_1}{r}\right]\left[1 - \Phi(t_0, T_1, \eta_{t_0}, \eta_B(T_1))\right] - (1-\tau^d)\frac{C_1}{r} \cdot p_B(t_0, T_1, \eta_{t_0}, \eta_B(T_1)) + D_{C_1,T_1}^{-} \tag{3.54}$$

Fig. 3.3. Portfolio composition and interpretation of the first debt issue



Equation (3.54) can be interpreted as a portfolio of barrier options where portfolio amounts are determined by risk-free perpetual bonds and lump sum payments. As illustrated in Figure 3.3, the first term $((1-\tau^d)C_1)/r$ is the value of an infinite after-tax coupon stream to bond investors. Noting that $exp(-r(T_1-t))[1-\Phi(t_0, T_1, \eta_{t_0}, \eta_B(T_1))]$ is the value of a down-and-out barrier option, the second term represents the value of the bond's principal repayment at and the offsetting coupon stream after maturity T_1 if the barrier is not hit. Both terms represent the value of payments to a finite maturity corporate bond holder if the firm does not go bankrupt before or at maturity. The principal is repaid but the infinite after-tax coupon stream is lost. The second line in equation (3.54) is the value of the bond in the case of bankruptcy with $p_B(t_0, T_1, \eta_{t_0}, \eta_B(T_1))$ as the value of the down-and-in barrier option. In the case of bankruptcy the bond holder foregoes the future value of the perpetual bond starting at bankruptcy time and paying a coupon of $(1-\tau^d)C_1$ but he receives the residual value $(1-\tau^{eff})\min\left[(1-\alpha)V_B; \sum_{j=1}^J P_j\right] P_1/(\sum_{j=1}^J P_j)$. Note that the value of the down-and-in barrier option and the payments are incorporated in the definition of $D^-_{C_1,T_1}$ of equation (3.52).

If we let $T_1 \to \infty$ in equation (3.54), we see that the second term involving the principal vanishes because

$$\lim_{T_1 \to \infty} e^{-r(T_1 - t)} = 0.$$

Recall that

$$\lim_{T_1 \to \infty} p_B(t_0, T_1, \eta_{t_0}, \eta_B(T_1)) \to p_B(t_0, \infty, \eta_{t_0}, \eta_B(T_1))$$

and (3.54) reduces to the value of perpetual corporate debt, i.e. the value of the perpetual as long as the firm remains solvent, as given by (3.19), and the value of the perpetual when the firm declares bankruptcy, as found in equation (3.23)

$$D_{C_1,T_1} = D^+_{C_1,T_1} + D^-_{C_1,T_1}.$$

3.2.3.3 Value of the Second Maturing and Consecutive Bonds

For finding an explicit solution for debt issue $j = 2$, with the index j defined as in Subsection 3.2.3.2, the changed capital structure after repayment of the first debt issue must be considered. Denote the recovery value for the second issue between T_1 and T_2 by

$$V^-_{C_2,T_2} = \min\left[(1-\alpha)V_B(T_2); \sum_{j=2}^{J} P_j\right] \frac{P_2}{\sum\limits_{j=2}^{J} P_j} \cdot p_B(T_2, T_1, \eta_{t_0}, \eta_B(T_2)). \tag{3.55}$$

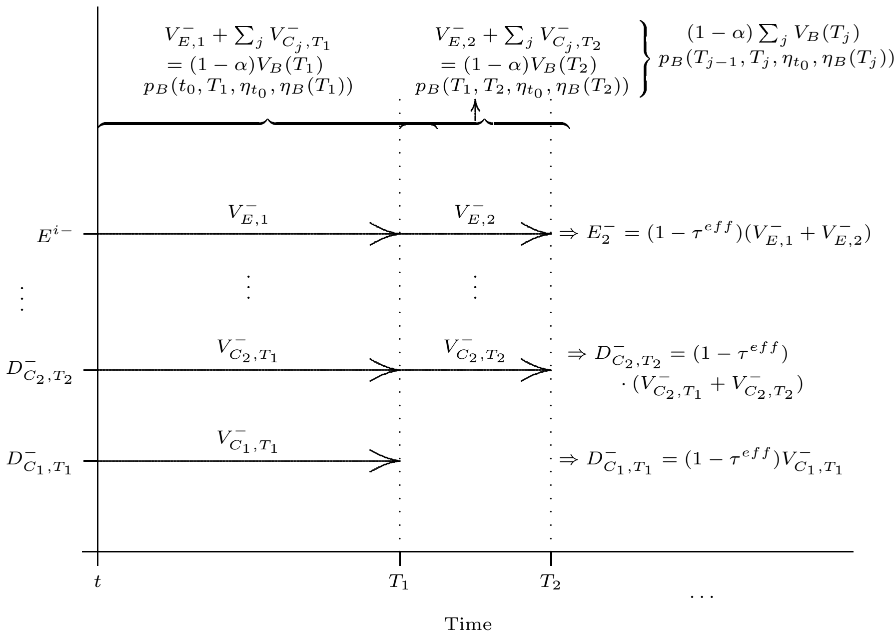
Adding this to the present value of the recovery value for the period from t_0 to T_1 yields the total value of recovery in the case of bankruptcy during the life of the bond $j = 2$, i.e.,

$$D^-_{C_2,T_2} = (1 - \tau^{eff})(V^-_{C_2,T_1} + V^-_{C_2,T_2}), \tag{3.56}$$

where $V^-_{C_2,T_1}$ is similar to (3.48) with P_1 replaced by P_2. Equation (3.54) still states the after-tax value of the debt issue. Using this procedure recursively, we can explicitly value all finite-maturity debt issues outstanding as of time t_0.

If we consider tax payments in the case of bankruptcy and bankruptcy losses, we find that the equity recovery value is

Fig. 3.4. Bankruptcy values for debt if the firm has issued more than one bond



$$V^-_{E,j=2} = \max\left[(1-\alpha)V_B(T_2) - \sum_{j=2}^{J} P_j; 0\right] \cdot p_B(T_1, T_2, \eta_{t_0}, \eta_B(T_2)). \tag{3.57}$$

Adding $V^-_{E,j=2}$ and $\sum_{j=2}^{J} V^-_{C_j,T_2}$ we obtain again

$$(1-\alpha)V_B(T_2)p_B(T_1, T_2, \eta_{t_0}, \eta_B(T_2)).$$

Thus, the values of bankruptcy losses for both periods, $]t, T_1]$ and $]T_1, T_2]$, as of t_0 are

$$\sum_{j=1}^{2} BC_j = \alpha \cdot V_B(T_1) \cdot p_B(t_0, T_1, \eta_{t_0}, \eta_B(T_1)) + \alpha \cdot V_B(T_2) \cdot p_B(T_1, T_2, \eta_{t_0}, \eta_B(T_2)) \tag{3.58}$$

and the recovery value for equity is accordingly

$$\sum_{j=1}^{2} E_j^- = (1-\tau^{eff}) \sum_{j=1}^{2} V_{E,j}^- . \tag{3.59}$$

The value of tax payments in the case of bankruptcy as of time t_0 amounts to

$$\sum_{j=1}^{2} G_j^- = \tau^{eff} \left(V_{E,j=1}^- + \sum_{j=1}^{2} V_{C_j,T_1}^- + V_{E,j=2}^- + \sum_{j=2}^{2} V_{C_j,T_2}^- \right) . \tag{3.60}$$

Figure 3.4 illustrates the splitting of the bankruptcy values into subperiods and claims. Vertically, the claims in the case of bankruptcy are listed in the order of maturity. The different maturities are shown at the bottom. As can be seen, in the first period, $]t_0, T_1]$, all security holders have claims to the firm. In the second period, $]T_1, T_2]$, the first claim has expired. The before-tax values V^- indicate the claims in each subperiod. By adding over subperiods and applying the tax regime, we find after-tax values of the different claims. By adding within the subperiod we find the total value of the bankruptcy claims, which add up to the present value of $(1-\alpha)V_B(T_j)$. Summing up, we find a finite maturity debt value of the issue $j = k$, where k denotes the kth bond maturing from today with

$$\begin{aligned} D_{C_k,T_k} = {} & e^{-r(T_k - t_0)} \left[P_k - (1-\tau^d)\frac{C_k}{r} \right] [1 - \Phi(t_0, T_k, \eta_{t_0}, \eta_B(T_k))] \\ & + (1-\tau^d)\frac{C_k}{r} [1 - p_B(t_0, T_k, \eta_{t_0}, \eta_B(T_k))] + D_{C_k,T_k}^- \end{aligned} \tag{3.61}$$

where

$$D_{C_k,T_k}^- = (1-\tau^{eff}) \sum_{j=1}^{k} V_{C_k,T_j}^- \tag{3.62}$$

and

$$V_{C_k,T_l}^- = \min \left[(1-\alpha)V_B ; \sum_{j=l}^{J} P_j \right] \frac{P_k}{\sum_{j=l}^{J} P_j} p_B(t_0, T_l, \eta_{t_0}, \eta_B(T_l)) \tag{3.63}$$

where $l = 1, \ldots, k$ and $T_0 = t_0$.

If the price of a risk-free bond with equivalent features than the corporate bond is denoted by

$$B_{C_j,T_j} = (1-\tau^d)\frac{C_j}{r}\left(1-e^{-r(T_j-t_0)}\right) + P_j e^{-r(T_j-t_0)}, \qquad (3.64)$$

equation (3.61) can be rewritten as

$$D_{C_k,T_k} = B_{C_k,T_k} - B_{0,T_k}\left[P_k - (1-\tau^d)B_{C_k,\infty}\right]\Phi(t_0,T_k,\eta_{t_0},\eta_B(T_k)) \\ -B_{C_k,\infty}\, p_B(t_0,T_k,\eta_{t_0},\eta_B(T_k)) + D^-_{C_k,T_k}, \qquad (3.65)$$

where $B_{0,T}$ represents the risk-free discount factor, and $B_{C_k,\infty}$ the value of a risk-free perpetual bond with continuous coupon payments C_j. The difference between a government bond and a credit risky corporate bond negatively depends on the probability of bankruptcy at and before maturity, the bankruptcy time, and positively on the recovery value.

Continuing the arguments concerning the recovery values for equity as stated in equation (3.57) for $j = 3, \ldots, J$, we find

$$E^- = \sum_{j=1}^{J} E_j^- + E_\infty^- \cdot 1_{\{T_j<\infty\}} \\ = (1-\tau^{eff})\left[\sum_{j=1}^{J} V^-_{E,j} + V^-_{E,\infty}\cdot \mathbf{1}_{\{T_j<\infty\}}\right]. \qquad (3.66)$$

where

$$V^-_{E,k} = \max\left[(1-\alpha)V_B(T_k) - \sum_{j=k}^{J} P_j; 0\right] p_B(t_0,T_k,\eta_{t_0},\eta_B(T_k)). \qquad (3.67)$$

In equation (3.66), $V^-_{E,\infty}$ denotes the claim of equity owners in the case of bankruptcy for the period after the last debt issue has been repaid if no perpetual debt has been issued.

Similarly, we can develop the present value of bankruptcy losses and taxes in the case of bankruptcy.

$$BC = \sum_{j=1}^{J} \alpha \cdot V_B(T_j)\cdot p_B(T_{j-1},T_j,\eta_{t_0},\eta_B(\infty)) \qquad (3.68)$$

$$G^- = \tau^{eff}(1-\alpha)\sum_{j=1}^{J} V_B(T_j)p_B(T_{j-1},T_j,\eta_{t_0},\eta_B(T_j)) \qquad (3.69)$$

The value of equity is again given as the difference between the value of the solvent firm V^+ and the sum of all debt values of the firm,

$\sum_j D^E_{C_j,T_j}$. $D^E_{C_j,T_j}$ differs from D_{C_j,T_j} with respect to the treatment of taxes and the recovery value of debt investors. Usually, tax authorities allow the deduction of interest payments on debt as a cost factor from taxable income, thus reducing the tax burden of the firm. Principal repayment does not enjoy this favorable treatment. As a result debt value is no longer proportional to after-tax value of the firm as in the case of perpetual debt.[17] If we take equation (3.61) and replace $(1-\tau^d)$, the personal tax rate of an investor, with the corporate tax rate $(1-\tau^c)$, and ignore the recovery value $D^-_{C_j,T_j}$ we find the value $D^{E+}_{C_j,T_j}$. Additionally we have to ensure that equity holders do not receive a tax subsidy on bond redemptions.[18] Therefore, the value of equity is

$$E = (1-\tau^e)\cdot\left[(1-\tau^c)\cdot V^+ - \sum_{j=1}^{J} D^{E+}_{C_j,T_j}\right] + E^-$$
$$-\tau^e \sum_{j=1}^{J} P_j \cdot e^{-r(T_j-t_0)}[1-\Phi(t_0,T_k,\eta_{t_0},\eta_B(T_k))], \quad (3.70)$$

where the last line describes the tax adjustment mentioned above.

To complete the analysis we would like to state the value of all tax payments which must account for the tax deductability of coupon payments at the corporate level in (3.70). We add the reduced tax burden of the firm to obtain[19]

[17] All terms including the coupon payments C_j in (3.61) are affected by taxation. The term involving the principal repayment in case of the firm staying solvent until maturity does not involve a tax rate.

[18] See Subsection 2.1 for a discussion of this issue.

[19] To implement Tax Regime 2, we delete in equation (3.70) the last line and deduct it instead from the governments claim in equation (3.71). Thus we shift value form the government to the equity claim.

$$
\begin{aligned}
G = {} & \tau^{eff} V_+ + \sum_{j=1}^{J} \tau^d \frac{C_j}{r} \left[\Phi(t_0, T_j, \eta_{t_0}, \eta_B(T_j)) - p_B(t_0, T_j, \eta_{t_0}, \eta_B(T_j))\right] \\
& -\tau^e \sum_{j=1}^{J} (1-\tau^c) \frac{C_j}{r} \left[\Phi(t_0, T_j, \eta_{t_0}, \eta_B(T_j)) - p_B(t_0, T_j, \eta_{t_0}, \eta_B(T_j))\right] \\
& +G^- \\
= {} & \tau^{eff} V^+ + G^- \\
& + \left[\tau^d - \tau^e (1-\tau^c)\right] \sum_{j=1}^{J} \frac{C_j}{r} \\
& \cdot \left[\Phi(t_0, T_j, \eta_{t_0}, \eta_B(T_j)) - p_B(t_0, T_j, \eta_{t_0}, \eta_B(T_j))\right]. \qquad (3.71)
\end{aligned}
$$

Note, that we can again decompose the values of equity in equation (3.70), tax payments in equation (3.71), and bankruptcy cost in equation (3.68) into portfolios of barrier options.

3.2.4 Term Structure of Credit Spreads

One of the key sources of information about firms is the term structure of credit spreads that can be derived as the difference between the term structure of the firm's debt and the term structure of government bonds. Recall, that the price of risk-free debt was defined in equation (3.64) by

$$B_{C_j,T_j} = (1-\tau^d) \frac{C_j}{r} \left(1 - e^{-r(T_j - t)}\right) + P_j e^{-r(T_j - t)}.$$

If prices for several government bond issues are outstanding for the valuation date t_0, a term structure of risk-free zero coupon bond yields can be estimated.[20] By the same argument, the term structure of credit risky bond yields might be estimated by estimating the corporate spot rate y_t in

$$D_{C_j,T_j} \equiv \int_{t_0}^{T_j} C_j e^{-y_s (s - t_0)} ds + P_j e^{-y_{T_j} (T_j - t_0)}, \qquad (3.72)$$

The credit spread can be obtained by

$$YS_s = y_s - r_s. \qquad (3.73)$$

[20] Svensson (1994)'s algorithm can be used to estimate zero rates from government coupon bonds.

Another way of comparing risky to risk-free debt is by analyzing par yields. Par yields reflect those coupons that the government or a firm announce to pay in order to issue a new bond at par.

Denote the government's par coupon by PR. Then replacing C_j in equation (3.64) with PR and solving for the par coupon, we find

$$PR = \frac{r \cdot 100}{1 - \tau^d}. \tag{3.74}$$

The value for finite-maturity debt as given in equation (3.54) can be solved in the same way. Denoting the corporate par yield by PY_j, we have

$$PY_j = \frac{r \cdot 100 \cdot [1 - DO(T_j, \eta, \eta_B)] - D^-_{0,T_j}}{(1 - \tau^d) \cdot [1 - DO(T_j, \eta, \eta_B) - p_B(t_0, T_j, \eta_{t_0}, \eta_B(T_j))]} \tag{3.75}$$

with

$$DO(T_j, \eta, \eta_B) = e^{-r \cdot (T_j - t_0)}[1 - \Phi(t_0, T_j, \eta_{t_0}, \eta_B(T_j))].$$

Subtracting equation (3.74) from equation (3.75), we calculate the par yield spread PYS_j by

$$PYS_j = \frac{r \cdot 100 \cdot p_B(t_0, T_j, \eta_{t_0}, \eta_B(T_j)) - r \cdot D^-_{0,T_j}}{(1 - \tau^d) \cdot [1 - DO(T_j, \eta, \eta_B) - p_B(t_0, T_j, V, V_B)]}. \tag{3.76}$$

3.3 The Case of Geometric Brownian Motion

Many traditional firm value models such as the ones of Goldstein et al. (2001), Leland (1994), Leland and Toft (1996), Duffie and Lando (2001) among others do not assume arithmetic Brownian motion for EBIT or firm value as we do here but assume geometric Brownian motion instead. Although this assumption is debateable since it implies that EBIT cannot become negative[21] we want to extend the analysis to the case of geometric Brownian motion to be able to compare results for both model assumptions.

[21] Goldstein et al. (2001) show that if the EBIT is assumed to follow a geometric Brownian motion with constant parameters, the firm value follows the same rule. Although the converse conclusion might not be true in general, we cannot neglect the fact that if the firm value follows a geometric Brownian motion, one solution for the driving EBIT-process would be that EBIT follows geometric Brownian motion as well. This directly leads to our criticism in Section 3.1.

Luckily, only a few adjustments to our general framework are necessary, because the general structure of equation (3.70) for the value of equity, of equation (3.61) for the value of the k-th finite maturity debt issue as well as the equations for the value of tax payments (3.71) and of bankruptcy losses (3.68), respectively, can be maintained. We only need to change the derivation of firm value, as compared to equation (3.4), its process and the equations in Subsection 3.2.3.1.

3.3.1 The General Case

Assume that the EBIT of the firm follows a geometric Brownian motion[22]

$$\frac{d\bar{\eta}}{\bar{\eta}} = \bar{\mu}_\eta dt + \bar{\sigma}_\eta dz^{\mathcal{P}}, \tag{3.77}$$

where $\bar{\mu}_\eta$ and $\bar{\sigma}_\eta$ denote the constant instantaneous drift and volatility of the process, under the physical measure $\mathcal{P}$. Again we need a risk premium $\bar{\theta}$ to change the measure to the equivalent martingale measure $\mathcal{Q}$, which leads to the risk-neutral process of EBIT

$$\frac{d\bar{\eta}}{\bar{\eta}} = \left(\bar{\mu}_\eta - \bar{\theta} \cdot \bar{\sigma}_\eta\right) dt + \bar{\sigma}_\eta dz^{\mathcal{Q}}.$$

To simplify notation, denote the risk-neutral drift by $\bar{\mu} = \bar{\mu}_\eta - \bar{\theta} \cdot \bar{\sigma}_\eta$. Then, total firm value amounts to[23]

$$\bar{V}_t = \frac{\bar{\eta}_t}{r - \bar{\mu}}. \tag{3.78}$$

Applying Itô's lemma we find the firm value process to be

$$\frac{d\bar{V}}{\bar{V}} = \bar{\mu} dt + \bar{\sigma}_\eta dz^{\mathcal{Q}}. \tag{3.79}$$

Note that the logarithm of the EBIT $\bar{\eta}$ and the firm value $\bar{V}$ follow the process

$$d\ln(\bar{\eta}) = d\ln(\bar{V}) = \left(\bar{\mu} - \frac{\bar{\sigma}_\eta^2}{2}\right) dt + \bar{\sigma}_\eta dz^{\mathcal{Q}}. \tag{3.80}$$

[22] Trying to keep the notation comparable to the case of arithmetic Brownian motion, we will use a bar to indicate respective GBM-parameters.

[23] See e.g. Shimko (1992) for guidance how to solve for the firm value.

This changes the solution of a homogenous ordinary differential equation

$$\frac{\bar{\mu}}{r}\bar{\eta}F_{\bar{\eta}} + \frac{\bar{\sigma}_\eta^2}{2r^2}\bar{\eta}^2 F_{\bar{\eta}\bar{\eta}} - rF = 0$$

to

$$F = \bar{A}_1 \cdot (\bar{\eta})^{-\bar{k}_1} + \bar{A}_2 \cdot (\bar{\eta})^{-\bar{k}_2} \tag{3.81}$$

with

$$\bar{k}_{1/2} = \frac{\left(\bar{\mu} - \frac{\bar{\sigma}_\eta^2}{2}\right) \mp \sqrt{\left(\bar{\mu} - \frac{\bar{\sigma}_\eta^2}{2}\right)^2 + 2r\bar{\sigma}_\eta^2}}{\bar{\sigma}_\eta^2} \tag{3.82}$$

Additionally, we may use equations (3.43) for the probability of hitting the bankruptcy level $\bar{\eta}_B$ before time T, $\Phi(t_0, T, \ln(\bar{\eta}_{t_0}/\bar{\eta}_B(T)), -(\bar{\mu} - \bar{\sigma}_\eta^2/2))$, and (3.45) for the finite maturity Arrow-Debreu bankruptcy price

$$\bar{p}_B\left(t_0, T, \ln\left(\frac{\bar{\eta}_B(T)}{\bar{\eta}_{t_0}}\right), -\left(\bar{\mu} - \frac{\bar{\sigma}_\eta}{2}\right)\right)$$

by adjusting the drift with $\bar{\mu} - \bar{\sigma}_\eta/2$ and using $\ln(\bar{\eta}_t)$ and $\ln(\bar{\eta}_B)$ instead of η_t and η_B. The resulting formulas for the first subperiod are

$$\bar{\Phi}(t_0, T, \bar{\eta}_{t_0}, \bar{\eta}_B(T_1)) = N(\bar{h}_1) + \left(\frac{\bar{\eta}_{t_0}}{\bar{\eta}_B}\right)^{-\frac{2\bar{\mu}}{\bar{\sigma}_\eta^2}+1} N(\bar{h}_2) \tag{3.83}$$

with

$$\bar{h}_{1/2} = \frac{-\ln\left(\frac{\bar{\eta}_{t_0}}{\bar{\eta}_B}\right) \pm \left(\bar{\mu} - \frac{\bar{\sigma}_\eta^2}{2}\right) \cdot (T_1 - t_0)}{\bar{\sigma}_\eta\sqrt{T_1 - t_0}}, \tag{3.84}$$

and

$$\bar{p}_B(t_0, T_1, \bar{\eta}_{t_0}, \bar{\eta}_B(T_1)) = \left(\frac{\bar{\eta}_{t_0}}{\bar{\eta}_B}\right)^{-\bar{k}_1} N(\bar{q}_1) + \left(\frac{\bar{\eta}_{t_0}}{\bar{\eta}_B}\right)^{-\bar{k}_2} N(\bar{q}_2) \tag{3.85}$$

with

$$\bar{q}_{1/2} = \frac{-\ln\left(\frac{\bar{\eta}_{t_0}}{\bar{\eta}_B}\right) \mp \sqrt{\left(\bar{\mu} - \frac{\bar{\sigma}_\eta^2}{2}\right)^2 + 2r\bar{\sigma}_\eta^2} \cdot (T_1 - t_0)}{\bar{\sigma}_\eta\sqrt{T_1 - t_0}}. \tag{3.86}$$

The Arrow-Debreu bankruptcy price for a subinterval can be found by the same argument as in Subsection 3.2.3.1:

$$\begin{aligned}\bar{p}_B(T', T, \bar{\eta}_{t_0}, \bar{\eta}_B(T)) = {} & \bar{p}_B(t_0, T, \bar{\eta}_{t_0}, \bar{\eta}_B(T)) \\ & - \bar{p}_B(t_0, T', \bar{\eta}_{t_0}, \bar{\eta}_B(T')). \end{aligned} \tag{3.87}$$

3.3.2 The Perpetual Debt Case

In the perpetual debt case all formulas of Subsection 3.2.2 stay intact. Only, the infinite maturity Arrow-Debreu price is the limit of equation (3.85) as $T \to \infty$

$$\bar{p}_B(t_0, \infty, \bar{\eta}_{t_0}, \bar{\eta}_B(\infty)) = \left(\frac{\bar{\eta}_{t_0}}{\bar{\eta}_B}\right)^{-\bar{k}_2}. \tag{3.88}$$

Equations (3.83), (3.85), (3.87), and (3.88) fully define the credit risk framework of Subsection 3.2.3.3 in terms of geometric Brownian motion.

To complete the comparison to the case of arithmetic Brownian motion, consider the optimality arguments for the perpetual continuous coupon and the bankruptcy barrier. Given the capital structure, the optimal bankruptcy EBIT in the GBM-model is

$$\bar{\eta}_B = \frac{\bar{k}_2(r - \bar{\mu})\bar{C}}{r(1 + \bar{k}_2)}. \tag{3.89}$$

As in the case of arithmetic Brownian motion, the capital structure is fully determined by the choice of a coupon $\bar{C}^*$ when maximizing the sum of debt and equity. In contrast to the case of arithmetic Brownian motion, $\bar{C}^*$ can be calculated explicitly as

$$\bar{C}^* = \left(\bar{\xi}\frac{r(1 + \bar{k}_2)}{\bar{k}_2(r - \bar{\mu})}\right)\eta_{t_0} \tag{3.90}$$

with

$$\bar{\xi} = \left[\frac{(1-\tau^{eff})-(1-k)(1-\tau^d)}{(1-\tau^{eff})-(1-k)\left[(1-\tau^d)(1-\bar{k}_2)+\bar{k}_2(1-\alpha)(1-\tau^{eff})\right]}\right]^{\frac{1}{\bar{k}_2}}. \tag{3.91}$$

Note that the optimal coupon of equation (3.90) is proportional to the initial EBIT $\bar{\eta}_{t_0}$. As a result the optimal bankruptcy barrier of equation (3.89) and the infinite maturity bankruptcy claim in equation (3.88) becomes $(\bar{\xi})^{(\bar{k}_2)}$ which is constant. Therefore, the debt $D_{\bar{C}^*,\infty}$ and equity prices E share the same homogeneity property of degree 1 with respect to the initial EBIT. This proportionality of security prices is convenient when discussing dynamic capital structure decisions because knowing the barrier value when to restructure in terms of the then prevailing EBIT is sufficient.[24]

[24] See, e.g. Goldstein et al. (2001), Christensen et al. (2000) who exploit the property elegantly. See Flor and Lester (2004) for a more formal proof.

In the special case $k = 1$, the coupon $\bar{C}^*$ increases to

$$\bar{C}^* = \bar{V}\frac{r(1+\bar{k}_2)}{\bar{k}_2}. \tag{3.92}$$

The asset substitution effect in the GBM-setting is discussed in Leland (1994) who reports that asset substitution incentives for equity owners unequivocally exist if the bankruptcy barrier is endogenously chosen. The debt covenant case is similar to the ABM-case.

3.4 A Numerical Extension of the Basic Setting

The valuation framework for corporate securities would not be complete if derivative securities are ignored. In most cases derivatives on a firm's equity or debt securities are complicated so that closed form solutions are not available. However, numerical methods have been developed which can also be exploited in the setting of this chapter.

Two methods are proposed in the following two subsections. In Subsection 3.4.1 a standard trinomial tree is used to approximate the EBIT-process. The implementation of the economic model of Chapter 2 is discussed and the extension of valuing derivative securities.

Subsection 3.4.2 proposes a method of direct numerical integration. The analytical solution of the finite maturity debt model as of Subsection 3.2.3 relies on finite maturity bankruptcy probabilities. A byproduct of the derivation of these bankruptcy probabilities can be used as the state density to find security prices quicker than in a trinomial tree.

Without loss of generality, both numerical methods are described for arithmetic Brownian motion only. They can be readily applied to geometric Brownian motion because the logarithm of the state variable follows the arithmetic Brownian motion of equation (3.80) and replacing the firm value formula of the ABM-firm of equation (3.4) with that of the GBM-firm of equation (3.78).

3.4.1 A Lattice Approach for the Corporate Securities Framework

3.4.1.1 The Approximation of the EBIT-Process

The stochastic factor in the Corporate Security Framework is the firm's EBIT η. In subsection 3.2 it is assumed that EBIT follows an arithmetic Brownian motion under the equivalent risk-neutral martingale measure $\mathbb{Q}$

$$d\eta = \mu dt + \sigma_\eta dz^{\mathbb{Q}},$$

where μ and σ_η are the risk-neutral drift and volatility of the EBIT-process. The EBIT-process is driven by a standard Wiener process $z^{\mathbb{Q}}$ under the risk-neutral probability measure $\mathbb{Q}$. All parameters are assumed to be constant.

A risk-neutral valuation tree can be constructed in which the discount rate is the risk-free interest rate r. The stochastic process is approximated numerically by a trinomial tree with time steps Δt and EBIT-step size

$$\Delta\eta = \lambda\sqrt{\sigma_\eta \Delta t}. \tag{3.93}$$

λ denotes a EBIT-spacing parameter.

The probabilities at each node to reach the following up-, middle-, and down-state nodes are

$$\pi_u = \frac{\sigma_\eta^2 \Delta t + \mu^2 \Delta t^2}{2\Delta\eta^2} - \frac{\mu\Delta t}{2\Delta\eta}, \tag{3.94a}$$

$$\pi_m = 1 - \frac{\sigma_\eta^2 \Delta t + \mu^2 \Delta t^2}{\Delta\eta^2}, \tag{3.94b}$$

$$\pi_d = \frac{\sigma_\eta^2 \Delta t + \mu^2 \Delta t^2}{2\Delta\eta^2} + \frac{\mu\Delta t}{2\Delta\eta}. \tag{3.94c}$$

In equations (3.94) the two parameters λ and Δt can be chosen freely.

To have a good approximation of the EBIT-process 900 to 1,100 steps are needed that determine Δt subject to the maturity of the tree. Kamrad and Ritchken (1991, p. 1643) suggest a value of $\lambda = 1.2247$ which they show to have the best convergence properties on average in their application in multi-state variables option pricing.

Recall that in the Corporate Securities Framework of Chapter 2, the firm declares bankruptcy whenever total firm value V hits a barrier V_B. The total firm value can be calculated explicitly by discounting all future EBIT-payments with the constant risk-free interest rate r which yields in the ABM-case (see equation (3.4))

$$V = \frac{\mu}{r^2} + \frac{\eta_t}{r}.$$

EBIT is distributed to all claimants of the firm. So, total firm value of an EBIT-model does not only include the market value of debt and equity, but also bankruptcy losses, and taxes to the government. This alters the notion of the bankruptcy barrier, as well as losses in the case of bankruptcy compared to traditional firm value models. The bankruptcy barrier can equivalently be defined in terms of an EBIT-value by equation (3.4).

The literature on barrier options valuation in lattice models, observes pricing problems because the barrier usually lies between two nodes. If Δt is decreased, the value of the barrier option might not converge because the barrier usually changes its distance to adjacent nodes. Boyle and Lau (1994) demonstrate the oscillating pattern of convergence. To overcome the deficiency Boyle and Lau (1994) suggest to adjust the steps $\Delta\eta$ such that the barrier is positioned just above one layer of nodes.[25] To mimic the algorithm, a λ closest to 1.225 is chosen to ensures that the $]t_0, T_1]$-bankruptcy barrier lies just above one node level. With this parameter constellation, the security values of the debt and equity issues converge sufficiently well to their analytical solutions.

3.4.1.2 Payments to Claimants and Terminal Security Values

In each node, EBIT is distributed among the claimants of the firm (see Figure 3.5). Payments to claimants are different in the case of bankruptcy. Therefore, Figure 3.5 exhibits two separate EBIT-distribution algorithms for the firm being (i) solvent and (ii) insolvent. The bankruptcy decision is modeled by testing if the expected future firm value in the current node is lower or equal to the bankruptcy level V_B.[26]

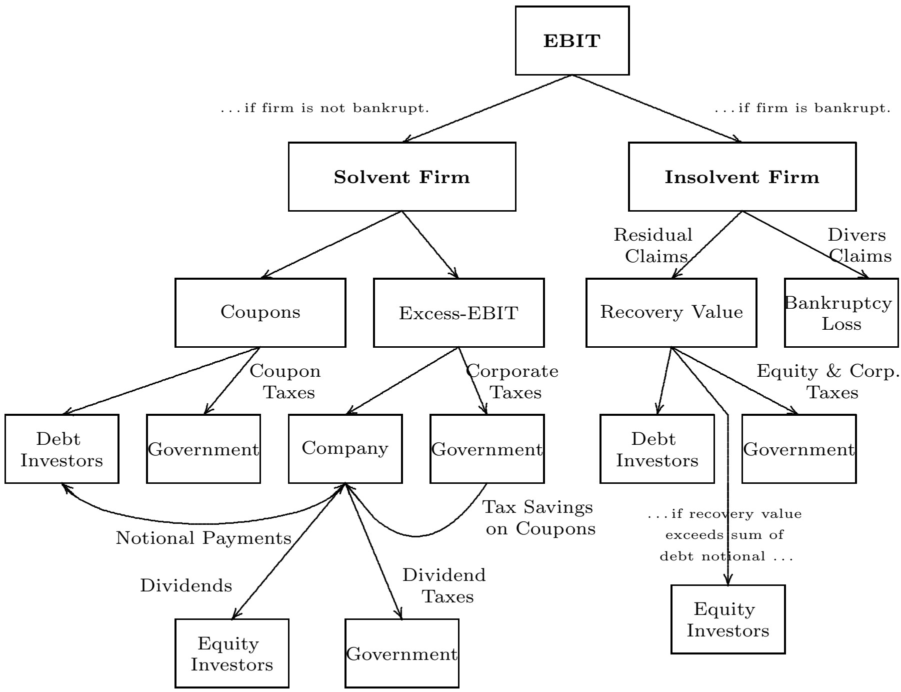
If the firm is solvent, the claims to EBIT are divided between debt and equity investors. Debt holders receive the contracted coupon payments and at maturity the notional amount. The rest of the EBIT remains with the firm.

The government imposes a tax regime which reduces the payments to the different security holders further. Debt investors pay a personal income tax rate τ^d on their coupon income. The firm pays a corporate income tax rate τ^c on corporate earnings – EBIT less coupon payments. What remains after adjusting for cash flows from financing transactions – the issue of new debt or the repayment of old debt – is paid out to equity investors as a dividend. The dividend is taxed at a personal

[25] See e.g. Kat and Verdonk (1995) or Rogers and Stapleton (1998) for other methods to overcome the convergence problem.

[26] This decision criterion can be generalized to equity holder's optimal decisions whether to honor current and future obligations of other claimants of the firm. The general splitting procedure of EBIT among claimants is not affected.

Fig. 3.5. Splitting the EBIT in a typical node among claimants



dividend tax rate of τ^e. Equity holders face the classical double taxation of corporate income.[27]

If the dividend is negative, equity investors have to infuse capital into the firm. We apply the tax system directly to these capital infusions as well so that negative EBIT and negative dividends result in an immediate tax refund. However, we exclude capital infusions due to debt repayments from the equity investor's tax base.

In a bankruptcy node, the current firm value is split among bankruptcy claimants: debt holders, government and a bankruptcy loss. The remaining value is then first distributed to debt holders proportionally to their outstanding notional. The distribution to debt holders is limited to the total notional amount. It might be possible that equity holders receive the excess portion of the residual value in bankruptcy if the bankruptcy barrier V_B allows. All bankruptcy claims except the

[27] Full double taxation can be exclude by altering the equity holders tax base from dividend income.

loss portion which is excluded from tax considerations are treated like equity for tax purposes, so that the final corporate earnings and final dividend are taxed accordingly.

To improve our numerical values we implement lattices not until the end of the longest lasting debt issue but use the analytical formulas derived in Sections 3.2 and 3.3 at each terminal node of the trinomial tree.

Note that the analysis can be extended to Tax Regime 2 and 3 easily by changing the cash flows from equity investors and the firm to the government according to the assumptions outlined in Subsection 2.1.5.

3.4.1.3 Security Valuation

All security prices are derived from the EBIT-process. Having determined all cash flows to the claimants and terminal security values, expectations in all other nodes before are discounted at the risk-free interest rate. This leads to values for the total firm value, the market value of all debt issues, and equity, and finally the value of tax payments.

The same procedure can be used to price options on equity, since we have equity values at each single node. Call option values at option maturity are

$$C_T = \max(E_T - X, 0),$$

where E_T denotes the price of equity at a specific node and X the exercise price. From these terminal values, we move backwards through the tree using the risk-free interest rate and the probabilities implied by the EBIT-process in equations (3.94).

3.4.2 Numerical Integration Scheme

If only European derivatives are studied which depend solely on their value at option maturity and if bankruptcy of the firm knocks out the derivative, its prices can be calculate computationally more efficiently.

The value of a derivative $Y_{t_0}(\eta, T)$ as of time t_0 with maturity T can be calculated as its expected payoff at maturity under the risk-neutral probability measure.[28]

[28] See e.g. Cochrane (2001).

$$\begin{aligned} Y_{t_0}(\eta, T) &= e^{-r(T-t_0)} E_{t_0}^{\mathbb{Q}}[Y_T(\eta_T, T)] \\ &= e^{-r(T-t_0)} \int_{-\infty}^{\infty} Y_T(\eta_T, T) \\ &\qquad \cdot (1 - \phi_T(t_0, T, \eta_{t_0}, \eta_T, \eta_B(T))) d\eta_T, \end{aligned} \quad (3.95)$$

where $\phi_T(t_0, T, \eta_{t_0}, \eta_T, \eta_B(T)) = P(\eta_T \in d\eta, \tau > T)P(\tau > T)$ denotes the joint probability of reaching a level of η_T at the derivative's maturity and the firm going bankrupt before T when starting today at η_{t_0}. This probability is deduced in Subsection 3.2.3.1 as a byproduct of the derivation of the bankruptcy probabilities (equation (3.43)). Recall that this bankruptcy probability for the interval $]t_0, t_j = T]$ is

$$\Phi(t_0, T, \eta_{t_0}, \eta_B(T)) = 1 - P_\nu(X_{t_j} \leq z_{t_j}, M_{t_j} < y_{t_j}),$$

with[29]

$$P_\nu\left((-1)^{p_j(\mathcal{A}_i)} X_{t_j} \leq z_{t_j} - 2\sum_{i=1}^{j}(-1)^{p_j(\mathcal{A}_i)-1} y_{t_i} \mathbf{1}_{\{i \in \mathcal{A}_i\}}\right) \\ = e^{z_n^* \frac{\nu}{\sigma^2}} N_n\left(\frac{z_{t_j} - z_j^* - \Omega_{j,n}\nu}{\sigma\sqrt{t_j}}, \Omega(\mathcal{A}_i)\right).$$

Differentiating this equation with respect to z_T, results in the desired density

$$\phi_T(t_0, T, \eta_{t_0}, \eta_T, \eta_B(T)) = \frac{\partial \Phi(t_0, T, \eta_{t_0}, \eta_T)}{\partial z_T}.$$

For the special case where the bankruptcy barrier is constant or the derivative's maturity lies before the first capital restructuring, the density simplifies to[30]

$$\phi(t_0, T, a, b) = \frac{\exp\left\{\frac{\mu a}{\sigma_\eta^2} - \frac{\mu^2 T}{2\sigma_\eta^2}\right\}}{\sigma_\eta \sqrt{T}} \left[n\left(\frac{-a}{\sigma_\eta\sqrt{T}}\right) - n\left(\frac{2b-a}{\sigma_\eta\sqrt{T}}\right)\right]. \quad (3.96)$$

In equation (3.96), $n(\cdot)$ denotes the standard normal density, a the starting value of the state variable and b its terminal value.

In general, the integral of equation (3.95) can only be solved analytically if the payoff function $Y_T(\eta, T)$ is well behaved. For option

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[29] Respective definitions of variables, sets, and notational conventions are given in Subsection 3.2.3.1.

[30] See Harrison (1985).

prices in the Corporate Securities Framework with several finite maturity debt issues closed form solutions cannot be derived.[31] However, the value of the derivative at maturity $Y_T(\eta_T, T)$ can be calculated analytically for each η_T since security values in the Corporate Securities Framework can be calculated explicitly. Therefore, we use numerical methods[32] to evaluate the integral in equation (3.95). Using the call option payoff $Y_T(\eta_T, T) = C_T$ yields the desired call option prices on the firm's equity.

Note that it is easier to differentiate numerically equation (3.96) because the resulting multivariate normal densities would include 2^N terms of both the multivariate normal distribution function and its density where N denotes the number of barriers. Depending on the accuracy of the approximation of the multivariate normal distribution, a numerical differentiation of hitting probabilities can build up considerable approximation errors that prevent the numerical integration algorithms for equation (3.95) from converging in reasonable time. To overcome the numerical problems, the hitting probability of equation (3.43) can be used to calculate sufficient data points to be able to spline the distribution function. It is numerically more efficient to spline the distribution function because the error accumulating numerical differentiation of the hitting probability is avoided. Moreover, the distribution function is monotonously increasing and has therefore an easier shape than the density. By differentiating the spline, equation (3.96) can be extracted with much higher accuracy. The probabilities can be found by evaluating the spline at the respective ending values.[33]

3.5 Summary

This chapter is devoted to demonstrate the flexibility of the Corporate Securities Framework developed in Chapter 2. Therefore, specific

[31] Toft and Prucyk (1997) analyze call prices on leveraged equity in a setting where only one perpetual debt issue is outstanding. This simplifies the analysis and allows the derivation of explicit formulae.

[32] Some numerical methods are sensitive to changes in the boundary values if the integrand only has values different from zero over closed interval. We therefore integrate from the bankruptcy-EBIT η_B to an upper bound of $\eta_{t_0} + 8\sigma_\eta\sqrt{T}$. Above the upper bound probabilities $\phi_T(t_0, T, \eta_{t_0}, \eta_T, \eta_B(T))$ are virtually zero and no value is added to the integral.

[33] Note that a similar method is used to extract implied densities from traded option prices. There the strike/implied volatility function is splined to extract the distribution function of equity prices at option maturity. See e.g. Brunner and Hafner (2002) and the references therein.

assumptions for the EBIT-process are made. In Section 3.1 it was argued that the process assumption has to be considered carefully and that the standard assumption of geometric Brownian motion might be economically debateable because it restricts EBIT-values considerably. Arithmetic Brownian motion appears a more natural choice.

Section 3.2 illustrates the ABM-case. Thereby, results from the literature have been reproduced which have only been derived for the GBM-case. In the perpetual debt ABM-model, the asset substitution problem is shown to only affect firms whose firm value dropped below the value of risk-free debt. Higher firm values allow equity holders to optimize EBIT-risk. Furthermore, the model is extended to the case of a complex capital structure. Subsection 3.2.3.1 derives closed form solutions of bankruptcy probabilities and prices of bankruptcy claims if the bankruptcy barrier changes deterministically over time. These new results are needed to propose analytical solutions for debt and equity prices.

To relate the ABM-setting to the existing literature, Section 3.3 reproduces well known results of the existing literature on GBM-firm value models within the setting of this chapter. We would like to stress that there is no reason to use GBM as the benchmark, although it has favorable mathematical properties.

Finally, Section 3.4 discusses the valuation of derivatives on corporate securities in the Corporate Securities Framework. Two numerical methods are proposed because closed form solutions are difficult to find. Thereby, the traditional trinomial tree approach may act as a benchmark method. Additionally to being able to price derivatives on corporate securities, the trinomial lattice is flexible enough to consider more complex tax structures such as Tax Regime 3 and to introduce optimal bankruptcy decisions by equity owners given the current capital structure. As a second numerical method, a direct numerical integration procedure is proposed to evaluate the expected derivative value at maturity under risk-neutral probability measure. The latter method is usually faster and the approximation error can be controlled for.

An empirical implementation of our model can be applied to a much wider range of firms than previous models. Allowing for a complex capital structure and deriving analytical solutions for debt and equity issues, we can use time series of a range of corporate securities for which market prices are readily available. Therefore, we are not dependent upon accounting data or ad-hoc estimates of parameters to implement the corporate finance framework. Elaborating on this insight, we propose a Kalman filter to estimate model parameters in Chapter 5.

4

Numerical Illustration of the ABM- and GBM-Model

This chapter is devoted to numerical illustrations of the analytical solutions derived in Chapter 3. Learning about the sensitivity of the models to parameter changes may guide us to sort out model features that might be important for empirical testing. Specifically, we illustrate the sensitivity of corporate security prices of firms with a complex capital structure where EBIT follows an arithmetic and geometric Brownian motion. Section 4.1 exemplifies the base model that is analytically and numerically solvable. Section 4.2 uses the numerical procedures proposed in Section 3.4 to value options on the firm's equity. Thereby, we focus on the equity value, its return distribution, and the structure of implied volatilities. The chapter is summarized in Section 4.3.

4.1 A Base Case Example

In this section we demonstrate the behavior of our model in detail and provide evidence on how sensitively different claims react to parameter changes. We start with a discussion of the base case parameters in Subsection 4.1.1. The analytical ABM-EBIT-model of Sections 3.2 is illustrated in Subsection 4.1.2.1. Subsection 4.1.2.2 presents the numerical extensions where bankruptcy is triggered optimally by equity owners and the government does not refund taxes when corporate earnings or dividends are negative (Tax Regime 3).

In Subsection 4.1.3, the exposition of the numerical example for the GBM-EBIT-model of Section 3.3 follows the same structure as the subsection on the ABM-version. However, we contrast the comparative static analysis with the one conducted for the ABM-model to avoid duplicate explanations.

4.1.1 The Economic Environment and the Base Case Firm

Each firm in the economy faces a constant risk-free interest rate of $r = 5\%$, which is a widely used interest rate level in numerical examples throughout the literature. The government is assumed to tax corporate earnings at a tax rate $\tau^c = 35\%$, income from equity investments and coupon income at $\tau^d = \tau^e = 10\%$. The corporate tax rate is at the upper edge of what is observed in the European Union. The personal tax rates are chosen to reflect that smaller investors are tax exempt on their investment income in many countries or evade taxes by shifting capital abroad so that only the net effect of taxation on corporate securities' prices is modeled.

The firm specific factors that are common among GBM- and ABM-firms are its current EBIT-level of $\eta_{t_0} = 100$, and its loss in the case of bankruptcy of $\alpha = 50$ %, i.e. only $1 - \alpha = 50$ % of the available bankruptcy firm value $V_B(t)$ can be distributed to financial bankruptcy claimants. Although the bankruptcy loss ratio α appears high[1], consider that α is measured with respect to total firm value in the case of bankruptcy and does not refer to a loss rate of market value of firm's assets. The bankruptcy level $V_B(T_j)$ is chosen so as to recover 50% of the outstanding notional of all debt issues.[2] In the past, the firm has chosen a financing structure as shown in Table 4.1. It has four debt issues outstanding where the short-, medium-, and long-termed issues have a maturity of 2, 4, and 10 years, respectively, and a notional of 600. Only the perpetual debt issue has a face value of 1,250.

The last column of Table 4.1 shows the bankruptcy level $V_B(T_j)$ in the respective subperiod. Due to the parameter constellation and the demand of a 50 % recovery in bankruptcy, bankruptcy levels and cumulative debt outstanding coincide.[3]

The only missing parameters are those for the ABM- and GBM-processes, respectively. If we choose a risk-neutral ABM-EBIT-drift

[1] Alderson and Betker (1995) estimate the mean percentage of total values lost in liquidation to be 36.5 % of a sample of 88 firms liquidated in the period from 1982 to 1993. Gilson (1997) reports a mean percentage liquidation cost of 44.4 %. His sample contained 108 firms recontracting their debt either out-of-court or under Chapter 11 in the period 1979-1989. Our α must be higher because it refers to a firm value representing all value from future EBIT-payments in a bankruptcy node.

[2] This is in line with usual assumptions. A standard reference is Franks and Tourus (1994) who report that on average 50.9 % of face value of total debt is recovered by debt holders.

[3] The bankruptcy level is calculated by $V_B(T_j) = \sum_k P_k(T_j)\mathbf{1}_{\{T_k<T_j\}}\rho/(1-\alpha)$, where ρ denotes the recovery fraction of a debt issue in bankruptcy.

Table 4.1. Financing structure of the base case firm

j	$\mathbf{P_j}$	In % of V	$\mathbf{T_j}$	$\mathbf{C_j}$	$\mathbf{V_B(T_j)}$
4	1,250	20.83 %	∞	6 %	1,250
3	600	10 %	10	5.5 %	1,850
2	600	10 %	4	5 %	2,450
1	600	10 %	2	4.5 %	3,050
$\sum$	3,050	50.83 %			

under the measure $\mathbb{Q}$ of $\mu = 10$ and a standard deviation of $\sigma_\eta = 40$, a GBM-process with parameters $\bar{\mu} = 3\ 1/3\ \%$ and $\bar{\sigma}_\eta = 18\ \%$ results in approximately the same security values as the ABM-case.

4.1.2 The Arithmetic Brownian Motion Firm

With this parameter combination and using equation (3.4), we find a total firm value of $V = 6,000$. The values of securities before and after taxes for the base case firm are shown in Table 4.2, Panel A. The table displays for each issued claim the going concern values $V^+_{\{\cdot\}}$ (Columns 2 to 5), the bankruptcy values $V^-_{\{\cdot\}}$ (Columns 6 to 8), and the sum of the two (Columns 9 to 11).[4] These claims are then split into the government's claim on taxes and the claim which remains with the investors. The Column *Tax Savings* indicates the tax advantage to interest payments on issued debt. The firm receives a tax credit at the marginal tax rate $\tau^c = 35\%$ on the coupon payments, whose value is implicit in $D^E_{C_j,T_j}$ of equation (3.70), but shown separately in the fourth column. This tax credit is then deducted from the taxes paid by equity owners.

Note that equity has no value in the case of bankruptcy. The maximum recovery value in each subperiod before tax $(1-\alpha)V_B(T_j)$ never exceeds the total outstanding debt notional.

The value of the loss portion in default is

$$\sum \alpha V_B(T_j) p_B(T_{j-1}, T_j, \eta_{t_0}, \eta_{T_j}) = 28.99$$

(equation (3.68)). So we have a value to claimants of $6,000 - 28.99 = 5,971.01$. If firms were allowed to report their financial statements at

[4] The total values of debt, equity and government taxes are calculated according to equations (3.61), (3.70), (3.71), respectively. The splitting into solvent, insolvent, and respective taxes is straightforwardly calculated from these equations.

Table 4.2. Security values in the ABM-Corporate Securities Framework. Panel A shows the splitting of values among solvent (Columns 2 to 5) and insolvent values (Columns 6 to 8). Columns 9 to 11 contain the sum of both values, Column 10 market values. Base case parameters: $\eta_{t_0} = 100$, $\mu = 10$, $\sigma_\eta = 40$, $r = 5\%$, $\tau^c = 35\%$, $\tau^d = \tau^e = 10\%$, $\alpha = 50\%$, $V_B(T_j)$ is determined that 50 % of the outstanding debt is recovered. Panel B shows changes in market prices when parameter values of the base case are changed.

Panel A: Base Case

j	V_j^+	Investors	Tax Savings	G_j^+	V_j^-	Investors	G_j^-	$V_j^+ + V_j^-$	Investors	$G_j^+ + G_j^-$
4.5 %, 2	592.39	587.25	15.41	5.14	0.95	0.60	0.35	593.34	587.85	5.49
5 %, 4	594.94	584.10	32.53	10.84	2.53	1.59	0.94	597.47	585.69	11.78
5.5 %, 10	611.17	585.47	77.10	25.70	6.10	3.84	2.26	617.27	589.31	27.96
6 %, ∞	1,453.41	1,308.07	436.02	145.34	19.41	12.23	7.18	1,472.82	1,320.30	152.52
$\sum$ Debt	3,251.90	3,064.88	561.06	187.02	28.99	18.27	10.73	3,280.90	3,083.15	197.75
Equity	2,690.11	1,321.71	1,221.54	146.86	0.00	0.00	0.00	2,690.11	1,321.71	1,368.40
BC					28.99	28.99		28.99	28.99	
$\sum$	5,942.01				57.99			6,000.00	4,433.85	1,566.15

Panel B: Comparative Statics of Market Values

	η_0		μ		σ_η		r (in %)		V_B (in %)		Number of Bonds		
	75	125	5.00	15.00	30.00	50.00	4.00	6.00	80 %	120 %	3	2	1
4.5 %, 2	584.06	588.87	456.44	589.15	589.12	581.62	600.57	548.43	588.94	583.06	0.00	0.00	0.00
5 %, 4	579.55	587.97	439.66	589.11	588.98	573.65	610.98	531.85	587.91	579.39	1,161.49	0.00	0.00
5.5 %, 10	580.61	593.53	427.25	597.50	597.02	568.67	645.66	510.50	592.58	582.18	585.40	1,716.99	0.00
6 %, ∞	1,297.75	1,332.39	919.79	1,349.19	1,347.30	1,257.62	1,674.47	1,034.98	1,328.06	1,303.80	1,311.62	1,284.74	3,084.09
$\sum$ Debt	3,041.97	3,102.76	2,243.14	3,124.96	3,122.41	2,981.56	3,531.67	2,625.75	3,097.49	3,048.44	3,058.52	3,001.73	3,084.09
Equity	1,016.85	1,632.11	275.31	2,572.44	1,312.98	1,345.47	2,741.24	590.83	1,322.12	1,317.80	1,339.15	1,397.70	1,508.77
BL	57.43	15.57	615.67	0.64	2.35	98.56	7.31	160.36	14.08	69.89	46.38	98.24	137.19
Taxes	1,383.75	1,749.56	865.88	2,301.97	1,562.26	1,574.41	2,469.77	1,067.51	1,566.32	1,563.87	1,555.95	1,502.33	1,269.94
Firm Value	5,500.00	6,500.00	4,000.00	8,000.00	6,000.00	6,000.00	8,750.00	4,444.44	6,000.00	6,000.00	6,000.00	6,000.00	6,000.00

discounted future income flow values, we would see this figure as a measure for total assets and liabilities of the firm.[5] In the last three columns of Table 4.2, Panel A, this value is disentangled by the financing structure of the firm. The firm would report an equity value of 2,690.11 and a total debt value of 3,280.90 on the liability side of the balance sheet. Future tax payments are not shown as separate liabilities. The before-tax values would therefore be reported.

The before-tax leverage ratio of 54.95 %, as can be seen from Figure 4.1, differs from what we see in market prices. Market prices of debt and equity account for the tax treatment of future claim payments. So we find an after-tax leverage ratio of 70.00 %, which is substantially higher than the before-tax leverage ratio. This asymmetry depends crucially on the tax regime. The government's claim absorbs much more of the equity than of the debt value. In any case, the government has a significant claim, that cannot be neglected for the valuation of a firm.

The leverage ratio after tax can be observed on markets if all securities on the liability side of a firm's balance sheets are traded. However, this is rarely the case. Most firms have bank debt and off-balance sheet financing which must be valued under additional assumptions.[6]

4.1.2.1 Comparative Statics of the ABM-Firm

In Table 4.2, Panel B, we compare the market values of the base case example in Panel A, Column 10 to the market values in different scenarios.

What would happen if parameters in the model change? We have to disentangle two effects:

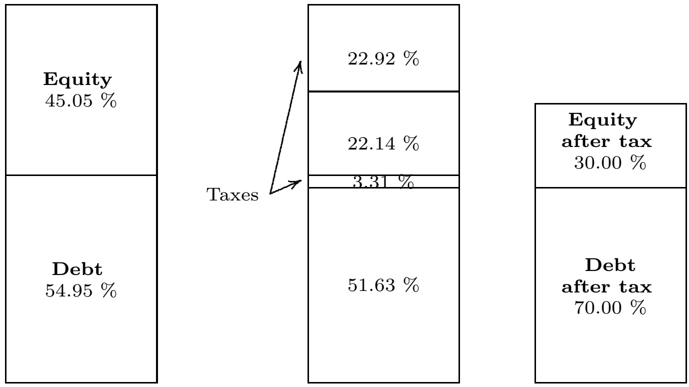
- Total firm value changes if one of the parameters in equation (3.4) is changed.
- Additionally, total firm value is distributed differently among claimants.

As an example, we take a decrease in the current EBIT η_{t_0} which is depicted in Columns 2 and 3 of Table 4.2's Panel B. As a first result,

[5] Accounting standards usually do not allow *future value* accounting. A historical cost accounting of financial statements might look quite different depending on past investments and depreciation rules.

[6] Empirical studies try to solve this important problem by using either book values (e.g. Delianedis and Geske (1999), Delianedis and Geske (2001)) or some ad-hoc appraisal. E.g. Jones, Mason and Rosenfeld (1984) apply the ratio of the notional of market valued debt to its value to non-traded debt classes thereby disregarding coupon differentials. Both methods are unsatisfactory and might lead to estimation biases. See also the discussion in Section 5.1.

Fig. 4.1. Leverage ratios in the base case before and after taxes



total firm value decreases. Since the bankruptcy barrier is unchanged, the firm moves further towards bankruptcy. The value of bankruptcy losses BC increases from 28.99 to 57.43 because the probability of going bankrupt has risen. Since total firm value has fallen from 6,000 to 5,500 and the value of bankruptcy losses has risen, the value of all other claims must decrease. We expect that short-maturity debt is affected less than long-maturity debt and equity. In fact, the longer the debt maturity the higher the change of market values. Note that debt holders are hedged naturally because the solvent and insolvent security values partly offset each other. Equity holders are affected most because the incremental payments are passed directly to them. Their losses amount to over 300 if EBIT decreases by 25. However, debt holders lose more by a 25 decrease of EBIT than they gain from a 25 increase of EBIT. Thus, security prices change asymmetrically. An increase of value is less pronounced for finite maturity securities but much larger for the perpetual and equity. The value of any debt issue has a reference value given by an equivalent but risk-free security. So increases of EBIT let debt values converge to their risk-free counterpart but contribute to the value of the equity holders' and the government's claim.

The effects of changing the risk-neutral EBIT-drift μ are much more pronounced compared to those of initial EBIT although they go into the same directions (see Columns 4 and 5 of Table 4.2's Panel B). First, a substantial firm value effect can be observed. Second, the bankruptcy probabilities change much more because not only the distance to default but also the parameters entering the formulas of Subsection 3.2.3.1

amplify each other. A higher drift reduces long-term bankruptcy probabilities. The value of the perpetual debt and equity gain most.

The following two columns display the comparative statics for EBIT-volatility. A shift of EBIT-volatility does not affect total firm value because the risk-neutral drift is left unchanged by assumption.[7] There exists an incentive for asset substitution for equity owners. Their claim value increases with the volatility. This is surprising because of the well known effect discussed in Subsection 2.1 that in the perpetual debt case a debt covenant mitigates the agency problem between debt and equity. In the case of a complex capital structure, there exist parameter constellations which confute this finding.

Most interestingly, perpetual debt is more sensitive to shifts in the volatility than equity itself. Recall that equity owners may abandon operations if the EBIT falls sufficiently low but profit from the higher upside potential. In contrast, debt holders lose higher coupon payments. Debt holders can recover some of this loss in bankruptcy proceedings, but in that case debt holders must pay full corporate and equity taxes. So the insolvent debt value of longer running bonds cannot compensate for the losses in the solvent value.

To better understand the dependence of equity value on the EBIT-volatility, Figure 4.2 depicts the equity value of an ABM-firm as a function of initial EBIT and the EBIT-volatility.[8] Given an initial EBIT, an increase of the volatility from very low levels rises equity value at first. The upward potential of gaining additional EBIT from higher risk outweighs the firm's bankruptcy cost value increase. If the volatility becomes too high, the threat of bankruptcy gets imminent and the value of bankruptcy losses dominate equity values. Therefore, the effect of a change of the EBIT-volatility on the equity value crucially depends on the firm's current state towards bankruptcy and its current risk profile.[9]

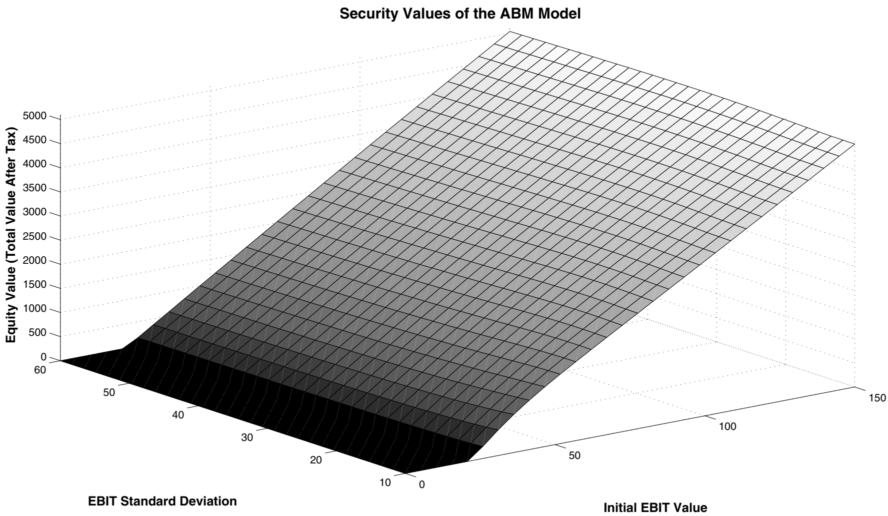
The value of securities is particularly sensitive to changes in the risk-free interest rate. As reported in Columns 8 and 9 of Panel B in Table 4.2, slight changes in the interest rate result in high swings of

[7] If we start with the physical process of equation (3.6), the risk-neutral drift will change because it depends on the risk-premium $\theta\sigma_\eta$. This would additionally involve the same effects as discussed in the case when the risk-neutral drift changes. For simplicity, this case is not considered here.

[8] The GBM-firm exhibits the same kind of pattern at of course different equity value levels. See Figure B.1 in Appendix B.

[9] Note that there exists an optimal EBIT-volatility for equity owners at which for a given initial EBIT equity value is maximized.

Fig. 4.2. Equity values after tax as a function of EBIT-volatility and current EBIT-value of an ABM-firm.



total firm value. The effects are comparable to those of the risk-neutral drift μ, however in the opposite direction.

A change of the level of the bankruptcy pattern documents minor effects except for long term debt. Increasing the bankruptcy levels to 120 % of the initial levels as of Table 4.1 rises the probability of the firm going bankrupt in the long run but less in the short run. However, the recovery values for all debt issues increase. Again, long-term debt issues suffer most.

In our framework the importance of modeling a complex capital structure can be analyzed. As first evidence, we alter the capital structure by merging the two shortest-termed debt issues. Thereby, the coupon of the longer lasting bond is maintained, thus increasing the total amount of coupon payments of the firm, and the bankruptcy barrier pattern is adjusted accordingly, i.e. is effectively increased in the subperiod with the now increased debt volume. The last three columns of Figure 4.2's Panels B compare the market values to the base case and show an interesting pattern. First, even those debt values that are not directly involved in the debt merger change their value substantially. Second, the total amount of debt value changes ambiguously because two effects counteract. On the one hand, the bankruptcy probability builds up because the bankruptcy barrier rises. On the other hand, the higher coupon payments increase the solvent debt value. In the first two debt mergers, the first effect dominates and total debt value decreases. In the last merger the second effect outweighs the higher bankruptcy probabilities. So all short-term debt holders would accept an offer to convert their debt issues into a perpetual debt issue. Third, equity value increases considerably throughout at the expense of the government's claim.[10]

4.1.2.2 Numerical Extensions of the ABM-EBIT-Model

The exercise of implementing the model numerically serves two purposes. First, the importance of taxation can be analyzed. Second, we want to answer the question of how important is an optimal bankruptcy decision of equity owners.

Therefore, we approximate the EBIT-process of equation (3.6) by a trinomial lattice[11] and distribute EBIT among different claimants at each node. The terminal values of the tree are set to the analytical formulas derived in Section 3.2.3. We choose the maturity of the tree to be 11 years, with 100 time steps per year.

Different Tax Regimes

Simulations[12] show that taxes can have a major impact on the value of the different securities. The equity value reduces little less than 20 % if switching from the most favorable (Tax Regime 2) to the least favorable (Tax Regime 3) tax regime, whereas switching from Tax Regimes 1 to 3 only induces a value reduction of around 7 %. Note that the tax regimes only affect the corporate's and equity investor's tax burden, debt holders are not concerned.

Therefore, we would like to stress the importance of modeling taxes explicitly. In all our scenarios, between 20 % to 30 % of total firm value is claimed by the government. Therefore, empirical tests of structural

[10] So, in our base case firm there is room for optimizing the capital structure.

[11] The method is the one proposed by Kamrad and Ritchken (1991) where the step size parameter $\lambda \approx 1.25$ is chosen such that the first bankruptcy barrier is perfectly hit. Recall that this procedure coincides with Boyle and Lau (1994)'s suggestion to overcome convergence problems of barrier option valuation. The implementation of the lattice is explained in detail in Subsection 3.4.1.

[12] Table B.1 is shifted to Appendix B because it does not increase the clarity of the exposition.

credit risk models should consider a tax regime. The precision of modeling a *realistic* tax regime is an issue. Our Tax Regimes 1 and 2 are analytically tractable and might give a good guideline.

Optimal Bankruptcy and Future Debt Issues

In the basic model, equity owners choose a bankruptcy pattern as a function of total debt outstanding. It would now be interesting how the analysis changes if bankruptcy is triggered optimally by equity holders.

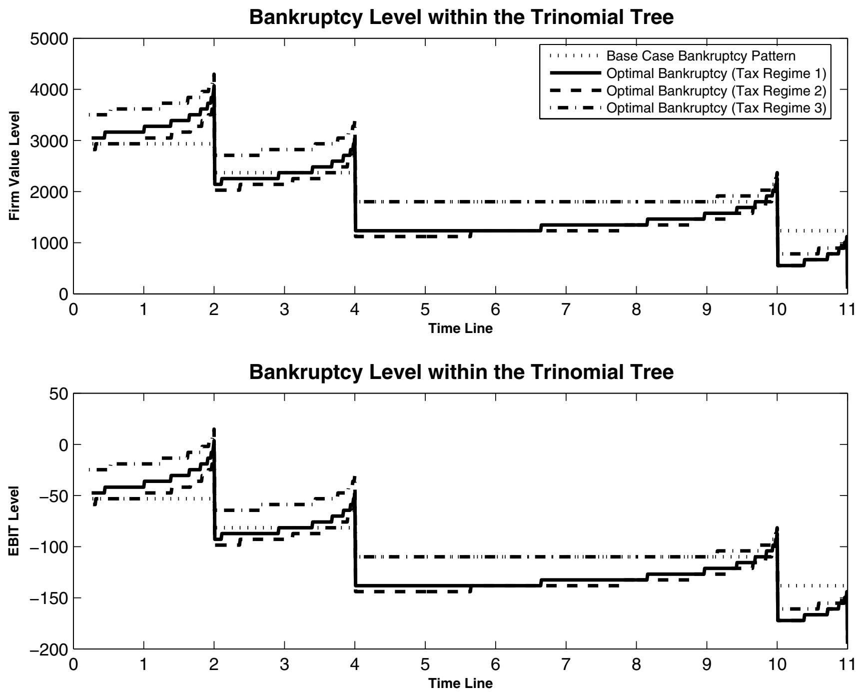
Comparing the base case under each of the tax regimes with the similar firm under optimal bankruptcy, market values of securities only change by small amounts. Short-term debt values may decrease. This implies that at some point in the future our base case bankruptcy pattern was set too low. Long-term debt prices and the equity value increases slightly as expected. However, our proposed modeling of the bankruptcy barrier seems reasonably accurate.

Figure 4.3 illustrates the decision of equity owners within the trinomial tree along the time line, where the different lines indicate the total firm value and the EBIT at which equity owners would no longer support the firm in the three tax regimes.

In a setting of a static capital structure and optimal bankruptcy trigger, the bankruptcy barrier is not constant along the time line. In the regions where the bankruptcy barrier appears relatively flat, capital outflows are determined by EBIT and coupon payments. Equity holders are willing to accept moderate cash infusions into the firm to meet the current obligations. Shortly before the repayment of a debt issue, equity owners bankruptcy decision are dominated by cash needs at debt maturity. As a result they need a sufficiently high level of EBIT, to justify the repayment of the debt issue. After the repayment, optimal bankruptcy levels fall to lower levels. Due to the reduced debt volume, total coupon payments of the firm are lower than before. The tax system influences the optimal bankruptcy levels. Tax Regime 2 is most favorable for equity holders because even the capital repayments are tax deductable. Therefore, the EBIT-level that equity holders demand to compensate for the debt repayment is lower than under the other tax regimes. In contrast, Tax Regime 3 which excludes the tax recovery of corporate losses rises the optimal bankruptcy barrier considerably.

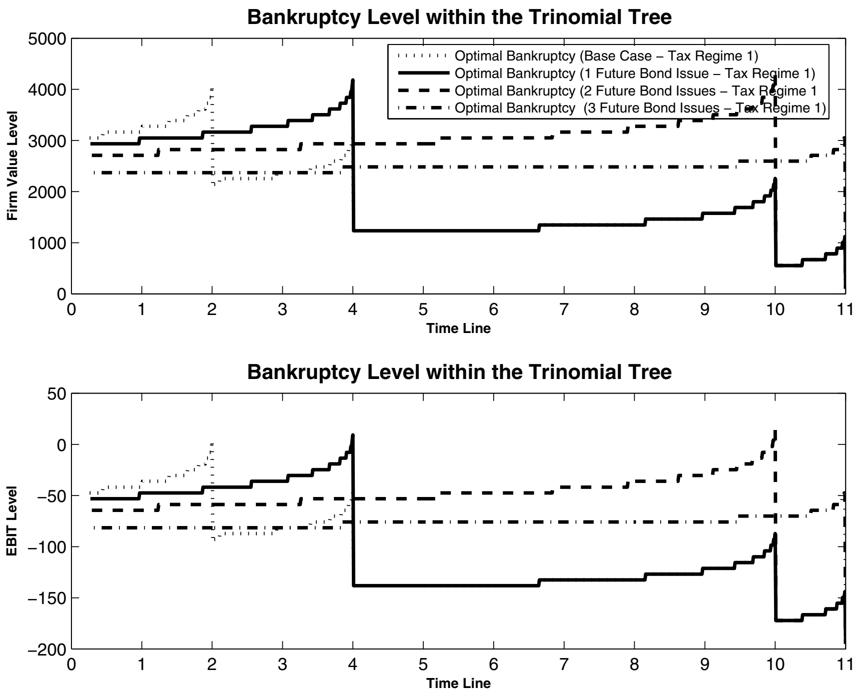
To assess if the current static capital structure setting is sufficient for the valuation of debt and equity, we need to test whether a future debt issue, instead of an equity issue, influences current values of the respective securities to an extent that it is relevant for asset pricing.

Fig. 4.3. Optimal bankruptcy level and corresponding EBIT-level within the trinomial tree approximation under different tax regimes (ABM-Corporate Securities Framework)



To investigate the price impact, each bond issue is replaced at its maturity by a new bond with a notional of the maturing debt and a 6 % coupon rate. The future bonds have almost zero present values. In contrast, equity values are influenced significantly. The crucial change concerns the future bankruptcy triggering of equity. Figure 4.4 indicates that the more future bonds the firm issues the less fluctuates the optimal bankruptcy level. One reason is that the notional amount of debt is reissued at the repayment date shifting equity holders' capital infusions due to debt repayments to times when the newly issued bonds mature. However, the bankruptcy level is higher after the first future debt issue due to higher coupon payments. The higher bankruptcy probability offsets some of the present value effect. The major component of the increase in equity value is future tax savings due to coupon deductability and the realization of tax benefits in those states

Fig. 4.4. Optimal bankruptcy level and corresponding EBIT-level within the trinomial tree approximation with future bond issues (ABM-Corporate Securities Framework)



where the firm goes bankrupt in the optimal bankruptcy case without a refinancing facility. Future bond issues reduce the present value of the government's claim. In general, the present value effect and corporate tax savings dominate and equity value rises.

Not all bond holders gain from the flattening of the bankruptcy barrier. In some cases, the unfavorable repayment schedule, i.e. the shifting of the peak of the bankruptcy barrier to later points in time harms existing bond holders. Equity holders are always better off.

In general, future debt issues cannot be ignored. However, we can conclude that the assumption of a constant bankruptcy barrier is reasonable if we allow for the refinancing of existing debt in the future. We have to account for the value of tax savings due to coupon payments in investors' equity value. Debt values are less affected by taxation.[13]

4.1.3 The Geometric Brownian Motion Firm

As mentioned in Subsection 4.1.1, a constant risk-neutral GBM-EBIT-drift of $\bar{\mu} = 3\ 1/3\ \%$ and a constant GBM-volatility of $\bar{\sigma}_\eta = 18\ \%$ yields almost similar market values of securities as in the ABM-case.[14] Comparing Panel A of Table 4.2 with Panel A of Table 4.3 shows that all four outstanding bonds are more valuable in the GBM-case than in the ABM-case. The small value gain can be attributed to a lower value of bankruptcy costs BC (17.88 vs. 28.99). BC can only be smaller in the GBM-case if bankruptcy probabilities are lower because losses in bankruptcy α, the risk-free interest rate r, and the pattern of bankruptcy barriers $V_B(T_j)$ are the same in both models. So the only difference is the probabilities with which the bankruptcy event occurs.

4.1.3.1 Comparative Statics of the GBM-EBIT-Model

In general, security values change into the same direction in both the analytical ABM- and GBM-EBIT-model if parameters are varied. However, there are differences in terms of the level of sensitivity.

Consider first the sensitivity to initial EBIT (Columns 2 and 3 of Table 4.2's and Table 4.3's Panel B). Firm value itself is more sensitive to EBIT-changes in the GBM-model as long as the risk-neutral EBIT-drift is positive.

$$\frac{\partial V}{\partial \eta_{t_0}} = \frac{1}{r} < \frac{\partial \bar{V}}{\partial \eta_{t_0}} = \frac{1}{r - \bar{\mu}} \Leftrightarrow 0 < \bar{\mu} < r$$

The difference of sensitivity of total firm value directly translates into much higher changes of security values in the GBM-case.

The same kind of reasoning is valid for changes of the risk-neutral drift $\bar{\mu}$ and the risk-free interest rate r (Columns 4 and 5 as well as 8 and 9 of Table 4.2's and Table 4.3's Panel B). Note that the GBM-firm

[13] Some of the effects of a future debt issue can be approximated analytically by manipulating equations (3.70) and (3.71).

[14] The GBM-parameter values can be determined by first choosing $\bar{\mu}$ so that firm values match under both process assumptions. Then, one can search for a $\bar{\sigma}_\eta$ that generates security market prices close to those of the ABM-case.

Table 4.3. Security values in the GBM-Corporate Securities Framework. Panel A shows the splitting of values among solvent (Columns 2 to 5) and insolvent values (Columns 6 to 8). Columns 9 to 11 contain the sum of both values, Column 10 market values. Base case parameters: $\eta_{t_0} = 100$, $\bar{\mu} = 3\ 1/3\ \%$, $\bar{\sigma}_\eta = 18\ \%$, $r = 5\ \%$, $\tau^c = 35\%$, $\tau^d = \tau^e = 10\ \%$, $\alpha = 50\ \%$, $V_B(T_j)$ is determined that 50 % of the outstanding debt is recovered. Panel B shows changes in market prices when parameter values of the base case are changed.

Panel A: Base Case

j	V_j^+	Investors	Tax Savings	G_j^+	V_j^-	Investors	G_j^-	$V_j^+ + V_j^-$	Investors	$G_j^+ + G_j^-$
4.5 %, 2	593.41	588.27	15.41	5.14	0.44	0.28	0.16	593.85	588.55	5.30
5 %, 4	597.68	586.81	32.59	10.86	1.16	0.73	0.43	598.84	587.55	11.29
5.5 %, 10	616.82	590.97	77.54	25.85	3.34	2.10	1.24	620.16	593.08	27.08
6 %, ∞	1,459.40	1,313.46	437.82	145.94	16.92	10.66	6.26	1,476.32	1,324.12	152.20
$\sum$ Debt	3,267.30	3,079.52	563.36	187.79	21.86	13.77	8.09	3,289.16	3,093.29	195.87
Equity	2,692.95	1,321.41	1,224.72	146.82	0.00	0.00	0.00	2,692.95	1,321.41	1,371.54
BC					17.88	17.88		17.88	17.88	
$\sum$	5,960.26				39.74			6,000.00	4,432.58	1,567.42

Panel B: Comparative Statics

	η_0		μ (in %)		σ_η (in %)		r (in %)		V_B (in %)		Number of Bonds		
	75	125	3.00	3.67	13.00	23.00	4.00	6.00	80 %	120 %	3	2	1
4.5 %, 2	572.49	589.13	583.14	589.13	589.15	583.13	600.58	503.14	589.13	584.03	0.00	0.00	0.00
5 %, 4	566.64	588.98	579.35	589.00	589.11	576.57	611.09	486.34	588.96	580.88	1,163.54	0.00	0.00
5.5 %, 10	566.91	596.52	580.97	596.69	597.56	571.17	646.96	466.49	596.41	583.88	588.11	1,690.42	0.00
6 %, ∞	1,256.24	1,337.26	1,286.71	1,339.97	1,349.12	1,238.03	1,682.87	943.01	1,336.16	1,299.13	1,313.09	1,261.24	2,917.63
$\sum$ Debt	2,962.28	3,111.89	3,030.18	3,114.80	3,124.93	2,968.90	3,541.50	2,398.98	3,110.66	3,047.93	3,064.74	2,951.66	2,917.63
Equity	419.75	2,260.88	711.47	2,260.14	1,312.50	1,358.63	6,675.90	234.39	1,318.02	1,325.14	1,342.11	1,426.48	1,564.37
BL	92.21	7.58	53.35	6.01	0.53	86.92	1.84	281.40	6.09	57.13	34.29	108.83	201.27
Taxes	1,025.76	2,119.65	1,205.01	2,119.05	1,562.03	1,585.54	4,780.76	835.22	1,565.23	1,569.80	1,558.86	1,513.02	1,316.74
Firm Value	4,500.00	7,500.00	5,000.00	7,500.00	6,000.00	6,000.00	15,000.00	3,750.00	6,000.00	6,000.00	6,000.00	6,000.00	6,000.00

value change is identical for an increase of the risk-neutral drift and a decrease of the interest rate by the same amount.[15]

$$\frac{\partial \bar{V}}{\partial \bar{\mu}} = \frac{\eta_{t_0}}{(r-\bar{\mu})^2} = -\frac{\partial \bar{V}}{\partial r}$$

However, the probabilities of bankruptcy and bankruptcy prices are both functions of μ and r and so are directly influenced as well.[16] By the effect of interest rates on the GBM-security prices we can therefore conclude that the GBM-model is again more sensitive to parameter changes where the major part of the value change is due to the firm value effect. Especially long-term securities benefit from increases of the risk-neutral drift and a decrease of risk-free interest rates. Note further that the ABM-firm accepts much higher interest rates without going bankrupt than the GBM-firm.

As in the ABM-model, the GBM-firm value is unaffected by changing the volatility (see Columns 6 and 7 of Table 4.2's and Table 4.3's Panel B). The 5 % percentage point change of GBM-volatility affects each security differently compared to the absolute move of 10 in the ABM-case. Although the changes in the two models are on different bases and not entirely comparable, note that the slope of the term structure of bankruptcy probabilities becomes much steeper in the GBM-model if volatility is increased. Whereas bond prices for short term bonds of the GBM-firm don't fall as much as those of the ABM-firm (583.13 vs. 581.62), the opposite is true for long-term bonds (1,238.03 vs. 1,257.63). The difference between two corporate bonds' prices of equal characteristics is only attributable to bankruptcy probabilities over the life of the bond. As in the ABM-model, equity owners of the GBM-firm have an incentive for asset substitution. Note that the low risk ABM- and GBM-firms have almost identical security prices. Firm values are identical and bond prices become nearly risk-free. Therefore, equity prices converge given the same tax regime.

Columns 10 and 11 of Table 4.2's and Table 4.3's Panel B depict the case of a change of the bankruptcy barrier. The effects on security prices are similar to those of changes of the EBIT-volatility. Again, the

[15] In contrast, the derivatives of the ABM-firm value with respect to the risk-neutral drift is $1/r^2$, and with respect to the interest rate is $-\eta_{t_0}/r^2 - 2\mu/r^3$. Therefore, the major difference of sensitivities with respect to the risk-neutral drift and the risk-free interest rate between the ABM- and the GBM-firm is due to the firm value effect.

[16] Bankruptcy probabilities are a function of r because the pattern of bankruptcy barrier is defined in terms of a value covenant for debt $V_B(T)$. Changing r, changes the equivalent EBIT-barrier $\eta_B(T)$ by equations (3.4) and (3.78).

GBM-firm's term-structure of bankruptcy probabilities steepens if the bankruptcy barrier is increased compared to the ABM-case.

The last three columns of Table 4.2's and Table 4.3's Panel B display the first qualitative dissimilarities of the comparative statics analysis between ABM- and GBM-firms. Whereas the debt holders of the ABM-firm would accept the offer of equity holders for an exchange of their bond issues with the perpetual bond, debt holders of the GBM-firm would unequivocally reject the offer. They demand a higher coupon than the offered 6 %, implying that the term structure of credit spreads in the GBM-model is steeper than in the ABM-model.

4.1.3.2 Numerical Extension of the GBM-EBIT-Model

Security values of the GBM-firm behave qualitatively similar to those in the ABM-model under the different tax regimes and optimal bankruptcy.[17]

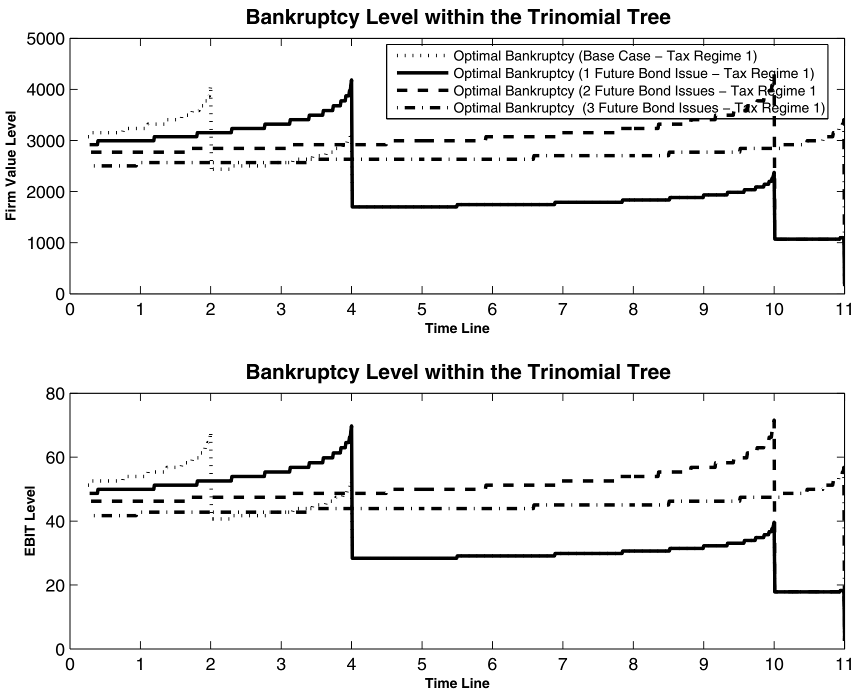
- Taxes are an important part of firm value.
- The optimal bankruptcy barrier is not constant.
- The barrier becomes close to constant if future debt issues are considered.
- However, the structure of the optimal bankruptcy barrier is slightly different. Comparing Figure 4.4 for the ABM-firm and Figure 4.5, the GBM-EBIT cannot become negative. As such, the optimal GBM-bankruptcy EBIT is much higher than the ABM one. In terms of firm value, the optimal bankruptcy barrier is equal at the peaks on the refinancing dates. Before the refinancing dates the GBM-firm value barrier is lower but has the same exponentially growing shape as the ABM-firm value barrier. The growth rate in the ABM-model is higher than in the GBM-model.

4.2 Valuing Equity Options

One of the frequently discussed issues in the asset pricing literature is why the theoretical option prices in the Black and Scholes (1973)/Merton (1974) framework cannot be observed empirically. Implied volatilities of observed option prices calculated in the Black/Scholes setting are not constant. They are higher for lower strike prices than for higher

[17] The detailed figures can be found in Table B.2 in Appendix B.

Fig. 4.5. Optimal bankruptcy level and corresponding EBIT-level within the trinomial tree approximation with future bond issues under Tax Regime 1 (GBM-Corporate Securities Framework)



ones. The convex relationship is commonly referred to as an option's volatility smile or smirk.[18]

[18] The literature on implied option volatilities is huge. Early evidence of the existence of implied volatility smiles is MacBeth and Merville (1979) who base their work on studies of Latané and Rendleman (1975) and Schmalensee and Trippi (1978). Emanuel and MacBeth (1982) try to relax the stringent volatility assumption of the Black and Scholes (1973) to account for the volatility smile but fail by a constant elasticity of variance model of the stock price. More recent studies of Rubinstein (1994), Jackwerth and Rubinstein (1996), and Jackwerth and Rubinstein (2001) take the volatility smile as given and exploit option prices to extract implied densities of the underlying asset. See also Buraschi and Jackwerth (2001) who report that after the 1987 market crash, the spanning properties of options decreased. They conclude that more assets are needed for hedging option prices which hints to additional risk factors such as stochastic volatility.

Several extensions of the Black/Scholes framework have been suggested to account for these empirical observations which can be categorized in two groups. First, a pragmatic stream of the literature introduced volatility structures and thus changed the physical distributional assumptions for the underlying. Although this procedure yields satisfying results for equity option trading, economic intuition is still lacking which underpins the use of volatility structures.[19]

Second, an economic stream of the literature tried to explain why the pricing kernel, defined as the state price function at option maturity, is different to the one implied by the Black and Scholes (1973) model. Arrow (1964) and Debreu (1959), Rubinstein (1976), Breeden and Litzenberger (1978), Brennan (1979) and others relate state prices to investor utility and the state dependent payoffs of securities and disentangle the effect of the investor's utility function from the probability distribution of the underlying asset. Therefore, the pricing kernel – and the value of securities – depends on assumptions about the utility function of the investor and on the distribution of the security.[20] The pricing kernel is valid for all securities in an economy. However, when taking an individual firm's perspective, the simple pricing kernel needs to be augmented by additional risk factors such as default or liquidity.

Using the economic framework of Chapter 2 and the analytical solution of Chapter 3 a simpler economic explanation might be suggested: The implied Black/Scholes-volatility smile of equity options can be explained as a result of the specific ability of equity holders to declare bankruptcy. This feature introduces dependence on the particular path of EBIT and changes the distribution of equity due to the conditioning on survival until option maturity. Contract design of equity, especially the limited liability, changes the local volatility of equity which in turn depends on the current state of the firm and influences the pricing kernel and equity option's implied volatilities. Our approach is related to Geske (1979)'s compound option approach. However, in contrast to Geske (1979) whose underlying of the compound option might be interpreted as a Merton (1974)-like firm with only one finite maturity zero

[19] See e.g. Rubinstein (1994) who adopts this procedure to the binomial model by allowing an arbitrary distribution of the underlying at option maturity. Heston and Nandi (2000) derive closed form solutions for options where the volatility of the underlying which follows a GARCH process.

[20] Franke, Stapleton and Subrahmanyam (1999) analyze the pricing kernel directly by comparing pricing kernels of investors with changing degree of risk aversion at different levels of investor's wealth. Franke et al. (1999) find the pricing kernel to be convex to accommodate for the dependence of risk aversion and investor's wealth leading to convex implied Black/Scholes-volatilities.

bond outstanding, the framework of Chapter 2 allows for a complex capital structure. As will be shown later, the debt structure influences the slope and the level of the implied volatility smile.[21]

In a related empirical study, Bakshi, Kapadia and Madan (2003) link the volatility smile to the distribution of equity returns. They show that a higher skewness and a lower kurtosis of equity returns result in steeper volatility smiles. Empirical evidence supports their hypothesis. However, Bakshi et al. (2003) do not offer an economic explanation why individual stock's returns should be skewed. In our EBIT-based firm value framework, the leverage ratio depends on the current state of the firm with respect to bankruptcy. Firms far from bankruptcy and with low leverage ratios exhibit a risk-neutral equity distribution that reflects the properties of the assumed EBIT-process. The function of implied equity option volatilities with respect to strike prices are at a low level but steep. The closer the firm is to bankruptcy skewness and kurtosis of equity values increase. The implied volatility level rises significantly but the smile becomes flatter, at least in the ABM-case. However, we stress that higher moments of equity returns might be misinterpreted in the presence of bankruptcy probabilities. Moreover, a key determinant of implied volatility structures is the firm's capital structure.

Toft and Prucyk (1997) value equity options in the restrictive Leland (1994) framework. The setting of Chapter 2 extends the Toft and Prucyk (1997) analysis. Their model is a special case of our framework if we restrict the capital structure to perpetual debt, the tax structure to corporate taxes only, and if we assume that free cash flow after taxes follows a geometric Brownian motion instead of EBIT following an arithmetic or geometric Brownian motion. We are able to analyze the pricing of options under different assumptions for the EBIT-process and of firms that have a complex capital structure. Toft and Prucyk (1997) shed some light an complex capital structures when they proxy a debt covenant in the perpetual debt case by the amount of a firm's short term debt. Our model allows to analyze firms with short term debt and long term debt directly.

Although security prices have analytical solutions as shown in Chapter 3, options written on the equity value can only be valued numeri-

[21] One might argue that these observations do not carry over to index options because the index cannot go bankrupt. However, the index consists of firms that can go bankrupt. The bankruptcy probability cannot be diversified away. E.g. pick an index of two firms with uncorrelated bankruptcy probabilities of 1 % per year. Only with a probability of 98.01 % both firm survive the next year. With almost 2 % probability the index exhibits at least one bankruptcy.

cally. Section 3.4 provides two numerical methods which will be used in this section: first, a trinomial lattice is used to approximate the EBIT-process and equity options are valued by backward induction. Second, we calculate the expected option value at option maturity by numerical integration which is possible because we investigate European style equity options. Additionally, both methods allow a fairly good approximation of the first four central moments of the equity value and its return distribution at option maturity. Therefore, we can directly compare our simulation results to empirical findings of Bakshi et al. (2003).

In the following subsections, equity option prices in our Corporate Securities Framework are compared to the Black/Scholes framework by calculating implied Black/Scholes volatilities. Firms with EBIT following an arithmetic and a geometric Brownian motion are analyzed, we can show that the general structure of implied volatilities does not only depend on the distributional assumption but primarily on the firms being allowed to go bankrupt and its capital structure.

A plot of implied volatilities against strike prices is used to visualize the functional form of the option prices. A plot of the unconditional partial equity density at option maturity helps for an in-depth analysis of the implied volatility structures.[22]

A comparative static analysis is performed for the EBIT-volatility, the risk-neutral EBIT-drift, the risk-free interest rate, the EBIT-starting value, and the financing structure of the firm. The following subsections summarize and interpret the results of the simulation.

To ease the exposition, the ABM- and GBM-firm is assumed to have a simpler capital structure than in Subsection 4.1.1. Since short term debt seems to have a major impact on option prices, both firms have issued only one short term bond with a maturity of $T_1 = 1$ year, a notional of $P_1 = 1,850$, and a coupon of $C_1 = 4.5$ %, and one perpetual bond with a notional of $P_2 = 1,250$ paying a coupon of $C_2 = 6$ %. As well, the loss of firm value in default is changed to $\alpha = 70$ % (65 %) in the ABM-case (GBM-case) which yields bankruptcy barriers of $V_B(T_1) = 5,083.33$ $(4,375.14)$ and $V_B(T_2) = 2,083.33$ $(1,785.71)$ if 50 % of the total notional of debt is to be recovered in the case of bankruptcy. The other parameter values are those of the base case in Subsection 4.1.1.

[22] The equity density is called partial because it only starts at an equity value of zero and the intensity of the zero value is not displayed in the graphs. As a result, the shown densities need not integrate to one. The "missing" probability mass is attributable to bankruptcy.

The tax structure is fixed. The way of determining the pattern of bankruptcy levels in the subperiods is left constant throughout the analysis.

4.2.1 Comparison of Numerical Methods

The trinomial lattice approach was implemented by using 900 steps[23] until option maturity and $\lambda \approx 1.75$. Due to the short option maturity a higher step size was needed to better grasp the knock-out feature of the option. Since the first bankruptcy barrier is most important, it appeared useful to choose λ to hit the first barrier exactly. For firms close to bankruptcy, λ is set close to 1.25 to get enough non-bankrupt nodes.

For the numerical integration approach, the risk-neutral distribution function was approximated by 200 points from the bankruptcy-EBIT to 8 standard deviations above the expected EBIT at option maturity. The accuracy of the multivariate normal distribution was set to 1e-8 so that each point had a maximum accumulated error of not more than 1e-6. The spline of the equity value distribution function at option maturity has an even lower error because it tends to eliminate errors of different sign. The numerical integration is performed with an error of 1e-6.

The trinomial tree approach and the numerical integration method give identical option prices up to minor approximation errors due to the numerical methods. Table 4.4 summarizes the differences of option prices and implied volatilities for the base case scenario of $\eta_{t_0} = 100$ and the option maturity being 6 months. The table reports in Panel A the relative option price difference and in Panel B the relative difference of implied volatilities. We choose to report relative differences to account for level effects.

Despite the fact that option prices and implied volatilities are very sensitive to the approximation, the differences between option prices are small given that they range from -0.0179 % to 0.2584 % of the respective price of the numerical integration approach. The respective range for implied volatilities is -2.1731 % and 0.1078 % of the implied volatilities of the numerical integration approach. Especially the mean differences MD are very comforting with only 0.0035 %. Comparing the maximum and the minimum pricing differences in Panel A, both approaches tend

[23] In the tree approach option prices were hard to approximate when the bankruptcy barrier changed during the option's life. If the option maturity falls after the short-term debt maturity, 1,100 steps were needed to get reasonable accuracy.

Table 4.4. Summary of relative differences of equity option prices (Panel A) and implied volatilities (Panel B) of the $\eta_0 = 100$, $T_O = 0.5$ scenario sets between the numerical integration and the lattice approach. The table reports the minimum and maximum relative differences as well as the mean absolute difference (MAD), the mean difference (MD), and the standard deviation of the mean difference (SDev). All numbers are in % of the values of the numerical integration approach.

Panel A: Relative Option Price Differences

Scenario	Obs.	Min	Max	MAD	MD	SDev.
All ABM and GBM scenarios	609	-0.0179	0.2584	0.0087	0.0035	0.0223
ABM only	315	-0.0151	0.1697	0.0079	0.0022	0.0182
GBM only	315	-0.0179	0.2584	0.0094	0.0048	0.0252
OTM-options	290	-0.0179	0.2584	0.0138	0.0072	0.0313
ITM-options	290	-0.0109	0.0366	0.0040	0.0002	0.0058

Panel B: Relative Implied Volatility Differences

Scenario	Obs.	Min	Max	MAD	MD	SDev.
All ABM and GBM scenarios	609	-2.1731	0.1078	0.0430	-0.0342	0.1408
ABM only	315	-0.0355	0.0559	0.0102	-0.0032	0.0133
GBM only	315	-2.1731	0.1078	0.0760	-0.0662	0.1904
OTM-options	290	-0.2532	0.0690	0.0248	-0.0175	0.0373
ITM-options	290	-2.1731	0.1078	0.0621	-0.0518	0.1985

to prices in-the-money options and options of the ABM-EBIT-model equally well whereas higher price differences for the GBM-model and out-of-the-money options can be observed. However, the worst mean differences of 0.0072 % for out-of-the-money options is still remarkably good.

Looking at implied volatilities in Panel B, the pattern of relative differences is almost similar. ABM-option implied volatilities differences are again smaller than those of GBM-option implied volatilities. However, in-the-money option implied volatilities show higher differences than out-of-the-money implied volatilities. The effect is not surprising because – in contrast to out-of-the-money options – implied volatilities of in-the-money options tend to be particularly sensitive to the approximated option price and the expected equity value. However note, that the mean differences of implied volatilities are sufficiently low, as well.

4.2.2 Equity Values and Their Densities at Option Maturity

Analyzing the densities of equity values becomes quickly difficult if the firm has a complex capital structure. To differentiate between different effects and to compare different scenarios Tables 4.5 and 4.7 summarize the first four moments of the equity distribution (Columns 9 to 12) and its return distribution (Columns 13 to 15) at option maturity[24] for all scenarios and for ABM- and GBM-firms, respectively. Panel A depicts the case of a 6 month option, Panel B the one of a 9 month option. The first three rows cover the base case firm with different initial EBITs η_0. Rows 4 and 5 illustrate the case of changes of the risk-neutral EBIT-drift μ, followed by two rows of the case of changed EBIT-volatility σ_η, risk-less interest rates r, and two rows of different option maturities T_O. The last four rows depict scenarios with different maturities for the short-term bond T_1. Figures 4.10 and 4.12 depict the equity value densities of the ABM- and GBM-firm in the 6 scenarios: a shift of the initial EBIT (Panel A), the risk-neutral drift (Panel B), the EBIT-volatility (Panel C), the risk-free interest rate (Panel D), the option maturity (Panel E), and the maturity of the short-term bond (Panel F). In the accompanying Figures 4.11 and 4.13, the density plots of equity values are translated into the corresponding equity return density plots.

4.2.2.1 General Comments

Equity is the residual contract to EBIT with a right to abandon future obligations. This has several effects on the risk-neutral density of equity values for a future point T_O if a good-state firm approaches bankruptcy. We start the discussion with the ABM-case.

One of the decisive elements of the moments of the equity value distribution is the position of the expected equity value. From Figure 4.6 we gain the insight that the expected equity value moves from the center of the distribution further towards zero if initial EBIT approaches zero. The position of the expected equity value in the partial density influences equity value moments considerably. The change of the moments of the equity value distribution is illustrated in Figure 4.7. As can be seen, the distribution and its moments undergo different phases.

(1) If an ABM-firm is far from bankruptcy, the density of equity is almost normal. Given a capital structure, finite maturity debt values at option maturity are rather insensitive to changes of EBIT[25] because debt has an upper value limit, i.e. its risk-free counterpart.

[24] The first four central moments shown are calculated by numerical integration of the respective expectation similar to option prices in equation (3.95). Without loss of generality, we define the return distribution of equity values with respect to its expected value. Therefore, the expected values of the return distribution are zero in all scenarios.

[25] The sensitivity of a security at option maturity with respect to the EBIT-level is important because we integrate over the product of the equity values which depends on the EBIT prevailing at option maturity, and the probability of oc-

Table 4.5. Unconditional central moments of the equity and its return distribution at option maturity as well as LS-regression results of the form $\ln(\sigma) = \beta_0 + \beta_1 \ln(X/E_t)$ in the ABM-Corporate Securities Framework. The base-EBIT $\eta_0 = 100$ and parameters are changed as displayed in the first 6 columns. The bankruptcy barrier is set such that the recovery of debt holders is 50 % of total debt outstanding and losses in the case of bankruptcy are $\alpha = 70$ %. T_1 denotes the maturity of the short bond, $\Phi(T_O)$ the bankruptcy probability up to option maturity T_O.

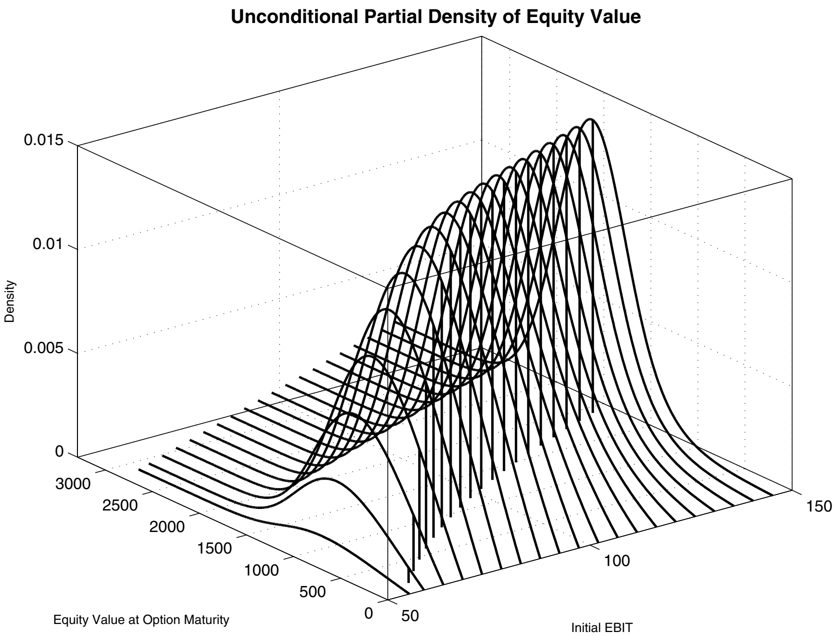
Panel A

$\eta(0)$	μ	σ_η	r	V_B	T_1	T_O	$\Phi(T_O)$	$E^Q(E(T_O))$	$\sigma(E(T_O))$	$\zeta(E(T_O))$	$\kappa(E(T_O))$	$\sigma(r_E(T_O))$	$\zeta(r_E(T_O))$	$\kappa(r_E(T_O))$	$exp(\beta_0)$	β_1	p-value
75.0	10.00	40	5.00%	5083.33	1.00	0.500	40.2354%	640.63	453.47	1.7795	3.7381	49.79%	0.8273	4.0334	1.6562	-0.5396	0.00
100.0							7.8111%	1205.89	415.28	0.4427	2.9036	40.40%	-2.0186	11.1134	0.6396	-0.9108	0.00
125.0							0.7781%	1625.25	403.06	-0.2763	3.4487	30.39%	-2.4926	15.0766	0.3688	-0.8365	0.00
100.0	8.00						81.1339%	185.78	384.86	2.8430	8.7371	71.49%	2.4950	6.4728	3.7620	-0.2015	0.00
100.0	15.00						0.0000%	2662.99	358.18	-0.0118	3.0187	13.83%	-0.6551	3.6748	0.1934	-0.5102	0.00
100.0		30					1.8075%	1311.98	319.59	-0.3454	3.6215	30.44%	-2.7058	16.7824	0.3839	-1.0831	0.00
100.0		50					16.1224%	1101.28	487.79	1.1189	3.1794	45.03%	-1.2774	8.3216	0.9108	-0.7493	0.00
100.0			4.00%				0.0000%	2761.85	448.08	-0.0091	3.0106	16.90%	-0.8020	3.9919	0.2327	-0.5055	0.00
100.0			5.50%				92.3149%	81.74	292.62	4.2235	18.9662	69.87%	3.7564	14.3446	5.0367	-0.1265	0.00
100.0						1.000	18.5796%	1250.11	465.23	1.7271	4.0329	29.93%	1.1891	2.8677	0.6105	-0.8920	0.00
100.0						2.000	18.5988%	2230.36	768.89	1.5186	3.4094	29.37%	0.6149	4.4332	0.6818	-0.5958	0.00
100.0					0.25		1.6363%	2364.47	370.51	0.2903	3.0306	15.88%	-0.4819	3.8408	0.5898	-1.0769	0.00
100.0					0.75		7.8111%	1231.06	390.92	0.4954	3.1376	36.47%	-2.0377	12.5244	0.5971	-0.9953	0.00
100.0					1.50		7.8111%	1171.37	435.42	0.4780	2.7821	43.75%	-1.9311	9.8993	0.6810	-0.8060	0.00
100.0					3.00		7.8111%	1126.61	447.80	0.5648	2.8050	46.44%	-1.8343	9.0008	0.7166	-0.7134	0.00

Panel B

$\eta(0)$	μ	σ_η	r	V_B	T_1	T_O	$\Phi(T_O)$	$E^Q(E(T_O))$	$\sigma(E(T_O))$	$\zeta(E(T_O))$	$\kappa(E(T_O))$	$\sigma(r_E(T_O))$	$\zeta(r_E(T_O))$	$\kappa(r_E(T_O))$	$exp(\beta_0)$	β_1	p-value
75.0	10.00	40	5.00%	5083.33	1.00	0.750	47.6579%	654.46	523.34	1.8376	3.8319	51.68%	1.3333	3.3410	1.5753	-0.4889	0.00
100.0							13.7522%	1228.50	446.94	1.0660	3.3341	37.03%	-1.3127	10.2378	0.6298	-0.9340	0.00
125.0							2.5796%	1650.77	451.70	0.1342	3.2643	32.12%	-2.1618	13.3058	0.3627	-0.8067	0.00
100.0	8.00						83.9805%	188.91	427.49	2.9499	9.3417	72.39%	2.6412	7.1579	3.2852	-0.1823	0.00
100.0	15.00						0.0006%	2705.31	438.66	-0.0135	3.0249	16.90%	-0.8294	4.3224	0.1919	-0.5079	0.00
100.0		30					4.5342%	1337.25	342.38	0.2403	3.4897	29.56%	-2.2579	15.7406	0.3760	-1.0760	0.00
100.0		50					23.9330%	1120.98	546.56	1.5499	3.7139	42.51%	-0.3310	6.6534	0.8989	-0.7486	0.00
100.0			4.00%				0.0009%	2798.91	548.75	-0.0102	3.0119	20.87%	-1.0339	4.9413	0.2312	-0.5045	0.00
100.0			5.50%				93.4312%	83.15	319.18	4.3925	20.3326	68.91%	4.0033	16.2126	4.2848	-0.1183	0.00
100.0						0.500	7.8111%	1205.89	415.28	0.4427	2.9036	40.40%	-2.0185	11.1144	0.6396	-0.9108	0.00
100.0						1.000	18.5796%	1250.11	465.23	1.7271	4.0329	29.93%	1.1891	2.8677	0.6105	-0.8920	0.00
100.0						2.000	18.5988%	2230.36	768.89	1.5186	3.4094	29.38%	0.6147	4.4440	0.6818	-0.5958	0.00
100.0					0.25		1.6366%	2394.83	467.62	-0.0612	4.0098	19.89%	-0.8657	5.1310	0.4958	-0.9898	0.00
100.0					0.75		13.7522%	1253.98	403.03	1.5582	3.8167	27.33%	0.8803	2.7616	0.5762	-0.9417	0.00
100.0					1.50		13.7522%	1193.54	480.26	0.9597	3.0557	42.37%	-1.4494	9.1353	0.6789	-0.8137	0.00
100.0					3.00		13.7522%	1148.22	501.85	0.9993	3.0741	46.01%	-1.4099	8.2204	0.7164	-0.7091	0.00

Fig. 4.6. Equity value densities of 6 month equity options in the ABM-Corporate Securities Framework as a function of η_0. Expected equity values are indicated by solid lines. Path probabilities are obtained by differentiating the splined distribution function of EBIT at option maturity.



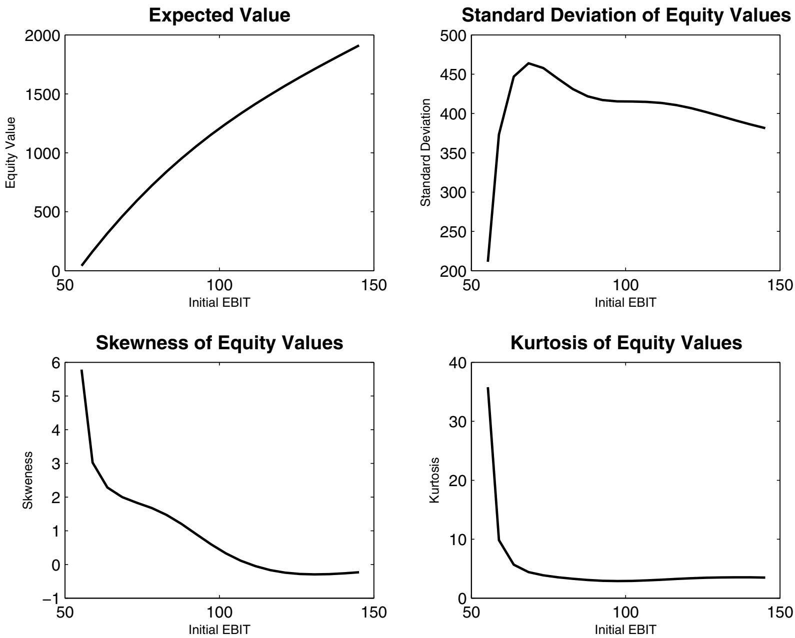
Equity of firms far from bankruptcy gain with each increase of EBIT a constant amount making the equity distribution symmetric and driving excess-kurtosis to zero.

Taking the high risk-neutral drift and low interest rate scenarios in Table 4.5 as an example, the skewness is slightly below 0 and kurtosis slightly above 3.

(2) The abandonment option bounds the value of equity from below at zero. The equity density will therefore exhibit a mass concentration at zero which is equal to the bankruptcy probability for a given time in the future. As a result, the continuous part of the unconditional distribution of equity value only integrates to $1 - \Phi(t_0, T, \eta_{t_0}, \eta_B(T)) \leq 1$. The bankruptcy probability pulls the

currence which is a function of both the initial EBIT and the EBIT at option maturity, from the bankruptcy barrier to infinity.

Fig. 4.7. Equity value density moments of 6 month equity options in the ABM-Corporate Securities Framework as a function of η_0. The moments are obtained by numerical integration.



expected equity value towards zero which implies that debt issues leave the region where they are insensitive to bankruptcy. However, finite maturity debt is still a sticky claim despite the slightly higher bankruptcy probability because they receive some recovery in bankruptcy. The stickiness of a debt issue depends on its maturity. Thereby, shorter maturity bonds are less sensitive than longer-term bonds. As a result, equity value suffers higher losses than debt if EBIT decreases but benefits more if EBIT increases. The asymmetry increases when approaching bankruptcy. It causes equity value skewness to decrease and excess-kurtosis to increase. The behavior of equity value skewness directly depends on the redistribution of assets for different realizations of EBIT at option maturity. If the firm moves closer to bankruptcy, equity value suffers from higher values of bankruptcy losses while finite maturity debt values are less affected. The probability of low equity values increases more than the normal distribution would predict, which skews the equity distribution to the right.[26]

This type of the equity value distribution is illustrated by the low risk and the high EBIT-value scenarios in Table 4.5.

(3) If the firm moves further towards bankruptcy, the bankruptcy event starts to dominate the shape and moments of the distribution. Equity skewness increases rapidly from its intermediate lower values. Equity kurtosis decreases to values even below 3. These effects are only driven by the mass concentration of the equity value distribution at 0 and have no direct economic interpretation. Pick as an example the base case 100-EBIT-firm and the cases of the short-term debt with longer maturities (Table 4.5). All these scenarios have in common that they have a modest bankruptcy probability, but a relatively high standard deviation.

(4) Close to bankruptcy, equity values are dominated by the bankruptcy event. Since the left tail of the distribution is no longer existing, equity value standard deviation decreases, skewness increases further, and kurtosis can reach levels significantly above 3. Good examples of these cases are the low 75-EBIT, the low risk-neutral drift, the high interest rate and the high risk firm (Table 4.5).

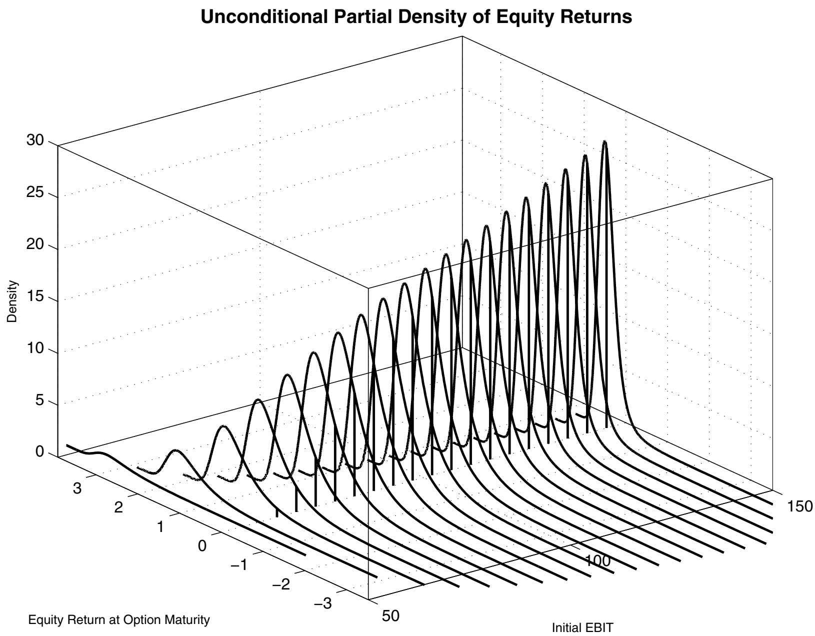
The risk-neutral density of continuous equity returns can be directly gained from the equity value density.[27] Figure 4.8 illustrates the equity return densities for initial EBIT ranging from 50 to 150. The closer the firm moves towards bankruptcy, the more moves the peak of the return density into the positive quadrant although the return is defined relative to its expected value.[28] In contrast to the equity value distribution, the equity return distribution has support over the whole real line because as the equity value at some future point approaches 0, its continuous return goes to minus infinity. However, the return distribution integrates only to $1 - \Phi(t_0, T, \eta_{t_0}, \eta_B(T)) \leq 1$, as well. Therefore, bad state firms (low initial EBIT, low risk-neutral drift, or high risk-less interest rate) have a positive expected return given that they survive

[26] Intuitively, we can argue directly within the trinomial tree: Fix the EBIT-tree's probabilities at each node. The vertical spacing of EBIT at one point in time is constant. However, the equity values at adjacent nodes are not equally spaced. The spacing of equity values relative to its current node's value increases close to the bankruptcy node because debt holders are senior claimants to the remaining total firm value. Therefore, the local relative volatility of equity increases close to bankruptcy. Economically this is intuitive since the equity value in a bankruptcy node is 0 to which a positive value of equity is pulled to.

[27] See the note on the conversion of densities in Appendix A.1.

[28] See the zero-return line of each return density.

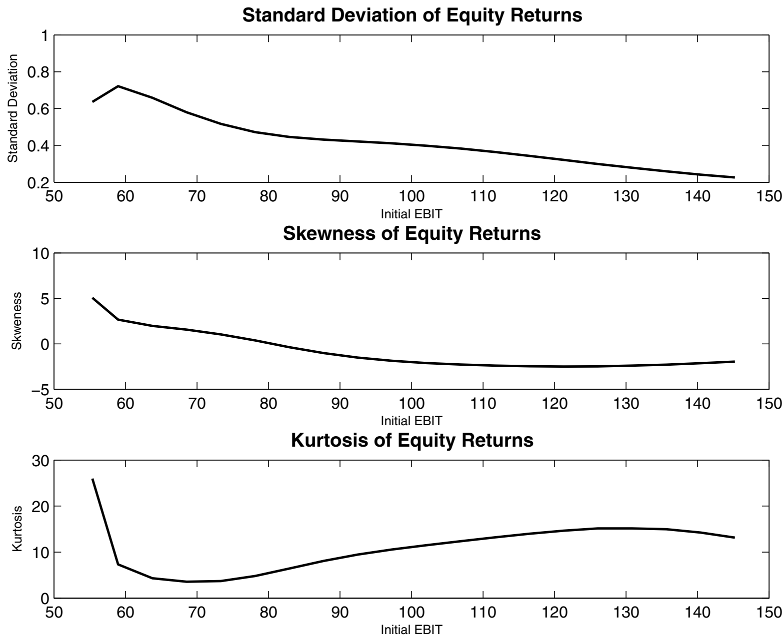
Fig. 4.8. Equity return densities of 6 month equity options in the ABM-Corporate Securities Framework as a function of η_0. The 0-returns are indicated by solid lines. Path probabilities are obtained by differentiating the splined distribution function of EBIT at option maturity.



until option maturity, and a positive skewness (see also Panels B and E of Figures 4.11 and 4.13, respectively) alongside with high kurtosis. It is exactly this bankruptcy probability that complicates the interpretation of return distributions and its moments. In some cases, the unconditional moments even become quite misleading if one compares them to moments of a regular distributions.

Comparing Figures 4.7 and 4.9, the return moments follow an almost similar pattern as the equity value moments described above although at a different level and with changes at a different scale. However, some additional notes might be warranted. The equity return standard deviation depends directly on the relative level of expected equity value and its standard deviation. The equity return standard deviation is driven by two effects: (i) the equity standard deviation increases in absolute terms and (ii) the expected equity value decreases. Both effects increase return standard deviation until equity value standard deviation

Fig. 4.9. Equity return density moments of 6 month equity options in the ABM-Corporate Securities Framework as a function of η_0. The moments are obtained by numerical integration.



drops low enough to allow equity return standard deviation to fall as well. Therefore, equity return standard deviation starts to drop much closer to bankruptcy than the equity value standard deviation.

Return skewness is expected to be negative in general because a decrease of EBIT always results in a relatively larger decrease of equity value than an increase. Therefore, return skewness is much lower than equity value skewness and only the bad state firms encounter positive return skewness because of the missing left tail. By the same argument, the tails of the return density are *thicker* than normal thus return kurtosis is much higher than the equity value kurtosis for good state firms. The swings of equity return kurtosis are more pronounced. Close to bankruptcy, the increase of return skewness and kurtosis is dampened by the relatively higher return standard deviation.

The equity value distribution of the GBM-firm runs through the same four stages described for the ABM-firm.[29] There are two distinct differences, though: First, equity value skewness never decreases below 0. Second, equity value kurtosis always exceeds 3. Both moments show the same swings the closer the firm moves towards bankruptcy. In fact, the ABM- and GBM-firm are indistinguishable if they are close to bankruptcy. Note further, that the equity return distributions show much similarity so that it becomes difficult to tell which process drives EBIT if the return distribution is the only kind of information.

Equity values are distributed neither normally nor log-normally in the ABM- and GBM-case. However, all densities *look normal* for good-state firms.

Table 4.6. The first four central moments of the equity value and its return distribution depending on the current state of the firm with respect to bankruptcy and the distributional assumption. The return distribution is centered around its expected value.

Equity Value Distribution

Firm State	(1) Excellent		(2) Good		(3) Medium		(4) Bad	
EBIT follows	ABM	GBM	ABM	GBM	ABM	GBM	ABM	GBM
$E^{\Omega}(E_T)$	very large		large		medium		small	
$\sigma(E(T))$	small		rel. small		medium		rel. large	
$\zeta(E(T))$	≈ 0	> 0	≤ 0	> 0	> 0	$>> 0$	$>> 0$	$>>> 0$
$\kappa(E(T))$	≈ 3	> 3	≤ 3	> 3	≥ 3	$>> 3$	$>> 3$	$>>> 3$

Equity Return Distribution

	(1) Excellent	(2) Good	(3) Medium	(4) Bad
$\sigma(r_E(T))$	very small	small	medium	large
$\zeta(r_E(T))$	< 0	< 0	$<< 0$	> 0
$\kappa(r_E(T))$	> 3	$>> 3$	> 3 (decr.)	$>> 3$

To summarize the general findings of this subsection, Table 4.6 gives an overview of the first four central moments of the equity value and its return distribution.

4.2.2.2 Comparative Statics

After the general discussion about equity value and return densities and before going into details, recall from Section 4.1 that parameter changes influence the equity density due to either a reduction of total firm value and/or a redistribution of a constant total firm value among different claimants. By equations (3.4) and (3.78), total firm value does not

[29] See Figures B.2 and B.3 in the appendix for the equity value densities as a function of initial EBIT and its moments. Figures B.4 and B.5 display the respective equity return densities and moment series.

Table 4.7. Unconditional central moments of the equity and its return distribution at option maturity as well as LS-regression results of the form $\ln(\sigma) = \beta_0 + \beta_1 \ln(X/E_t)$ in the GBM-Corporate Securities Framework. The base-EBIT $\eta_0 = 100$ and parameters are changed as displayed in the first 6 columns. The bankruptcy barrier is set such that the recovery of debt holders is 50 % of total debt outstanding and losses in the case of bankruptcy are $\alpha = 65$ %. T_1 denotes the maturity of the short bond, $\Phi(T_O)$ the bankruptcy probability up to option maturity T_O.

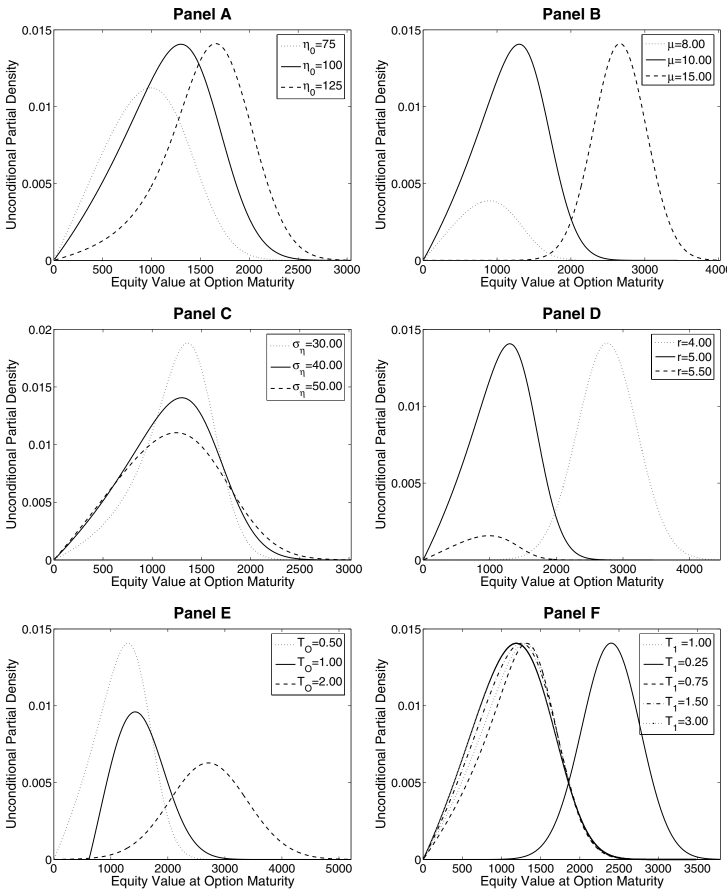
Panel A

$\eta(0)$	μ	σ_η	r	V_B	T_1	T_O	$\Phi(T_O)$	$E^{\mathbb{Q}}(E(T_O))$	$\sigma(E(T_O))$	$\zeta(E(T_O))$	$\kappa(E(T_O))$	$\sigma(r_E(T_O))$	$\zeta(r_E(T_O))$	$\kappa(r_E(T_O))$	$exp(\beta_0)$	β_1	p-value
75.0	3.33%	18%	5.00%	4357.14	1.00	0.500	78.6057%	144.36	290.58	3.1360	12.8248	70.22%	2.4493	6.4458	3.5985	-0.2102	0.00
100.0							1.0069%	1323.58	504.69	0.4842	4.4591	45.73%	-1.8564	8.4684	0.5634	-0.4179	0.00
125.0							0.0015%	2312.42	623.62	0.4734	5.1688	28.57%	-0.9775	4.6390	0.3826	-0.2763	0.00
100.0	3.00%						26.3453%	580.39	376.12	1.9185	6.5375	53.45%	-0.4574	5.9861	1.3027	-0.5139	0.00
100.0	3.67%						0.0014%	2323.50	624.04	0.4731	5.1476	28.43%	-0.9680	4.6012	0.3813	-0.2751	0.00
100.0		13%					0.0310%	1364.97	361.03	0.2650	3.8795	29.02%	-1.3621	7.2263	0.3786	-0.3665	0.00
100.0		23%					4.7142%	1272.28	617.97	1.0015	6.6668	56.75%	-1.7032	7.1600	0.7669	-0.4233	0.00
100.0			4.00%				0.0000%	6789.70	1244.04	0.8086	10.6568	18.33%	-0.3932	3.4415	0.2547	-0.1192	0.00
100.0			5.50%				63.1049%	285.36	371.07	2.4781	7.8316	63.15%	1.8390	4.4066	2.6452	-0.3150	0.00
100.0						1.000	6.3537%	1375.84	659.58	1.1645	4.3383	48.69%	-0.8721	3.6766	0.5582	-0.4000	0.00
100.0						2.000	6.3623%	2511.92	1021.12	1.2887	4.9195	39.57%	-0.6929	4.7161	0.4763	-0.6091	0.00
100.0					0.25		0.0319%	2405.18	502.35	0.3630	3.2449	21.39%	-0.6037	3.4751	0.3716	-0.6644	0.00
100.0					0.75		1.0069%	1324.15	497.10	0.4528	3.3109	44.28%	-1.7827	8.2907	0.5556	-0.3955	0.00
100.0					1.50		1.0069%	1321.51	512.43	0.3935	3.2055	47.09%	-1.8875	8.3964	0.5757	-0.4317	0.00
100.0					3.00		1.0069%	1321.20	523.53	0.3973	3.1576	48.32%	-1.8775	8.1756	0.5892	-0.4262	0.00

Panel B

$\eta(0)$	μ	σ_η	r	V_B	T_1	T_O	$\Phi(T_O)$	$E^{\mathbb{Q}}(E(T_O))$	$\sigma(E(T_O))$	$\zeta(E(T_O))$	$\kappa(E(T_O))$	$\sigma(r_E(T_O))$	$\zeta(r_E(T_O))$	$\kappa(r_E(T_O))$	$exp(\beta_0)$	β_1	p-value
75.0	3.33%	18%	5.00%	4357.14	1.00	0.750	82.1484%	148.58	336.05	3.4339	17.1215	71.26%	2.6329	7.3133	3.1836	-0.1868	0.00
100.0							3.3795%	1349.92	594.06	0.8910	6.0670	50.56%	-1.6599	7.4386	0.5603	-0.4092	0.00
125.0							0.0370%	2347.46	770.62	0.6018	6.2762	35.93%	-1.2563	5.7652	0.3819	-0.2762	0.00
100.0	3.00%						35.5460%	593.90	447.57	2.2563	8.8675	52.62%	0.4202	5.0979	1.2833	-0.4877	0.00
100.0	3.67%						0.0349%	2362.69	771.83	0.6005	6.2314	35.69%	-1.2421	5.7264	0.3805	-0.2749	0.00
100.0		13%					0.2768%	1392.10	441.14	0.4041	4.1405	35.62%	-1.5152	7.5855	0.3766	-0.3580	0.00
100.0		23%					10.3782%	1297.22	719.96	1.6434	12.5294	57.68%	-1.3839	6.5952	0.7651	-0.4225	0.00
100.0			4.00%				0.0000%	6867.45	1536.65	0.9031	12.2042	22.53%	-0.5064	3.3914	0.2549	-0.1352	0.00
100.0			5.50%				68.9919%	292.03	433.74	2.6543	9.8096	65.84%	2.0388	4.7540	2.4232	-0.2772	0.00
100.0						0.500	1.0069%	1323.43	503.60	0.4160	3.2699	45.69%	-1.8622	8.4936	0.5637	-0.4189	0.00
100.0						1.000	6.3537%	1375.84	659.58	1.1645	4.3383	48.69%	-0.8721	3.6766	0.5582	-0.4000	0.00
100.0						2.000	6.3623%	2511.92	1021.12	1.2888	4.9192	39.56%	-0.6937	4.7278	0.4763	-0.6090	0.00
100.0					0.25		0.0319%	2437.30	625.27	0.6554	7.6604	26.50%	-0.7502	3.7543	0.3425	-0.2048	0.00
100.0					0.75		3.3795%	1350.65	585.25	0.9715	6.2502	46.16%	-1.0368	3.8813	0.5528	-0.3836	0.00
100.0					1.50		3.3795%	1347.98	605.72	0.8322	5.7932	53.05%	-1.7603	7.5811	0.5733	-0.4328	0.00
100.0					3.00		3.3795%	1347.66	620.26	0.8105	5.5245	54.87%	-1.7673	7.3857	0.5883	-0.4292	0.00

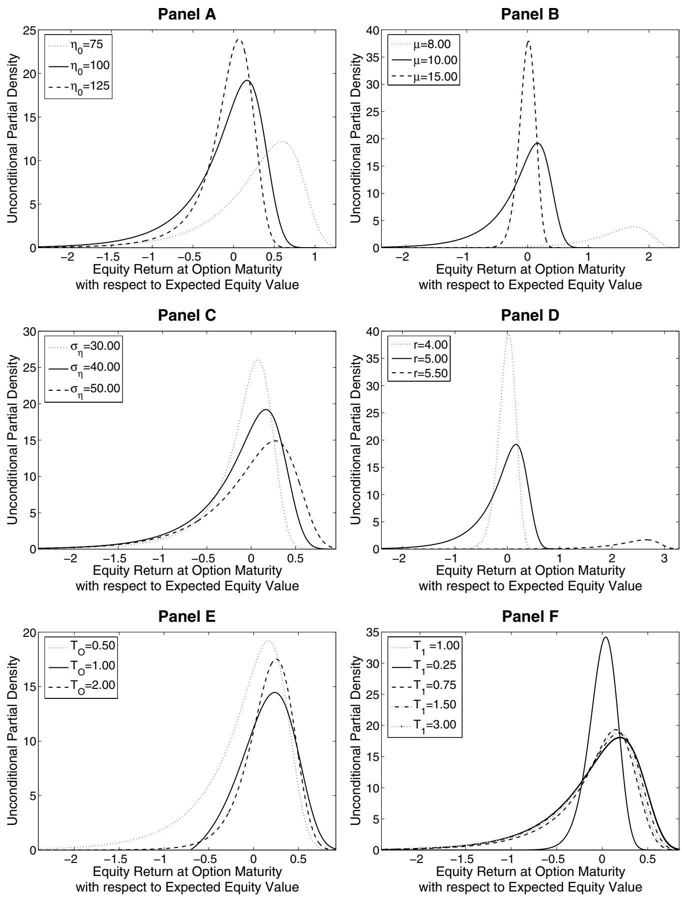
Fig. 4.10. Unconditional partial densities of equity in the ABM-Corporate Securities Framework with $\eta_0 = 100$ at $T_O = 0.5$: Parameter changes are indicated in the legend. The bankruptcy barrier V_B is set so that 50 % of the outstanding notional is recovered in bankruptcy and bankruptcy losses are $\alpha = 70$ %. Path probabilities are obtained by differentiating the splined distribution function of EBIT at option maturity.



depend on the volatility of the EBIT-process if the risk-neutral drifts are left unchanged.[30] A change of the volatility that leads to a change of

[30] Note that the risk-neutral EBIT-process under the measure $\mathcal{Q}$ is modeled directly. If we had started with the EBIT-process under the physical measure $\mathcal{P}$, μ_η would be the drift of the physical EBIT-process and the risk premium θ_η is needed to change the probability measure to the risk-neutral measure $\mathcal{Q}$. Then, a change of σ_η alters the risk-neutral drift by means of the risk premium. Here, we use the

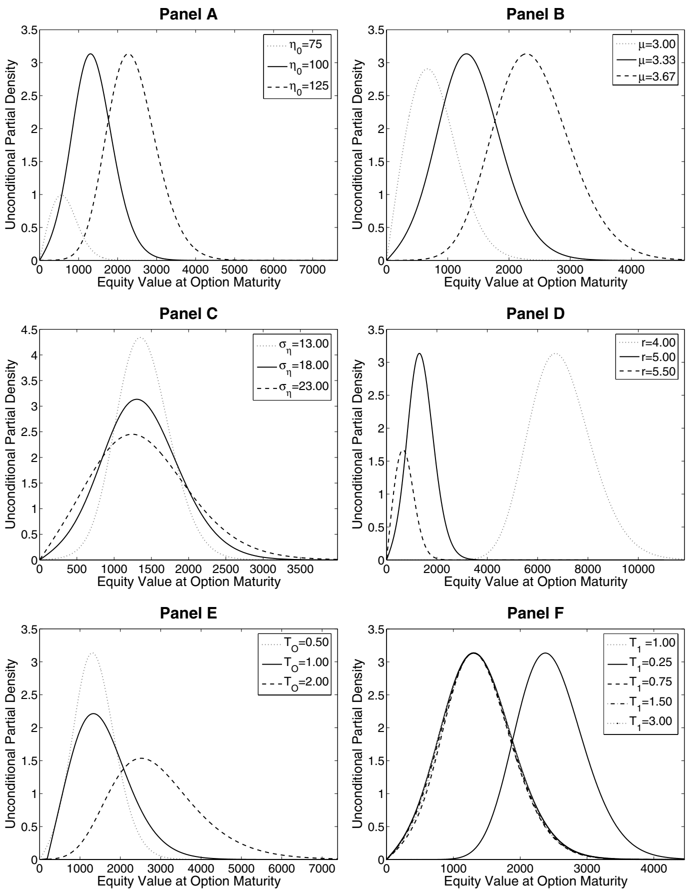
Fig. 4.11. Unconditional partial return densities of equity in the ABM-Corporate Securities Framework with $\eta_0 = 100$ at $T_O = 0.5$: Parameter changes are indicated in the legend. The bankruptcy barrier V_B is set so that 50 % of the outstanding notional is recovered in bankruptcy and bankruptcy losses are $\alpha = 70$ %. Path probabilities are obtained by differentiating the splined distribution function of EBIT at option maturity.



equity value results from a redistribution of firm value, i.e. a change of the probability of bankruptcy. A change of the financing structure does not change total firm value either but has effects on the distribution

implicit assumption that a change of the volatility induces a change of the risk premium so that the risk-neutral drift remains unaffected. See Subsection 2.1.1 for a detailed discussion.

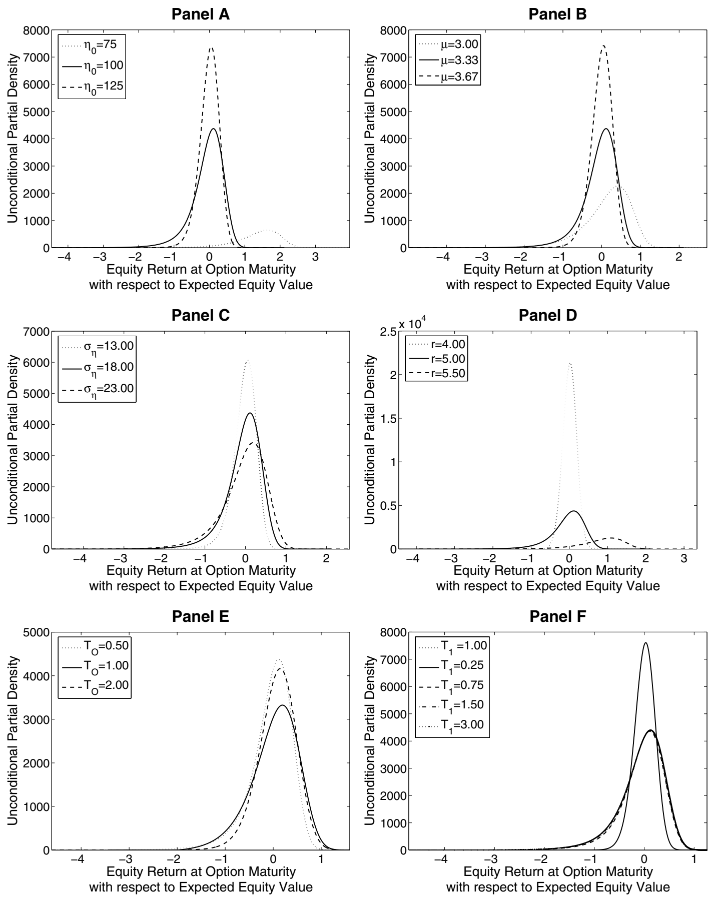
Fig. 4.12. Unconditional partial densities of equity in the GBM-Corporate Securities Framework with $\eta_0 = 100$ at $T_O = 0.5$: Parameter changes are indicated in the legend. The bankruptcy barrier V_B is set so that 50 % of the outstanding notional is recovered in bankruptcy and bankruptcy losses are $\alpha = 65$ %. Path probabilities are obtained by differentiating the splined distribution function of EBIT at option maturity.



of value among different claimants. All other parameter changes, those of the risk-neutral drift, interest rates, and initial EBIT-value, change firm value *and* the firm's stance towards bankruptcy.

Pick first the cases where total firm value does not change. An increase of the EBIT-volatility decreases the expected equity values (see Panel C of Figure 4.10). The skewness of equity values changes from

Fig. 4.13. Unconditional partial return densities of equity in the GBM-Corporate Securities Framework with $\eta_0 = 100$ at $T_O = 0.5$: Parameter changes are indicated in the legend. The bankruptcy barrier V_B is set so that 50 % of the outstanding notional is recovered in bankruptcy and bankruptcy losses are $\alpha = 65$ %. Path probabilities are obtained by differentiating the splined distribution function of EBIT at option maturity.



negative to positive. The kurtosis starts above 3 decreases to below 3 and increases again. Thus, we observe a firm that moves through the first three types of equity densities described above. Skewness and kurtosis change according to the discussion for the type (2) firm.

A change of the financing structure extends short-term bond maturity and reduces the current and the expected equity value. For the

ABM-case, Panel F of Figure 4.10 shows that financing structures with $T_1 > 0.5$ result in overlapping equity densities. In the $T_1 = 0.25$-case, the expected equity value shifts considerably to the right. Four effects drive this result: First, the 3-month bond is repaid before option maturity. Equity holders do not face this payment at option maturity and so expected equity value rises. Second, due to the lower total debt outstanding the firm faces a lower bankruptcy barrier at option maturity than at t_0. If the debt maturity is increased for more than the option maturity, not only option holders have a higher probability of being knocked out[31] but also equity holders at option maturity face the higher bankruptcy barrier. Third, longer-maturity bonds pay coupons for a longer period, thus increasing debt value and decreasing equity value. Fourth, higher coupon payments imply higher tax savings due to the tax advantage of debt which shifts value from the government to equity holders. The decreasing expected equity values in Table 4.5 demonstrate that the coupon and bankruptcy probability effect dominate when the short-term bond maturity is increased.

The equity return standard deviations reflect exactly that behavior (Column 13 Table 4.5). The longer the short-term bond maturity, the higher becomes the return standard deviation. The higher moments of the equity and its return distribution are driven by the lower sensitivity of short-term bonds to changes in EBIT and the higher bankruptcy probability during the life of the short-term bond. Panel F of Figure 4.10 clearly displays that the bankruptcy effect dominates. For a given density value, equity values are lower in the increasing part of the equity densities the longer the maturity of the short-term bond. As a result, the equity value standard deviation and skewness increase whereas kurtosis decreases. The return moments show the same pattern. However, returns are skewed to the left and not to the right, as expected.

Table 4.7 and Panel F of Figures 4.12 and 4.13 depict the GBM-case which show the same effects.

Changing the option maturity T_O, i.e. the point in time at which we investigate the equity densities in the future, gives further insights into the dependence of the equity distribution on option maturity and the capital structure. Table 4.8 summarizes the moments of the ABM-equity value and its return distribution for maturities ranging from 3 months to 3 years where the standard deviations have been annualized by the $\sqrt{T}$-rule (see also Rows 10 and 11 of Table 4.5 and Panel E of

[31] Note that the probability of going bankrupt in the first three months is only 1.4385 %, whereas the bankruptcy probability until option maturity with longer lasting bonds is 7.8111 %. See Table 4.5.

Figures 4.10 and 4.11 for the respective graphs of the equity value and its return densities).

Table 4.8. ABM-equity value and its return moments of the equity distribution with maturities T_O from 0.25 to 3. Standard deviations are annualized by the $\sqrt{T}$-rule.

T_O	$E^{\mathbb{Q}}(E(T_O))$	$\sigma(E(T_O))$	$\zeta(E(T_O))$	$\kappa(E(T_O))$	$\sigma(r_E(T_O))$	$\zeta(r_E(T_O))$	$\kappa(r_E(T_O))$
0.25	1,182.12	702.77	-0.2432	2.9625	76.2450 %	-2.4349	12.5044
0.50	1,205.89	587.29	0.4427	2.9036	57.1358 %	-2.0185	11.1144
0.75	1,228.50	516.08	1.0661	3.3341	42.7505 %	-1.3139	10.2492
1.00	1,250.11	465.23	1.7271	4.0329	29.9259 %	1.1891	2.8677
1.50	2,179.92	557.87	1.6068	3.3781	21.2804 %	1.1930	2.9633
2.00	2,230.36	543.69	1.5187	3.4095	20.7674 %	0.6154	4.4277
3.00	2,331.06	525.84	1.4349	3.4630	19.9796 %	-0.1092	6.3654

The repayment of the short-term bond at $T_1 = 1$ has a large impact on the equity distribution. Expected equity value surges because the debt burden is reduced. Although standard deviations increase in absolute terms, the annualized standard deviations decreases but jumps up after the repayment date. This discontinuity is driven by the prevailing higher equity value and the fact that the left tail of the equity value distribution is lengthened i.e. the distance of the expected value to the bankruptcy level is extended. The return standard deviation decreases monotonically meaning that the relative riskiness of the firm decreases the longer it survives. In any case, scaling standard deviations of distributions at different points in time by the $\sqrt{T}$-rule is impossible.

The higher moments of the ABM-equity value distribution are clearly influenced by the proximity to bankruptcy. Skewness increases at first because the left tail of the distribution continues to be cut and probability mass is concentrated at 0 as bankruptcy becomes more probable. After the debt repayment, the bankruptcy barrier falls and the restitution of part of the left tail reduces skewness again. Kurtosis increases until the debt repayment date, jumps down at that date, and starts growing again. The peculiar behavior of the higher moments of the return distribution is experienced here as well. The last two columns of Table 4.8 exhibit a good example. The sudden jump of the bankruptcy level causes return skewness to change its sign and return kurtosis to drop below 3. Inspection of Panel E of Figure 4.11 does not reveal these facts!

Interpretation of the moments of the GBM-equity value and return distribution as a function of maturity can follow along the same

Table 4.9. GBM-equity value and its return moments of the equity distribution with maturities T_O from 0.25 to 3. Standard deviations are annualized by the $\sqrt{T}$-rule.

T_O	$E^{\mathbb{Q}}(E(T_O))$	$\sigma(E(T_O))$	$\zeta(E(T_O))$	$\kappa(E(T_O))$	$\sigma(r_E(T_O))$	$\zeta(r_E(T_O))$	$\kappa(r_E(T_O))$
0.25	1,297.07	728.76	0.1512	3.1387	63.5196 %	-1.5634	7.8315
0.50	1,323.43	712.20	0.4160	3.2699	64.6123 %	-1.8622	8.4936
0.75	1,349.74	684.04	0.7829	3.7046	58.3691 %	-1.6617	7.4388
1.00	1,375.84	659.58	1.1645	4.3383	48.6858 %	-0.8721	3.6766
1.50	2,449.05	700.41	1.2914	4.6809	26.8249 %	-0.2256	3.5146
2.00	2,511.92	722.04	1.2887	4.9194	27.9808 %	-0.6931	4.7179
3.00	2,640.78	756.49	1.3994	5.5727	29.2571 %	-1.1809	5.9715

lines as the ABM-case (see Table 4.9, Rows 10 and 11 of Table 4.5, and Panel E of Figures 4.10 and 4.11). The same decrease of equity value standard deviation at the debt repayment date can be experienced. Thereafter, annualized equity value standard deviations increase slightly as expected if EBIT follows a log-normal distribution. However, equity value skewness and kurtosis increases continuously with maturity. The equity return distribution shows a more interesting pattern. In contrast to the ABM-case, equity return standard deviation increases after debt repayment. However, higher return moments show an ambiguous pattern but skewness stays negative and kurtosis above 3 for all maturities.[32]

All other changes to the base case parameter set cause a change of total firm value per se which forces a redistribution of claim values. A reduction of the risk-neutral EBIT-drift, of the initial EBIT-level, and an increase of the risk-less interest rate results in a reduction of firm value and thus of the expected value of equity. The closer the firm moves towards bankruptcy, the higher equity value skewness, the higher the kurtosis, i.e. the firms are of type (3) and (4). Moves in the opposite direction makes the equity densities look *normal*, i.e. firms approach those of type (1). Panels B and D of Figure 4.10 for the ABM-case and Figure 4.12 for the GBM-case illustrate this truncation of the density on the left at bankruptcy.

Comparing our return moment pattern to those reported in Table 6 of Bakshi et al. (2003), the picture fits into our simulated moments: (i) Our return distribution exhibits positive return skewness if the firm

[32] Ait-Sahalia and Lo (1998) report in their Figure 7 moments of risk-neutral return distributions estimated non-parametrically from time series of option prices. Higher return moments of the S&P-500 index are instable with respect to maturity and show kinks. Effects that our simple example exhibits, as well.

is close to bankruptcy. In Bakshi et al. (2003), IBM is not a convincing candidate for this. However, ABM-equity return densities around debt repayments dates have higher skewness as well. (ii) Our return kurtosis is generally above 3. The exception again is the ABM-equity return density around debt repayment. American International, Hewlett Packard, and IBM have average kurtosis below 3. (iii) There is a tendency that an increase of skewness implies higher kurtosis, which Bakshi et al. (2003)'s Table 6 shows, as well. Exceptions in our model are only firms just before bankruptcy, which usually have no options traded on their equity. (iv) Our volatility and skewness is generally higher.

As a final remark, differences might be due to the replication procedure used by Bakshi et al. (2003) who need to average across option maturities to get a maturity-consistent time series of option prices. Since we can resort to the whole unconditional distribution of equity at maturity, Bakshi et al. (2003)'s results might be biased within our framework and therefore be not the best comparison. Additionally, we showed above that debt repayments before option maturity have a huge impact on the equity return distribution which might distort the results of Bakshi et al. (2003). However, it is the only study so far that analyzes individual stock option.

4.2.3 Equity Option Prices and Implied Black/Scholes Volatilities

In option markets it is observed that option's implied Black/Scholes volatility as a function of strike prices is monotonously falling. This specific functional form is usually referred to as the option's volatility smile. The economic literature did not present an easy explanation for this phenomenon but tried technical extensions such as stochastic volatility models which produced observed volatility smiles.[33]

In the Corporate Securities Framework proposed in Chapter 2, an inversion of the Black and Scholes (1973) formula is not possible because the prices of all security values depend only on the size of EBIT and the capital structure in a particular state at option maturity. Payments

[33] See Jackwerth (1999) for a literature overview. Stochastic volatility models have been studied before by e.g. Heston and Nandi (2000) and Heston (1993). Dumas, Fleming and Whaley (1998) provide evidence that an implied volatility tree in the sense of Rubinstein (1994) is not superior to the ad-hoc applied implied volatility curve in a Black/Scholes model. See also Grünbichler and Longstaff (1996) who analyze volatility derivatives where the volatility itself is assumed to follow a mean-reverting process. Fleming, Ostdiek and Whaley (1995) describe the properties of the quoted volatility index on the S&P 100.

during the life of the option to debt and equity holders are irrelevant. Therefore, we can calculate equity option prices by numerical integration and by finite difference methods. The implied volatilities of the option can be backed out from the more general form of the Black and Scholes (1973) option pricing formula which is based on the expected future value of the underlying asset.[34]

$$\begin{aligned} C_{t_0} &= e^{-r(T_O - t_0)} E^{\mathbb{Q}}\left[\max(E_{T_O} - X, 0)\right] \\ &= e^{-r(T_O - t_0)} \left[E^{\mathbb{Q}}\left[E_{T_O}\right] N(d_1) - X N(d_2) \right] \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{E^{\mathbb{Q}}\left[E_{T_O}\right]}{X}\right) + \frac{\sigma_{IV}^2}{2}(T_O - t_0)}{\sigma_{IV}\sqrt{T_O - t_0}} \\ d_2 &= d_1 - \sigma_{IV}\sqrt{T_O - t_0}, \end{aligned}$$

and σ_{IV} is the annual implied volatility of the logarithm of the underlying equity value E over the period $T_O - t_0$.

When using implied Black/Scholes volatilities as a benchmark, we compare each of our scenarios with the log-normal density of equity which Black and Scholes (1973) assumed for the underlying. If the Black and Scholes (1973) model were correct, the implied volatilities for all options must be equal.

From equation (4.1) it follows that high option prices imply high implied volatilities. If implied volatilities increase for lower strikes, the equity densities have more probability mass left of the strike price than the log-normal density. To see this, take a state contingent claim which pays 1 currency unit if the equity price has a certain level at maturity. As Breeden and Litzenberger (1978) point out, the price difference of two of such claims with different strike levels are related to the difference of the probability of the two states actually occurring. If the claim with the lower strike is worth more than the other as implied by the log-normal assumption implicit in the Black/Scholes model, this probability must be higher. Thus, we can conclude that the probability mass between the strikes of these two claims is higher than implied by the Black/Scholes distributional assumption.[35]

Bakshi et al. (2003) analyze the connection between the physical equity return distribution, its risk-neutral counterpart, and the resulting

[34] See, e.g. Hull (2000), p.268 ff.

[35] See Appendix A.2 for a more formal exposition.

equity option implied Black/Scholes volatilities on a single-stock basis. The analytical and empirical results strongly support the hypothesis that the more the risk-neutral return distribution is left-skewed, the higher the curvature of implied volatility smile. Furthermore, a higher kurtosis flattens the smile somewhat. Their Table 5 summarizes these results.

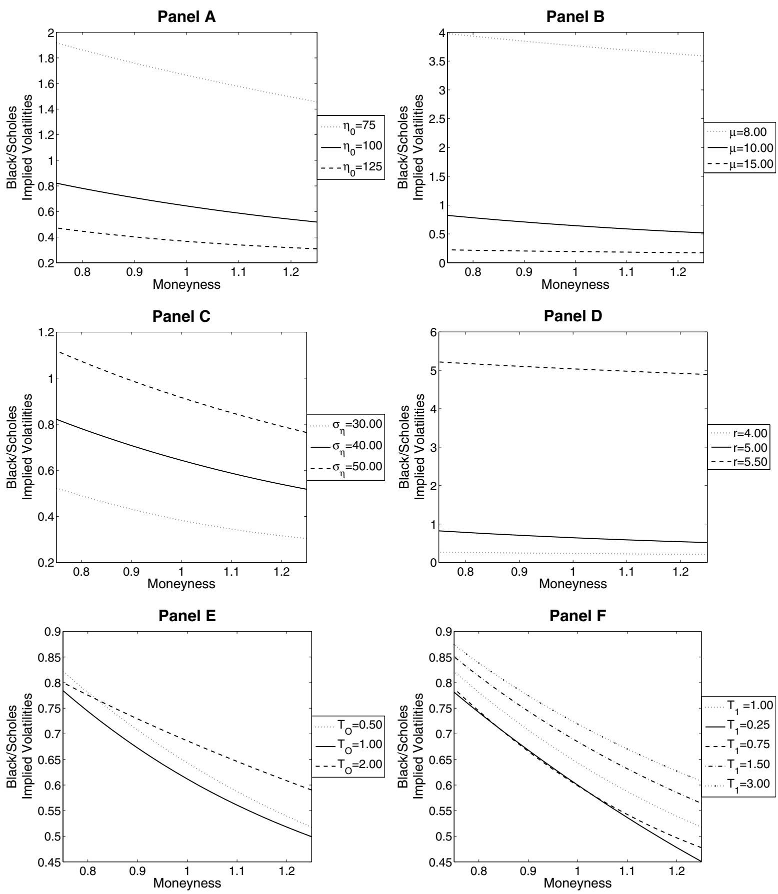
The simulation results of equity option prices in our ABM-Corporate Securities Framework support that behavior. The GBM-firm exhibits exceptions to their rule if the bankruptcy probability rises very quickly. However, we are able to give intuition to these findings. In our model, the equity return distribution results from the specification of the equity contract as being the residual claim to EBIT, a complex capital structure, and the distinct ability of equity owners to declare bankruptcy. This is the primary economic interpretation of the results found by Bakshi et al. (2003), and so completes their analysis on the economic level.

In all our examples, we find a downward sloping implied volatility curve. The level of implied volatilities and the curvature of the strike/volatility function depend on the current state of the firm towards bankruptcy expressed by the equity return standard deviation. If we detail the analysis by comparing our parameter settings, we find several stylized facts: (i) Implied volatilities of the GBM-case are higher than those of the comparable ABM-case if the firm is far from bankruptcy. The opposite is true close to bankruptcy because the GBM-volatility decreases with the EBIT-level thus changing the term structure of bankruptcy probabilities. (ii) The closer the firm is to bankruptcy, the higher the implied volatility of at-the-money options. (iii) The ABM-implied volatility structure gets flatter for higher at-the-money implied volatility levels. These firms have a high return distribution skewness and kurtosis. (iv) The GBM-firm exhibits steeper slopes at higher implied volatility levels.

It seems important to note that the linking of the level and shape of the volatility smile to equity return moments seems only suitable for good state firms. Following our discussion about higher moments of the return distribution, the bankruptcy probability might bias the relation detected by Bakshi et al. (2003). Their sample firms can be considered as being far from bankruptcy.

Figures 4.14, 4.15 display implied volatility curves against moneyness for ABM- and GBM-firms for 6 month option maturity. Hereby, moneyness is defined as the fraction of the strike to the current equity value. As expected, lower EBIT-starting values (Panels A), higher in-

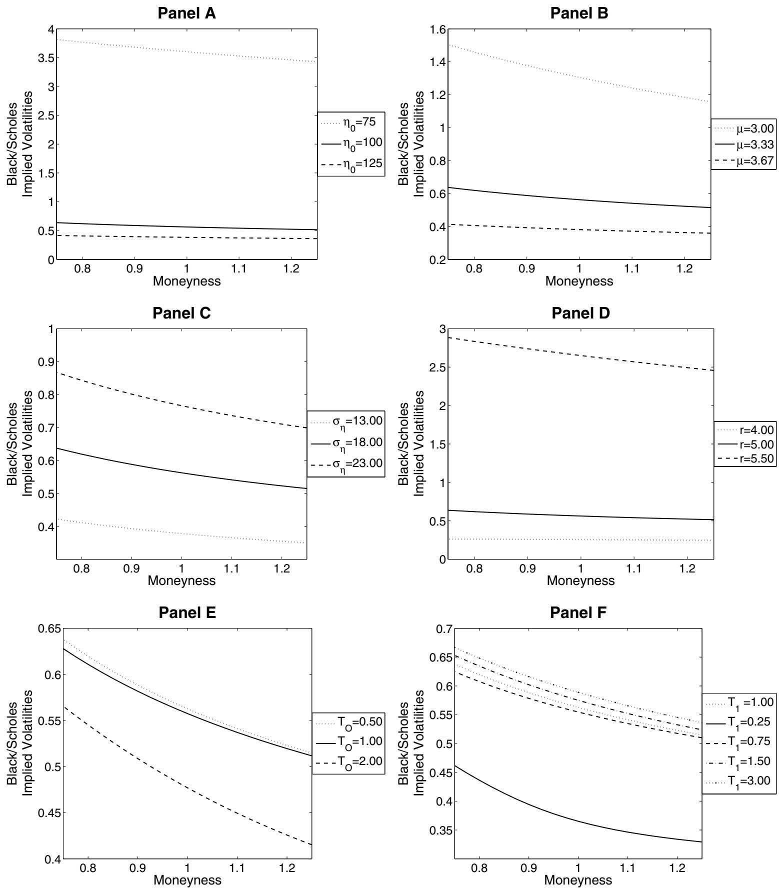
Fig. 4.14. Implied Black/Scholes volatilities of 6 month equity options in the ABM-Corporate Securities Framework with $\eta_0 = 100$: Parameter changes are indicated in the legend. The bankruptcy barrier V_B is set so that 50 % of the outstanding notional is recovered in bankruptcy and bankruptcy losses are $\alpha = 70$ %. Option prices are obtained by numerical integration.



terest rates (Panels D), lower risk-neutral drifts (Panels B), and higher EBIT-volatility (Panels C) take the firm closer to bankruptcy and thus show higher implied volatility levels.

The maturity of the option (Panels E) has only a major impact on the volatility smile if debt is repaid during the option's life. Then, the implied volatility smile becomes much flatter in the ABM-case and drops in the GBM-case. Recall from the last subsection that expected

Fig. 4.15. Implied Black/Scholes volatilities of 6 month equity options in the GBM-Corporate Securities Framework with $\eta_0 = 100$: Parameter changes are indicated in the legend. The bankruptcy barrier V_B is set so that 50 % of the outstanding notional is recovered in bankruptcy and bankruptcy losses are $\alpha = 65$ %. Option prices are obtained by numerical differentiation.



equity values increase due to the capital infusion by equity owners to repay debt.

The last three columns of Tables 4.5 and 4.7 are devoted to a regression analysis as performed by Bakshi et al. (2003) in their Table 3. The implied volatility is represented by the regression model

$$\ln(\sigma_{IV}) = \beta_0 + \beta_1 \ln\left(\frac{X}{E_0}\right). \tag{4.2}$$

In equation (4.2), $\exp(\beta_0)$ can be interpreted as the at-the-money implied volatility. β_1 is a measure of the steepness of the implied volatility curve. For example, the figures of Rows 10 and 11 confirm that increasing the option maturity continuously increases the ATM-implied volatility, but the 2-year option has the lowest slope.

Note that the at-the-money implied volatility is always higher than the standard deviation of the equity return density for good state firms. This reflects the fact that the expected future equity value lies above the current value of equity and so the at-the-money implied volatility with respect to the expected future equity value is lower.

For firms closer to bankruptcy implied volatilities can become large. At-the-money levels of 100 % and more are common.

The effects of changes of the financing structure need a more detailed discussion. As mentioned in the last subsection, equity return standard deviations depend on the schedule of debt maturities. Recall that the earlier the short-term debt matures, the more imminent becomes debt repayment. If the firm survives, the bankruptcy barrier is lowered and the equity value jumps upwards. Extending short-term debt maturity increases the period of coupon payments and the bankruptcy barrier in the extension period which decreases current and future equity value. As a result, equity return standard deviation and ATM-implied volatilities increase. As can be seen from Figure 4.14 Panel F, the slope of the implied volatility smile flattens. This graph also illustrates Toft and Prucyk (1997)'s debt covenant effect although they interpret a higher covenant, i.e. a higher bankruptcy barrier, as a substitute of the amount of short-term debt. Effectively, we increase the bankruptcy barrier as well. However, we have the additional effect of a reduced (expected) equity value because the maturity of short-term debt is extended. As is demonstrated here, the term structure of the corporate capital structure matters. The simple argument of a debt covenant is not enough to explain the richness of implied volatility smiles. Again, our framework gives a very intuitive and simple explanation.

4.3 Summary

This chapter illustrates the analytical solutions of Chapter 3 for an EBIT following an arithmetic and geometric Brownian motion of the Corporate Securities Framework of Chapter 2.

In Section 4.1 of this chapter, a standard comparative static analysis of the Corporate Securities Framework is performed. The numerical example shows the expected patterns of changes of security values with respect to parameter value changes. However, the behavior of security prices becomes much richer the more securities are added to the capital structure.

One of the interesting properties produces a rise of the EBIT-volatility, which increases equity values although the debt covenant employed in the analytical setting would have called for the opposite result. So, in a multiple debt setting, the comforting Leland (1994) result that debt covenants overcome asset substitution effects does not hold for all parameter settings. Additionally, there seems to be room for equity holders to optimize EBIT-risk.

From our numerical examples, several suggestions for direct empirical testing of structural credit risk models can be drawn. First, we find that taxes are an important part of firm value that cannot be neglected in an empirical test. Second, in the extended numerical version of our model where equity owners trigger optimal bankruptcy and where different tax regimes exist, we find that the three implemented tax regimes lead to similar equity and debt values whereas the optimal bankruptcy decision by equity holders increases the value of equity considerably. Third, we demonstrate that the optimal bankruptcy level is not constant but increases near the maturity of debt issues. Since in the basic setting equity holders infuse capital in periods of cash outflows to other claimants to keep invested capital constant, equity holders need a sufficiently high expected firm and equity value to be willing to keep the firm alive.

We do not analyze dynamic capital structure decisions. However, in a static selling of future debt issues, we find that future debt issues change the optimal bankruptcy behavior and cash flows of equity investors although the debt issue itself has no present value. A future debt issue postpones equity capital infusions. As a result, the optimal bankruptcy level flattens to an almost constant level that underpins the assumption of a constant bankruptcy barrier with an economically reasonable argument. Despite the higher bankruptcy level compared to the scenario without refinancing, equity holders gain from tax savings due to future coupon payments and tax loss recoveries. Therefore, future debt issues - at least in a static form - need to be considered for an empirical application.

In Section 4.2, we use a numerical implementation of the Corporate Securities Framework to price options on equity in order to explain the existence and the shape of volatility smiles. We compare our option prices to the Black/Scholes setting by calculating implied volatilities.

Since our distributional assumptions are different to those of the Black/Scholes environment, differences in the shapes of implied volatility curves have to be expected. However, it is interesting that our economically intuitive environment is sufficient to explain a behavior that needed much more elaborate mathematical techniques before. We find that the distributional assumption for EBIT is not crucial to the general downward sloping shape of implied volatilities but the features of the equity contract. Equity holders can stop infusing capital into the firm and declare bankruptcy. The option on equity forgoes. As a result, the unconditional distribution of equity values at option maturity is skewed and exhibits excess kurtosis. ABM-firms and GBM-firms run through several stages of distribution types. We categorized the stages by four generic types of distributions when a firm in an excellent condition moves towards bankruptcy. Equity return distributions of ABM- and GBM-firms behave surprisingly similar except for the level and the sensitivity. Good state firms exhibit small return standard deviation, a slightly negative skewness and a small excess kurtosis. If the firm moves towards bankruptcy the return standard deviation and kurtosis increase, skewness decreases. Close to bankruptcy the skewness turns positive at high levels of the standard deviation and kurtosis. The effects are due to the concentration of the probability mass of the equity value distribution at zero. As a result, moments of equity value and return distributions of firms close to bankruptcy must be interpreted carefully because the levels of higher moments might be misleading when compared to moments of regular distributions.

The effects on option prices are as follows: All curves of implied volatilities as a function of moneyness are convex and monotonously decreasing. Options on equity of firms close to bankruptcy generally show very high at-the-money implied volatilities with a decent slope. The further the firm's distance to default, the lower implied volatility levels and the steeper the slope. Financing decision can have a significant impact on implied volatilities. Repayment of debt around option maturities incurs sharp increases in implied volatilities due to the jumps by equity values. Switching to longer debt maturities might decrease equity values due to higher coupon payments which outweighs the present value effect of no immediate debt repayment. For good state firms, debt repayment can effectively decrease implied volatilities of longer lasting equity options because (i) expected equity value is increased (ii) and bankruptcy becomes less imminent once the debt burden is lowered.

In the GBM-case, the term structure of bankruptcy probabilities can influence equity values considerably and thus effectively increase the slope of implied volatilities the higher its level.

Our explanation of equity smirks is simply and intuitively linked to the economic condition of the firm and to its debt structure. We argue this as a significant progress to other studies of option smirks which use mathematically more elaborated but economically less intuitive concepts.

5

Empirical Test of the EBIT-Based Credit Risk Model

This chapter proposes a direct empirical implementation of the class of EBIT-based structural firm value models as discussed in Chapter 2. To date, only a mixture of accounting data and time series of equity prices have been used to estimate parameters of structural firm value models. We develop a Kalman filter that incorporates time series of bond prices which usually convey important information about a firm's economic condition. We suggest that the use of time series of all traded securities in an empirical study will improve the quality of the estimators of the latent EBIT-process.

In Section 5.1, we give a brief overview of empirical studies which are based on firm value models. Section 5.2 discusses estimation procedures used in the literature and proposes a suitable Kalman filter for the Corporate Securities Framework. Section 5.3 raises practical issues for the actual estimation. Finally, Section 5.4 collects the results of the simulation study. A brief summary is given in Section 5.5.

5.1 Existing Literature and Shortcomings

Tests of reduced-form asset pricing models can be designed and performed relatively easily. In contrast, direct implementations of structural credit risk models have been rare due to two reasons:

- Theoretical firm value models have been too restrictive to be applied to corporate data. To fit real world firms into e.g. the model of Merton (1974) or Black and Cox (1976), several ad-hoc adjustments to the firm's debt structure are needed. In most cases the debt structure is artificially reduced to a single debt issue by constructing a zero or coupon bond with a notional of the total debt outstanding and a maturity equal to the duration of the entire debt structure.[1] Empirical results may therefore be influenced by the adjustments. Only the proposed EBIT-based Corporate Securities Framework of Chapter 2 and Ericsson and Reneby (1998) represent more general frameworks for the valuation of corporate securities which seem appropriate for direct estimation. These models allow to exploit trading data of corporate securities, i.e. time series of equity and corporate bond prices, that are abundantly available, and need not rely solely on accounting data.

- Firm value models need more advanced econometric methods. Most firm value models operate with a latent variable. The dependence of corporate securities on the latent variable implies non-stationary parameters of the processes of corporate securities.[2]

As a result, most of the empirical literature on firm value models has been reduced to a test of stylized model behavior on aggregate levels such as industry, rating classes, or firm sizes. Predictions of firm characteristics such as leverage ratios, corporate bond coupons, corporate bond issue discounts, credit spreads were regressed on time series of equity and bond prices.

Fischer et al. (1989a) find that observed leverage ratio ranges can be explained by a firm value model where dynamic recapitalization is costly. Importantly, they point out that the current leverage ratio of a firm might stem from an optimal decisions made several periods ago. As a result, the observed leverage ratio might not be optimal but lie in a range of inertia where high recapitalization cost prevent the firm from adjusting the leverage.

Another aspect of corporate debt is the existence of call premia and issue discounts. In a similar framework as in their first paper, Fischer et al. (1989b) find empirical support that debt contracts are designed to mitigate agency conflicts between debt and equity holders. Call premia and issue discounts are related to firm risk, which the model predicts.

In a model with a constant capital structure, Longstaff and Schwartz (1995) extend the pricing model for corporate debt to include stochastic interest rates which are correlated with the firm value. Their empirical section strongly supports the incorporation of stochastic interest

[1] See e.g. Jones et al. (1984).

[2] See Section 5.2. Also Ericsson and Reneby (2002a) for a discussion of estimator design. Duan (1994) describes a more general setting of latent variable estimation.

rates in firm value models.[3] However, Delianedis and Geske (2001) and Huang and Huang (2003) report contradictory evidence.

One of the primary issues of the option pricing literature is the source of implied Black and Scholes (1973) volatility smirks as observed in option prices. Toft and Prucyk (1997) relate option prices to the leverage ratio of firms in the Leland (1994) model. Despite the simple capital structure, their regression results support a strong dependence of the pricing bias of equity options on the leverage of the firm. Interpreting short term debt as a kind of debt covenant they can even relate steeper smirk functions to firms which are mainly financed by short term debt.[4]

Recently, Brockman and Turtle (2003) contrast the performance of a barrier option approach of pricing corporate debt which is used in firm value models such as the framework of Chapter 2 and Ericsson and Reneby (1998) with the Merton (1974)-like approach where bankruptcy can only occur path-independently at the maturity of a debt issue. The cross section of a large sample of US firms indicates that the barrier option feature of corporate securities is statistically significant and outperforms the traditional static approach.

Summarizing, most tests of stylized facts support firm value models and attest predictive power with respect to distinct features.

Parameter estimation of firm value models until so far was restricted to the simple Merton (1974)-model and with unsatisfactory empirical methods.[5] Jones et al. (1984) were the first to parameterize the Merton (1974)-model by using accounting data and equity times series. They are not able to reproduce observed bond prices. Apart from the criticism of their parameter estimation methods, they have bond types in their sample that clearly contradict the stringent assumptions of the Merton (1974)-model, i.e. only a single zero-coupon bond is outstanding. Subsequent studies tried to overcome some of the problems in Jones et al. (1984). Delianedis and Geske (1999), Delianedis and Geske (2001), Eom, Helwege and Huang (2004), and Huang and Huang (2003) resort to a calibration method in order to extract the time series of asset values. Their primary focus was to explain why structural models

[3] Longstaff and Schwartz (1995) approximate the hitting probabilities and the hitting prices by an algorithm that extends Buonocore, Nobile and Ricciardi (1987)'s one dimensional version to a two-dimensional stochastic process. As Collin-Dufresne and Goldstein (2001) show, Longstaff and Schwartz (1995)'s approximations are incorrect which has an impact on the credit spreads shown in their paper.

[4] See also the critical remarks on the Toft and Prucyk (1997) in Section 4.2.

[5] See Section 5.2 for details.

fail to explain the whole credit spread. Other factors such as stochastic interest rates, stochastic recovery rates, differential tax treatments, illiquidity premia, and market risk factors were tried as explanatory variables additional to default risk. However, none of the studies was able to find a consistent and reasonable explanation.[6]

A very promising example of direct estimation of structural credit risk models is Ericsson and Reneby (2004) who are able to parameterize a structural model with a simple capital structure with Duan (1994)'s latent variable approach. Their estimates for individual firm's corporate bonds are remarkably well. However, their estimation method only uses time series of equity prices.

The Kalman filter suggested here is based on the idea to use as much time series information as possible to estimate the model parameters.[7] We will argue that our approach is simple but more accurate than any estimation method suggested so far.[8]

5.2 Estimation of Parameters of the Corporate Securities Framework

5.2.1 The Corporate Securities Framework Revisited

Recall from Chapter 3 that the state of the firm is assumed to be described by the firm's EBIT which is assumed to follow an arithmetic Brownian motion:

$$d\eta = \mu dt + \sigma_\eta dz^{\mathbb{Q}},$$

where μ is the risk-neutral drift of the EBIT-process under the equivalent martingale measure $\mathbb{Q}$, σ_η is the volatility of EBIT and $dz^{\mathbb{Q}}$ describes a standard Brownian motion.

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[6] Similar results are reported in Collin-Dufresne, Goldstein and Martin (2001). In a regression study they try to explain credit spreads by several economic and market wide factors, but conclude that the most important explanatory factor as of a principal component analysis is not among the variables tried out. In contrast to this evidence, Longstaff et al. (2004) can attribute the majority of the observed credit spread to default risk. However, they use credit default swap data and a reduced form approach in their study.

[7] Bruche (2004) suggests a more computationally demanding simulation approach which also can incorporate all kind of corporate securities.

[8] In financial applications, Kalman filters have primarily been used to estimate interest rate processes where the short rate is treated as a latent variable. See e.g. Geyer and Pichler (1999), Babbs and Nowman (1999).

The firm is assumed to go bankrupt if EBIT hits a lower barrier η_B, where a fraction α of the then available firm value V_B is lost.

The firm and investors are exposed to a tax system with three different kind of taxes. Debt holders' coupon payments are taxed at a tax rate τ^d. A corporate tax rate τ^c is applied to corporate earnings, i.e. EBIT less coupon payments. Corporate earnings after tax are paid out as a dividend, which in turn is taxed at the personal tax rate of equity owners τ^e.

The capital structure of the firm consists of J debt issues with model prices D_{C_j,T_j}, $j = 1 \dots J$, where C_j denotes the coupon level and T_j the time of maturity of debt issue j, and equity with a model price of E. Model prices of debt and equity are functions of a parameter vector $\Theta = \{\mu, \sigma_\eta, \alpha, \eta_B, \tau^c, \tau^d, \tau^e\}$, the risk-less interest rate r, and the current EBIT η_{t_0}.

Under the assumption that EBIT follows a geometric Brownian motion, only the process parameters $\bar{\mu}$ and $\bar{\sigma}_\eta$ are exchanged for μ, σ_η, to arrive at the parameter vector $\bar{\Theta}$. Without loss of generality, we restrict the exposition to the case of arithmetic Brownian motion.

5.2.2 Estimation Approaches Using Accounting Data

A very intuitive way of implementing structural credit risk models was first proposed by Jones et al. (1984). Jones et al. (1984) estimated parameter values of the classical Merton (1974)-model by using a mixture of accounting and market, i.e. equity time series, data. In the Merton (1974)-model, a time series of firm values $\hat{V}_n$, an estimate of the asset volatility $\hat{\sigma}_V$, and the interest rates $\hat{r}_n$, are needed. Jones et al. (1984) propose the following estimation procedures:

- The firm value time-series:
  Given the total liabilities of a firm as of quarterly reports, its firm value can be estimated as the sum of the market value of traded debt and equity. The remaining market value of non-traded debt is assumed to be valued proportionally to the value of traded debt. Alternatively, book values of non-traded assets could be employed with some netting of short-term liabilities with short-term assets.[9]
- Asset volatility:
  As a first guess, the standard deviation of the firm's asset can be derived from the constructed firm value changes directly. Alternatively, the theoretical relation

[9] See e.g. Delianedis and Geske (1999) and Delianedis and Geske (2001) for the book value approach and the netting of short-term assets with short-term liabilities.

$$\hat{\sigma}_V = \hat{\sigma}_E \frac{E}{\frac{\partial E}{\partial V}\hat{V}} \tag{5.1}$$

can be used given an estimate of the equity volatility $\hat{\sigma}_E$ and the estimate of the firm value time series $\hat{V}$. Equity volatility is estimated from time series of stock prices.

- Interest Rates:
  Forward par yields were extracted from government bond dirty prices.

The estimation procedure has been criticized to have several drawbacks. First, the construction of the firm value time series depends crucially on the assumption on non-traded debt values. Any of the proposed approximations can lead to biases of the general conclusion. Second, a standard argument against estimating the firm value volatility directly from the constructed firm value time series is that the resulting volatility estimate inherits the deficiencies of the construction process and that it is strictly backward looking. The estimate of the firm value volatility as of equation (5.1) is conceptually inconsistent because the constant asset volatility $\hat{\sigma}_V$ is calculated by assuming that $\hat{\sigma}_E$ is constant.[10] However, $\hat{\sigma}_E$ changes if $\hat{V}$ and consecutively E and $\frac{\partial E}{\partial V}$ change.

5.2.3 Calibration Approach

A widely accepted procedure to come up with estimates of the time series of the firm value and the asset volatility calibrates both parameters to data simultaneously. Instead of constructing a time series of firm values from balance sheet data, a second equation together with equation (5.1) is used, i.e. the theoretical equity value, to solve for the two unknowns $\hat{V}_n$ and $\hat{\sigma}_{V,n}$ at each observation time.[11] As Ericsson and Reneby (2002a) point out, the system of two equations suffers the same criticism as the approach using accounting data because equation (5.1) is still applied as if $\hat{\sigma}_E$ were constant.

5.2.4 Duan's Latent Variable Approach

In a simulation study Ericsson and Reneby (2002a) find that a maximum-likelihood estimator treating the firm value as a latent variable outperforms a least squares estimator used traditionally within the calibration

[10] See Ericsson and Reneby (2002a) or Bruche (2004).

[11] Delianedis and Geske (1999), Delianedis and Geske (2001), Eom et al. (2004), and Huang and Huang (2003) apply the calibration method in an academic context. Moodys/KMV is an example of commercial application, see Crosbie and Bohn (2003).

approach for the corporate security models suggested by Merton (1974), Briys and de Varenne (1997), Leland and Toft (1996), and Ericsson and Reneby (1998). Although the Ericsson and Reneby (1998) framework is comparable to the Corporate Securities Framework, a direct implementation as suggested by Ericsson and Reneby (2002a) seems problematic if more than the time series of equity prices is used to estimate the latent variable process.[12]

Ericsson and Reneby (2004) estimate the Ericsson and Reneby (2002a) model and propose an estimator for an equity time series only. Denote the n-th observed market price of equity by e_n with $n = 1, \ldots, N$. The log-likelihood function for the observed market prices is then

$$L(e; \Theta) = \sum_{n=2}^{N} \ln f\left(e_n | e_{n-1}; \eta_n, \Theta\right). \tag{5.2}$$

The N-dimensional vector e denotes the time-series of observed equity prices. $f(\cdot|\cdot)$ is defined as the conditional density of the price observations.

Note that η_n is not directly observable, it must be calculated for $n = 1, \ldots, N$ using the inverse function of equity prices given the parameter vector Θ. Denote this inverse transformation function by $\eta_n = E^{-1}(e_n, \Theta)$. Since we model security prices in terms of a latent EBIT-variable η, we need to change the variable of the conditional density from the observed equity price to the unobserved EBIT-levels.[13]

In our case of arithmetic Brownian motion, EBIT is normally distributed: $\Delta\eta = \eta_s - \eta_t \sim N(\mu\Delta t, \sigma_\eta^2 \Delta t)$ where $\Delta t = s - t$ is the time between two observations. Therefore, the conditional density can be transformed to

$$f\left(e_n | \, e_{n-1}, d_{n-1}; \eta, \Theta\right) = n(\eta_{n-1} + \mu\Delta t, \sigma_\eta^2 \Delta t)\big|_{\eta_n} \cdot \left.\frac{dE}{d\eta}\right|_{\eta_n}, \tag{5.3}$$

where d_{n-1} collects the time series of debt prices until t_{n-1} and $n(\mu, \sigma^2)$ denotes the normal density with parameters μ and σ. Maximizing equation (5.2) with the specification made in equation (5.3) results in an estimator for the time series of η_n and the parameter vector Θ.

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[12] Additionally, results from a simulation study might not directly apply to pricing data where market forces such as trading strategies and liquidity issues may reduce data quality.

[13] The method was proposed by Duan (1994) in the context of deposit insurance. The function $E^{-1}(e_n, \Theta)$ represents Duan (1994)'s inverse transformation function necessary to calculate the transformed maximum-likelihood function.

5.2.5 A Kalman Filter Approach

If more than one time series of corporate security prices is available for estimation, the one-dimensional maximum-likelihood estimator of equation (5.2) must be extended to allow for multi-dimensionality which cannot be accomplished easily. The inverse transformation function used in equation (5.3) needs a more general concept of optimal $\eta_n(e_n, d_n, \Theta)$ with respect to equity price e_n, debt prices d_n of the period n and the parameter vector Θ. It is not obvious how to specify this function, so that the maximum-likelihood estimator keeps its statistical properties.

Therefore, the use of a Kalman filter approach[14] seems more appropriate. A state representation seems especially suitable for EBIT-based structural credit risk models. EBIT cannot be observed directly. However, observations of time series of equity and debt prices are available which are in turn non-linear functions of the EBIT. The Kalman filter extracts for each period the best estimate of the latent state variable given the observed security prices of the period. In order to be applicable, the non-linear observation function must be linearized either by a Taylor series expansion as proposed in the literature on extended Kalman filters[15] or a derivative free approximation algorithm[16]. Both approaches are discussed next, starting with the Taylor series expansion.

As before, consider N discrete observations of equity and corporate bond prices. The Kalman updating scheme has a simple one-dimensional state vector η_n with a disturbance term $\Delta z_n^{\mathbb{Q}} \sim N(0, \Delta t)$, $n = 1, \ldots, N$.

$$\eta_n = \eta_{n-1} + \mu \Delta t + \sigma_\eta \Delta z_n^{\mathbb{Q}} \tag{5.4}$$

Given an estimate of last period's estimated EBIT $\hat{\eta}_{n-1}$, the best prediction of this period's EBIT, i.e. the conditional expectation of the n-th period's EBIT, is[17]

[14] See e.g. Maybeck (1979).

[15] See e.g. Haykin (2002).

[16] See Norgaard, Poulsen and Ravn (2000).

[17] Note that in this subsection the firm's equity E is denoted by the same letter as the expectation operator. To distinguish between the two symbols and to be precise on the information when taking expectations, E_n is reserved for the unconditional expectation in period n and $E_{n|n-1}$ denotes the expectation for period n conditional on information up to period $n-1$. The single letter E without subscript refers to equity.

$$\begin{aligned}\bar{\eta}_{n|n-1} &= E_{n|n-1}(\eta_n) \\ &= \hat{\eta}_{n-1} + \mu \Delta t. \end{aligned} \tag{5.5}$$

Denote by $\hat{\Sigma}_\eta(n-1) = E_{n-1}[(\eta_{n-1} - \hat{\eta}_{n-1})^2]$ the variance of last period's estimation error. Then, the variance of the prediction error of the n-th period becomes

$$\begin{aligned}\bar{\Sigma}_\eta(n|n-1) &= E_{n|n-1}\left[\left(\eta_n - \bar{\eta}_{n|n-1}\right)^2\right] \\ &= \hat{\Sigma}_\eta(n-1) + \sigma_\eta^2 \Delta t, \end{aligned} \tag{5.6}$$

where the fact was used that $\Delta z_{n-1}^{\mathbb{Q}}$ is uncorrelated with the prediction error. Assume next that at each observation time t_n, a stock price e_n and J bond prices $d_n = \{d_{j,n}\}$, $j = 1, \ldots, J$, are observed with observation errors $\varepsilon_{e,n}$ and $\varepsilon_{d,n} = \{\varepsilon_{j,d,n}\}$, respectively. Collect the prices and observation errors in $J+1 \times 1$ vectors $y_n = (e_n, d_n')'$ and $\varepsilon_n = (\varepsilon_{e,n}, \varepsilon_{d,n}')' \sim N(0, R)$. Since the equity and debt pricing functions depend non-linearly on EBIT, a Taylor series expansion around the predicted EBIT $\bar{\eta}_{n|n-1}$ is needed to linearize both functions for use in a Kalman filter.

$$\begin{aligned} y_n(\eta_n) &= Y(\eta_n, \bar{\eta}_{n|n-1}, \varepsilon_n) \\ &= \begin{pmatrix} E(\eta_n, \Theta, r_n, t_n) \\ D(\eta_n, \Theta, r_n, t_n) \end{pmatrix} + \begin{pmatrix} \varepsilon_{e,n} \\ \varepsilon_{d,n} \end{pmatrix} \\ &\approx \begin{pmatrix} E(\bar{\eta}_{n|n-1}, \Theta, r_n, t_n) + \frac{\partial}{\partial \eta} E(\cdot)\Big|_{\eta = \bar{\eta}_{n|n-1}} (\eta_n - \bar{\eta}_{n|n-1}) \\ \\ D(\bar{\eta}_{n|n-1}, \Theta, r_n, t_n) + \frac{\partial}{\partial \eta} D(\cdot)\Big|_{\eta = \bar{\eta}_{n|n-1}} (\eta_n - \bar{\eta}_{n|n-1}) \end{pmatrix} \\ &\quad + \begin{pmatrix} \varepsilon_{e,n} \\ \\ \varepsilon_{d,n} \end{pmatrix} \end{aligned} \tag{5.7}$$

In equation (5.7), the linearization of the observation error is additive because the derivative $\partial Y / \partial \varepsilon = I$, have mean zero, and do not aggregate over time.[18] The interest rate r_n and time t_n change deterministically while iterating through the filter implying that there is no correlation between interest rate and EBIT-changes, i.e. time and

[18] Note that more complicated observation error models can be implemented without difficulty. If observation errors have non-linear impact on prices and interrelate with other observation errors, or have non-zero means, the partial derivative would no longer be equal to the identity matrix and the observation error is linearized the same way as the latent state variable.

interest rates have no stochastic effect on EBIT. Given the linearized observation equation, the n-th period's security prices would be predicted to be

$$E_{n|n-1}(Y_n) = \bar{Y}_{n|n-1}(\bar{\eta}_{n|n-1}, \bar{\eta}_{n|n-1}, \mathbf{0}) \tag{5.8}$$

with an observation prediction error of

$$\begin{aligned} v_n &= y_n - \bar{Y}_{n|n-1}(\bar{\eta}_{n|n-1}, \bar{\eta}_{n|n-1}, \mathbf{0}) \\ &= G_\eta(\eta_n - \bar{\eta}_{n|n-1}) + \varepsilon_n. \end{aligned} \tag{5.9}$$

In equation (5.9), the $J+1 \times 1$-vector G_η collects the derivatives of the observation vector Y with respect to EBIT η. The observation prediction error has a variance of

$$\begin{aligned} \bar{\Sigma}_Y(n|n-1) = E_{n|n-1} \Big[&\big(y_n - \bar{Y}_{n|n-1}(\bar{\eta}_{n|n-1}, \bar{\eta}_{n|n-1}, \mathbf{0})\big) \\ &\big(y_n - \bar{Y}_{n|n-1}(\bar{\eta}_{n|n-1}, \bar{\eta}_{n|n-1}, \mathbf{0})\big)' \Big] \\ = G_\eta \bar{\Sigma}_\eta(n|n-1) G_\eta' + R \end{aligned} \tag{5.10}$$

and a covariance with the EBIT-prediction error of

$$\begin{aligned} \bar{\Sigma}_{Y\eta}(n|n-1) = E_{n|n-1} \Big[&\big(\eta_n - \bar{\eta}_{n|n-1}\big) \\ &\big(y_n - \bar{Y}_{n|n-1}(\bar{\eta}_{n|n-1}, \bar{\eta}_{n|n-1}, \mathbf{0})\big)' \Big] \\ = \bar{\Sigma}_\eta(n|n-1) G_\eta. \end{aligned} \tag{5.11}$$

Having observed the new observation vector y_n, the predictions of equations (5.5) and (5.8) can be updated. The best estimator of the current EBIT becomes

$$\begin{aligned} \hat{\eta}_n &= E_n(\eta_n | y_n) \\ &= \bar{\eta}_{n|n-1} + K_n v_n \end{aligned} \tag{5.12}$$

with the Kalman gain vector K_n defined by

$$K_n = \bar{\Sigma}_{Y\eta}(n|n-1) \bar{\Sigma}_Y(n|n-1)^{-1}. \tag{5.13}$$

The variance of the EBIT-estimator is

$$\begin{aligned} \hat{\Sigma}_\eta(n) &= Var(\eta_n | y_n) \\ &= \bar{\Sigma}_\eta(n|n-1) - K_n' \bar{\Sigma}_{Y\eta}(n|n-1). \end{aligned} \tag{5.14}$$

The Kalman filter recursion begins with

(1) an initial estimate of the state variable $\hat{\eta}_0$ and its variance $\hat{\Sigma}_\eta(0)$.
(2) Next, the predictions $\bar{\eta}_1$, $\bar{Y}_1$, $\bar{\Sigma}_\eta(1|0)$, $\bar{\Sigma}_Y(1|0)$, and $\bar{\Sigma}_{Y\eta}(1|0)$ are formed.
(3) With the observation y_1, the predictions are updated to $\hat{\eta}_1$, and $\hat{\Sigma}_\eta(1|0)$. The observation estimates can then be calculated by

$$\hat{Y}_n = \begin{pmatrix} E(\hat{\eta}_n, \Theta, r_n, t_n) \\ \\ D(\hat{\eta}_n, \Theta, r_n, t_n) \end{pmatrix}. \tag{5.15}$$

(4) Steps (2) and (3) are repeated until $n = N$.

The extended Kalman filter depends on the linearization of the observation equation (5.7) by a first order Taylor series expansion. Norgaard et al. (2000) criticize that the extended Kalman filter's Taylor series expansion is only accurate near the expansion point and that optimizing the parameter vector Θ might suffer numerical difficulties. In particular, approximation errors accumulate because in many applications analytical derivatives for Taylor series expansions are not available and numerical methods are employed. Moreover, a Taylor series expansion cannot capture the stochasticity of the underlying functions. Therefore, they propose a polynomial approximation of second order which offers advantages for its implementation because no explicit analytical derivative is needed but only function evaluations near but not too close to the expansion point. As a result, the differences taken never get so small as to cause numerical difficulties. This divided difference approach eases the implementation of filtering schemes and improves accuracy.

In particular, equation (5.7) is replaced by a second order Stirling interpolation which yields for the equity value

$$\begin{aligned} e_n = {} & E(\bar{\eta}_{n|n-1}, \cdot) + \frac{1}{2h}[E(\bar{\eta}_{n|n-1} + h, \cdot) - E(\bar{\eta}_{n|n-1} - h, \cdot)]\Delta\eta \\ & + \frac{1}{h^2}\left[E\left(\bar{\eta}_{n|n-1} + \frac{h}{2}, \cdot\right) - E\left(\bar{\eta}_{n|n-1} - \frac{h}{2}, \cdot\right)\right](\Delta\eta)^2 \\ & + \varepsilon_{e,n}, \end{aligned} \tag{5.16}$$

where h is the step size of the approximation and $\Delta\eta = (\eta_n - \bar{\eta}_{n|n-1})$.[19] Alternatively, we could have used the linearly transformed variable

$$Z = \frac{\eta}{s} \tag{5.17}$$

[19] For more complicated observation error models, $\varepsilon_{e,n}$ would be replaced by a sum of approximation terms involving h around the expected observation error.

with s being some constant for an approximation of the function

$$\tilde{E}(Z,\cdot) \equiv E(sZ,\cdot) = E(\eta,\cdot). \tag{5.18}$$

This transformation changes the power series approximation of the original function to

$$\begin{aligned} e_n = {} & E(\bar{\eta}_{n|n-1},\cdot) + \frac{1}{2h}[E(\bar{\eta}_{n|n-1} + hs,\cdot) - E(\bar{\eta}_{n|n-1} - hs,\cdot)]\Delta\eta \\ & + \frac{1}{h^2}\left[E\left(\bar{\eta}_{n|n-1} + \frac{h}{2}s,\cdot\right) - E\left(\bar{\eta}_{n|n-1} - \frac{h}{2}s,\cdot\right)\right](\Delta\eta)^2 \\ & + \varepsilon_e, \end{aligned} \tag{5.19}$$

which is strictly different to equation (5.16).[20] By picking s to be the square root of $\bar{\Sigma}_\eta$ and $h = 3$ as the kurtosis of the underlying distribution, the transformation optimally approximates the stochastic function $E(\eta_n,\cdot)$.[21] Similarly, all model bond prices D_{C_j,T_j} can be approximated.

In Subsection 5.4 evidence is presented that the Kalman filter outperforms the estimation procedures proposed by Ericsson and Reneby (2002a) because all observable variables are incorporated in the measurement of the state variable whereas Ericsson and Reneby (2002a) only use equity prices[22].

5.2.6 Parameter Estimation and Inference

The Kalman filter framework of the last subsection is convenient because it provides the necessary information for a maximum-likelihood estimator and inference. From equations (5.8) and (5.10), the observation error v_n, $n = 1,\ldots,N$ is normally distributed with

$$v_n \sim N(0, \bar{\Sigma}_Y(n|n-1)). \tag{5.20}$$

Therefore, the density of the estimated observation given the true observation is

$$\begin{aligned} f_{\bar{Y}_{n|n-1}}(y_n|\hat{Y}_{n-1},\hat{\eta}_{n-1},\Theta) = {} & \frac{1}{\sqrt{2\pi}}\left|\bar{\Sigma}_Y(n|n-1)\right|^{-\frac{1}{2}} \\ & \times \exp\left\{-\frac{1}{2}v_n'\bar{\Sigma}_Y(n|n-1)^{-1}v_n\right\} \end{aligned} \tag{5.21}$$

[20] Note that due to equation (5.18), the Taylor series expansion around $\bar{\eta}_{n|n-1}$ is invariant to linear transformations.

[21] See Norgaard et al. (2000) for the argument of optimally choosing h and s in the sense that the approximation error is minimized.

[22] See also Ericsson and Reneby (2004).

and the log likelihood[23] of the whole filter becomes

$$L(\Theta) = \sum_{n=1}^{N} log\left[f_{\bar{Y}_{n|n-1}}(y_n|\hat{Y}_{n-1}, \hat{\eta}_{n-1}, \Theta) \right]. \tag{5.22}$$

All functions of equation (5.21) depend on the parameter vector Θ which can be estimated by maximizing the likelihood function of equation (5.22).

$$\hat{\Theta}_{ML} = \sup_{\Theta} L(\Theta). \tag{5.23}$$

As an alternative to equation (5.23), another reasonable objective function can be used that has properties as to converge to a solution quicker and more robustly, i.e. the minimum of the sum of squared or absolute pricing errors

$$\hat{\Theta}_{SSQE} = \inf_{\Theta} \sum_{n=1}^{N} v_n' v_n, \tag{5.24}$$

$$\hat{\Theta}_{SAE} = \inf_{\Theta} \sum_{n=1}^{N} |v_n|' \mathbf{1} \tag{5.25}$$

Since all three estimators belong to the class of extremum estimators and as such to general method of moment estimators, they converge to the true parameter values Θ_0 as $N \to \infty$.[24] Therefore, irrespective of the objective function, the maximum-likelihood function of equation (5.22) can be used for inference. By the fact that observation error v_n are Gaussian, the estimated parameter $\hat{\Theta}$ is

$$\sqrt{N}(\hat{\Theta} - \Theta_0) \sim N(0, \hat{H}^{-1}) \tag{5.26}$$

where the estimated information matrix

$$\hat{H} = -\sum_{n=1}^{N} \left. \frac{\partial^2 log f_{\bar{Y}_{n|n-1}}(y_n|\hat{Y}_{n-1}, \hat{\eta}_{n-1}, \Theta)}{\partial\Theta\partial\Theta'} \right|_{\Theta=\hat{\Theta}}$$
$$= -\sum_{n=1}^{N} \hat{H}_n$$

[23] See e.g. Hamilton (1994), Chapter 13 for the details.
[24] See Mittelhammer, Judge and Miller (2000), Part III and Chapter 11.

converges to its true value H_0 in probability for $N \to \infty$.[25] For stock and bond prices, similar asymptotic distributions can be derived by the delta-method[26]

$$\sqrt{N}(\hat{E}(\eta_n, \cdot) - E(\eta_n, \cdot)) \sim N(0, \hat{H}_{E,n}) \tag{5.27}$$

with

$$\hat{H}_{E,n} = - \left. \frac{\partial E(\eta_n, \cdot)}{\partial \hat{\Theta}'} \right|_{\Theta = \hat{\Theta}} \hat{H}_n^{-1} \left. \frac{\partial E(\eta_n, \cdot)}{\partial \hat{\Theta}} \right|_{\Theta = \hat{\Theta}}$$

and

$$\sqrt{N}(\hat{D}(\eta_n, \cdot) - D(\eta_n, \cdot)) \sim N(0, \hat{H}_{D,n}) \tag{5.28}$$

with

$$\hat{H}_{D,n} = - \left. \frac{\partial D(\eta_n, \cdot)}{\partial \hat{\Theta}'} \right|_{\Theta = \hat{\Theta}} \hat{H}_n^{-1} \left. \frac{\partial D(\eta_n, \cdot)}{\partial \hat{\Theta}} \right|_{\Theta = \hat{\Theta}} .$$

5.3 Implementing the Corporate Securities Framework

The Corporate Securities Framework of Chapters 2 and 3 is structured modular where the firm value is distributed among all claimants of the firm's EBIT. To be fully specified the whole financing structure is needed to calculate security prices because the bankruptcy event interrelates all debt claims to one another. However, the value of all claims is not observed on a regular basis. Therefore, the model has to be restricted so that observations of only part of the issued securities suffices to estimate parameter values. Additional assumptions are needed.

Consider first the assumption that the total amount of debt outstanding is left unchanged, i.e. $\bar{P} = \sum_{j=1}^{J} P_j$ for all outstanding debt contracts $j = 1, \ldots, J$ and all future points in time. The constant total debt volume makes the recovery values of individual debt contracts independent of the bankruptcy time since each outstanding debt issue recovers a fraction of the firm value after bankruptcy losses and taxes, that is proportional to total debt outstanding. So, the recovery fraction depends only on the initial debt structure. Therefore, maturing debt contracts are refinanced by issuing new debt. As shown in Subsection 4.1.2.2, future debt issues flatten the optimal bankruptcy barrier.

[25] See Hamilton (1994), p. 389.

[26] See e.g. Campbell, Lo and MacKinlay (1997), Appendix A4.

It is safe to assume η_B to be time invariant which eases the calculation of bankruptcy probabilities and prices considerably. The value of a debt contract simplifies to

$$D_{C_k,T_k} = e^{-r(T_k-t_0)}\left[P_k - (1-\tau^d)\frac{C_k}{r}\right][1-\Phi(t_0,T_k,\eta_{t_0},\eta_B)] \\ +(1-\tau^d)\frac{C_k}{r}[1-p_B(t_0,T_k,\eta_{t_0},\eta_B)] + D^-_{C_k,T_k}.$$

Recall from Chapter 2 that $\Phi(t,T,\eta_t,\eta_B)$ and $p_B(t,T,\eta_t,\eta_B)$ denote the probability of going bankrupt in the period $]t,T]$ when EBIT starts at η_t facing the constant barrier η_B and the Arrow-Debreu price of a claim that pays one currency unit in bankruptcy, respectively. The time t_0-value of the funds that bond holders are able to recover in bankruptcy are

$$D^-_{C_k,T_k} = (1-\tau^{eff})\min\left[(1-\alpha)V_B;\bar{P}\right]w_k p_B(t_0,T_k,\eta_{t_0},\eta_B).$$

where $w_k = P_k/\bar{P}$ represents the fraction of bond k with respect to total debt $\bar{P}$.[27]

The original framework only allows for a single, constant risk-free interest rate. Even if we abstract from the stochasticity of risk-free interest rates, the daily change of interest rates should be considered in the estimation process and a suitable maturity within the term structure has to be chosen. Denote by B_{C_k,T_k} the value of a risk-free bond with the same features and tax treatment as the corporate bond D_{C_k,T_k}. The corporate bond price can be restated as

$$\begin{aligned} D_{C_k,T_k} = B_{C_k,T_k}&[1-\Phi(t_0,T_k,\eta_{t_0},\eta_B)] \\ &-B_{C_k,\infty}[p_B(t_0,T_k,\eta_{t_0},\eta_B) \\ &-\Phi(t_0,T_k,\eta_{t_0},\eta_B)] + D^-_{C_k,T_k}. \end{aligned} \tag{5.29}$$

If we allow for future starting bond issues, its value would be

$$\begin{aligned} D_{C_k,S_k,T_k} = B_{C_k,S_k,T_k}&[1-\Phi(t_0,T_k,\eta_{t_0},\eta_B)] \\ &-B_{C_k,\infty}[p_B(S_k,T_k,\eta_{t_0},\eta_B) \\ &-\Phi(S_k,T_k,\eta_{t_0},\eta_B)B_{0,S_k}] \\ &-\Phi(S_k,T_k,\eta_{t_0},\eta_B)B_{0,S_k}P_k I_{\{S_k>t\}} \\ &+D^-_{C_k,T_k}, \end{aligned} \tag{5.30}$$

[27] See Chapters 2 and 3 for a detailed derivation of the formulas and the exact definition of variables.

where S_k denotes the bonds issue time. The default probability during the bond's life is

$$\Phi(S_k, T_k, \eta_{t_0}, \eta_B) = \Phi(t_0, T_k, \eta_{t_0}, \eta_B) - \Phi(t_0, S_k, \eta_{t_0}, \eta_B),$$

and the Arrow-Debreu price of bankruptcy for the subperiod $]S_k, T_k]$ is

$$p_B(S_k, T_k, \eta_{t_0}, \eta_B) = p_B(t_0, T_k, \eta_{t_0}, \eta_B) - p_B(t_0, S_k, \eta_{t_0}, \eta_B).$$

The dependence of the corporate bond price in equations (5.29) and (5.30) on the choice of the term of the risk-free interest rate is reduced considerably. Since the whole term-structure of risk-free interest rates is known at each point in time, the price of the risk-free equivalent bonds can be calculated easily. However, the risk-free interest rate r still remains explicit when determining the equity value and might be set to a term between five and ten years which should approximately equal the average payback time of a firm's investments.

A firm's total liabilities $\bar{P}$ can be set to the reported amount as of the last available annual report. Note that this value usually represents the notional amount of debt outstanding and not its current value as of the balance sheet date. We might also consider $\bar{P}$ to be a constant or deterministically changing parameter determined within the estimation process.

5.4 The Simulation Study

5.4.1 Experiment Design

To evaluate whether the Kalman filter proposed in Section 5.2.5 is able to detect the parameters of the EBIT-process correctly, a simulation study was conducted for the ABM- and the GBM base case firm introduced in Section 4.1.1. With respect to the discussion in Subsection 5.3, where it was proposed to allow for future debt issues to simplify and speed up the Kalman filter evaluations, it is assumed that the firm refinances each maturing debt issues by a 6 % coupon bond with an identical notional and infinite maturity.[28] The model parameters are summarized in Table 5.1.

Using the drift and the volatility of the stochastic processes of Panel A in Table 5.1, for both the ABM- and the GBM-firm, 500 EBIT-paths of 50 daily observations have been simulated. For each

[28] As illustrated in Subsection 4.1.2.2, the refinancing bonds have a current value of approximately zero.

Table 5.1. Model parameters of the ABM- and GBM-firm in the Kalman filter simulation study

Panel A: Model Parameters

Economic Variables

r	5 %
τ^c	35 %
τ^d	10 %
τ^e	10 %
$\varepsilon_{e,n}$	$N(0, 0.005)$
$\varepsilon_{d,j,n}$	$N(0, 0.0025)$
	$\forall n, j$

Firm Specific Variables

	ABM	GBM
η_{t_0}	100	
α	50 %	
V_B	3,050	
μ	10	3.33 %
σ_η	40	18 %
♯ of issued stocks: 200		

Panel B: Initial Financing Structure

j	P_j	In % of V	S_j	T_j	C_j
7	600	10 %	10	∞	6 %
6	600	10 %	4	∞	6 %
5	600	10 %	2	∞	6 %
4	1,250	20.83 %	-1	∞	6 %
3	600	10 %	-1	10	5.5 %
2	600	10 %	-1	4	5 %
1	600	10 %	-1	2	4.5 %

EBIT-realization, the security values have been calculated according to pricing equations as of Chapter 3 and Section 5.3. The security values have then been translated into price quotes, assuming that the firm issued 200 stocks in total. All price quotes have been perturbed by an observation error drawn from the distribution of $\varepsilon_{e,n}$ and $\varepsilon_{d,j,n}$.

The stock price and the dirty prices of the currently issued bonds, i.e. bonds $j = 1, \ldots 4$, are observable. The parameter vector Θ was reduced to contain the process parameters μ and σ_η only.

In principle, we could have included the bankruptcy barrier V_B and/or the refinancing coupons C_j, $j > 4$ in the parameter vector, as well. If the bankruptcy barrier were included, we might have allowed equity holders to pick the bankruptcy barrier optimally assuming that the firm is already in the infinite maturity state. This appears viable because it was shown in Subsection 4.1.2.2 that the inclusion of future debt issues flattens the optimal bankruptcy barrier. However, for each pair of μ and σ_η, the flat optimal bankruptcy barrier is equivalent to a deterministically set constant bankruptcy barrier without the additional burden to calculate the optimal bankruptcy barrier. Therefore, for estimating the Kalman filter, the optimization of the bankruptcy barrier would only add complexity without making the simulation study more accurate. A similar argument is true for the refinancing coupons. If included in the parameter vector Θ, refinancing coupons would have to be determined by finding the par forward yields of the refinancing bonds. The refinancing coupons change for every single simulated EBIT due to changes of the term structure of bankruptcy probabilities. However, for each parameter choice of μ and σ_η, the Kalman filter delivers the best estimates of the latent EBIT times series given that the par forward yields of the refinancing bond issues have been determined by these best estimates. Due to the unique relationship between EBIT and refinancing coupons, the refinancing coupons can as well be fixed for the simulation study.

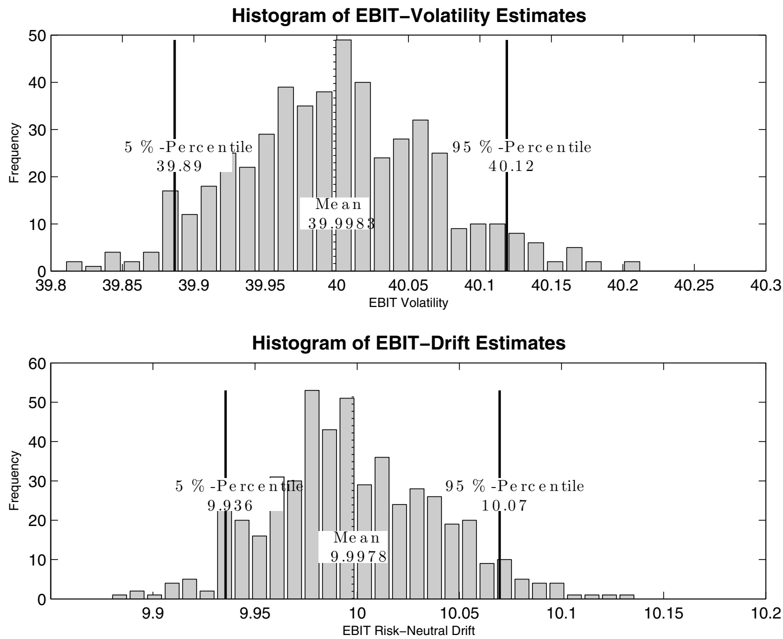
Preliminary tests have shown that the log-likelihood function of equation (5.22) encounters problems with starting values for μ and σ_η that are too far from the true parameters because the likelihood of the filter becomes close to zero. In contrast, the sum of absolute pricing differences of equation (5.25) is a less sensitive objective function for arbitrary starting values. It converges quickly to the global solution, i.e. the true parameters, as long as the starting parameters of the filter lie above the true parameters. If the starting parameters are chosen below the true parameters, the estimation procedure eventually converges to a local minimum whereas only very few runs could actually identify the true parameters. So, a two step estimation procedure seems appropriate: A first estimation run is performed by minimizing the sum of absolute pricing errors with starting values far above the true parameters. A second run maximizes the log-likelihood function by starting the estimation at the estimates of the first run.

5.4.2 Parameter Estimation Results

Figure 5.1 collects the estimated ABM-risk-neutral drift and volatility parameters of the first estimation step in histograms. In contrast to Ericsson and Reneby (2004), we are able to identify both the volatility and the risk-neutral drift.[29] Both estimators appear to be unbiased and the true parameters lie within the 90 % confidence bounds of the

[29] To overcome their difficulties, Ericsson and Reneby (2004) need to set the EBIT-drift to the risk-less interest rate adjusted for an exogenously estimated constant cash payout ratio per period.

Fig. 5.1. Histogram of the estimated ABM-EBIT risk-neutral drift and volatility parameter from 500 simulation runs á 50 periods by minimizing the sum of absolute pricing errors. True parameter values are $\mu = 10$ and $\sigma_\eta = 40$.



simulated parameter distribution. The confidence bounds itself are sufficiently small given that we only used 50 data points for each simulation. The second round of estimation produced even closer confidence bounds (see Figure 5.2).

Panels A and B of Table 5.2 compare the statistics of the parameter, state variable and security price estimates of the two estimation steps. The maximum-likelihood estimation improves parameter estimates considerably. We observe that the second step not only reduces the range of the estimates but also the mean errors and standard errors. Furthermore, the Jarque-Bera statistic for testing normality of the parameter estimation error distributions would reject the normality hypothesis for the risk-neutral drift in the first estimation step at the 5 % level. The normality of the maximum-likelihood estimator cannot be rejected. The mean estimation error and standard error of the state variable and security prices are small although differences can become large in single periods (see the MIN- and MAX-columns in

Fig. 5.2. Histogram of the estimated ABM-EBIT risk-neutral drift and volatility parameter from 500 simulation runs á 50 periods by maximizing the likelihood function. True parameter values are $\mu = 10$ and $\sigma_\eta = 40$.

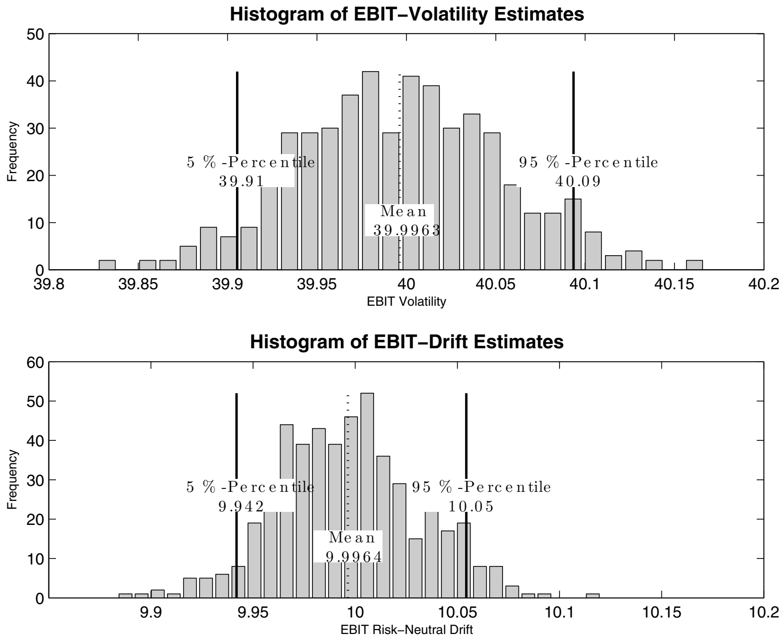


Table 5.2). Tests for normality of estimation errors of the state variable and long-term security must be rejected in the first estimation step at the 1 % level. The maximum-likelihood estimation improves the situation. At least, the Jarque-Bera statistic of the equity pricing errors can no longer reject normality.

The results for the GBM-simulation study are different. Figure 5.3 illustrates that the true parameters fall within the confidence bounds. The parameter estimates are symmetrically distributed but slightly biased upwards. However, the maximum-likelihood estimation does not improve the parameter estimates (Figure 5.4). The bias moves downwards and Table 5.3 confirms that standard errors increase substantially. The same observation can be made for the state variable estimates. Estimated security prices are less affected. Although normality of the estimated parameters cannot be rejected in the first estimation step, the Jarque-Bera statistic exceeds the critical value at all reasonable confidence values in the second step.

Table 5.2. Simulated distribution summary statistics of the estimated ABM-Corporate Securities Framework. The table reports mean errors ME, mean absolute errors MAE, maximum MAX and minimum errors MIN, as well as standard errors STD, skewness $SKEW$ and kurtosis $KURT$ of the distribution. The Jarque-Bera test statistics and its p-value are also provided. The simulation comprised 500 runs á 50 periods.

Panel A: Minimizing Absolute Pricing Errors

	Simulated Error Distribution							Jarque/Bera	
	ME	MAE	MAX	MIN	STD	$SKEW$	$KURT$	JBSTAT	P-Val.
$\hat{\sigma}_\eta - \hat{\sigma}_\eta$	-0.0017	0.0545	0.2132	-0.1904	0.0689	0.1732	3.0325	2.4921	0.2876
$\mu - \hat{\mu}$	-0.0022	0.0330	0.1363	-0.1206	0.0413	0.2612	3.0598	5.6995	0.0579
$\eta - \hat{\eta}$	-0.0363	0.7045	3.5273	-3.2213	0.8836	0.2089	3.0966	191.4799	0.0000
$E - \hat{E}$	0.0009	0.0531	0.2783	-0.2794	0.0673	-0.0047	3.1220	15.5345	0.0004
$D_{4.50\ \%,2} - \hat{D}_{4.50\ \%,2}$	0.0003	0.0392	0.2122	-0.2047	0.0492	-0.0165	3.0479	3.4935	0.1743
$D_{5.00\ \%,4} - \hat{D}_{5.00\ \%,4}$	0.0014	0.0363	0.1791	-0.1767	0.0459	-0.0077	3.0632	4.3822	0.1118
$D_{5.50\ \%,10} - \hat{D}_{5.50\ \%,10}$	0.0016	0.0337	0.1969	-0.1674	0.0421	-0.0140	3.0012	0.8234	0.6625
$D_{6.00\ \%,\infty} - \hat{D}_{6.00\ \%,\infty}$	0.0020	0.0313	0.1501	-0.1680	0.0394	0.0540	3.0307	13.1050	0.0014

Panel B: Maximizing Log Likelihood

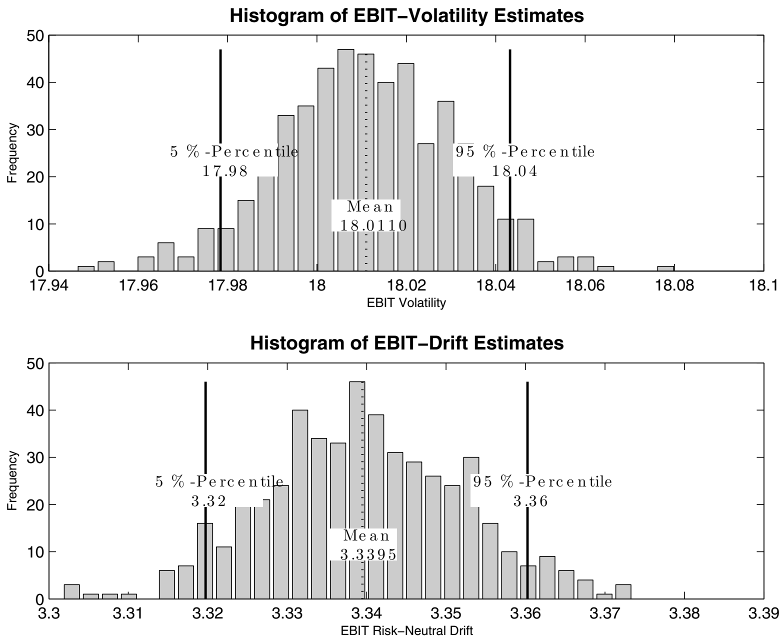
	Simulated Error Distribution							Jarque/Bera	
	ME	MAE	MAX	MIN	STD	$SKEW$	$KURT$	JBSTAT	P-Val.
$\hat{\sigma}_\eta - \hat{\sigma}_\eta$	-0.0037	0.0463	0.1669	-0.1730	0.0573	0.0755	2.8554	0.9789	0.6130
$\mu - \hat{\mu}$	-0.0036	0.0279	0.1202	-0.1164	0.0349	0.1090	3.1212	1.2295	0.5408
$\eta - \hat{\eta}$	-0.0634	0.6147	3.4626	-3.2222	0.7684	0.1173	3.1282	74.3952	0.0000
$E - \hat{E}$	0.0009	0.0534	0.2712	-0.2746	0.0669	-0.0048	3.0440	2.0871	0.3522
$D_{4.50\ \%,2} - \hat{D}_{4.50\ \%,2}$	0.0002	0.0392	0.2120	-0.2034	0.0492	-0.0171	3.0466	3.4541	0.1778
$D_{5.00\ \%,4} - \hat{D}_{5.00\ \%,4}$	0.0012	0.0364	0.1800	-0.1740	0.0458	-0.0092	3.0181	0.6841	0.7103
$D_{5.50\ \%,10} - \hat{D}_{5.50\ \%,10}$	0.0016	0.0337	0.1945	-0.1683	0.0421	-0.0143	2.9985	0.8556	0.6520
$D_{6.00\ \%,\infty} - \hat{D}_{6.00\ \%,\infty}$	0.0021	0.0314	0.1459	-0.1652	0.0393	0.0527	2.9983	11.5779	0.0031

The problems of the maximum-likelihood estimation in the GBM-case are most likely due to the small sample size of 50 time steps per simulation. In the GBM-case, the state variable $\bar{\eta}_n$ is log-normally distributed. This non-linearity of the state variable translates $\hat{\Theta}$ of equation (5.23) into a quasi-maximum-likelihood estimator where normality is achieved only asymptotically as $N \rightarrow \infty$. Nevertheless, the proposed Kalman filter seems to work well even for the GBM-EBIT model.

In a second experiment, we test whether the filter is sensitive to a correctly specified observation error. The observation error of security prices is omitted. The two estimation steps are conducted as described above including the assumption that prices are observed with the indicated errors as of Table 5.1, Panel A.[30] The estimators of the risk-neutral EBIT-drift and EBIT-volatility of both the ABM- and GBM-filter are biased. The confidence intervals of the parameter estimates of the ABM-filter still contain the true values in both

[30] See the Figures B.6 to B.9 and Tables B.3 to B.4 in the appendix.

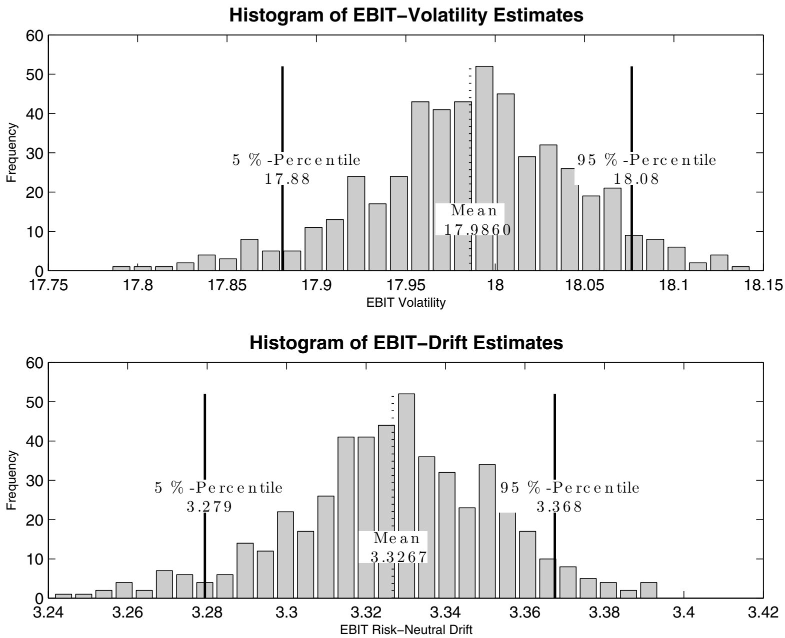
Fig. 5.3. Histogram of the estimated GBM-EBIT risk-neutral drift and volatility parameter from 500 simulation runs á 50 periods by minimizing the sum of absolute pricing errors. True parameter values are $\bar{\mu} = 3.\dot{3}$ % and $\bar{\sigma}_\eta = 18$ %.



steps. Similarly to the first simulation experiment, the first step estimation of the GBM-filter produces estimators biased upwards whereas the maximum-likelihood estimators are biased downwards. Confidence intervals do not contain the true values. The GBM-filter is sensitive to a misspecification of observation errors in small samples. A preliminary test of the GBM-filter for time series of $N = 200$ price observations per security confirms that the distribution of the estimators moves closer to normality with its center approaching the true parameter values. The ABM-filter seems to be robust against misspecifications.

Although further research seems to be warranted on the small sample properties of the objective functions (5.23), (5.24), and (5.25), our results are promising. The objective functions converge in a way that the parameter estimators are close to the true parameters of the sampled time series although we introduce a quite substantial pricing error of several basis points in bond prices and several percent in equity prices.

Fig. 5.4. Histogram of the estimated GBM-EBIT risk-neutral drift and volatility parameter from 500 simulation runs á 50 periods by maximizing the likelihood function. True parameter values are $\bar{\mu} = 3.\dot{3}$ % and $\bar{\sigma}_\eta = 18$ %.



5.5 Summary

In this chapter a state-space representation of the Corporate Securities Framework of Chapters 2 and 3 is proposed. In contrast to the existing literature, the estimation procedure is able to incorporate multiple time series of corporate security prices. Since the EBIT-process can be chosen freely, we can test which type of EBIT-process fits security price movements best.

In a simulation study, we show that estimators based on the Kalman filter can identify the true parameters of the EBIT-process. For the ABM- and the GBM-filter, 500 time series of 50 daily observations of stock and bond prices are simulated where each observation is measured with a zero-mean error. The estimation is performed in two step. The first parameter estimates were achieved by minimizing the sum of absolute pricing errors. This objective function allows for almost arbitrary starting values as long as the estimation does not start in a

Table 5.3. Simulated distribution summary statistics of the estimated GBM-Corporate Securities Framework. The table reports mean errors ME, mean absolute errors MAE, maximum MAX and minimum errors MIN, as well as standard errors STD, skewness $SKEW$ and kurtosis $KURT$ of the distribution. The Jarque-Bera test statistics and its p-value are also provided. The simulation comprised 500 runs á 50 periods.

Panel A: Minimizing Absolute Pricing Errors

	Simulated Error Distribution							Jarque/Bera	
	ME	MAE	MAX	MIN	STD	$SKEW$	$KURT$	JBSTAT	P-Val.
$\hat{\sigma}_\eta - \hat{\sigma}_\eta$	0.0001	0.0002	0.0008	-0.0005	0.0002	-0.0543	3.2203	1.1408	0.5653
$\mu - \hat{\mu}$	0.0001	0.0001	0.0004	-0.0003	0.0001	0.0127	2.9627	0.0635	0.9687
$\eta - \hat{\eta}$	0.3521	0.6970	3.1401	-2.6597	0.8080	0.0501	3.0901	18.8744	0.0001
$E - \hat{E}$	-0.0033	0.0382	0.2688	-0.2273	0.0489	-0.0049	3.4543	214.8005	0.0000
$D_{4.50\ \%,2} - \hat{D}_{4.50\ \%,2}$	0.0009	0.0384	0.2020	-0.1838	0.0482	-0.0177	3.0325	2.3919	0.3024
$D_{5.00\ \%,4} - \hat{D}_{5.00\ \%,4}$	0.0023	0.0588	0.2992	-0.3393	0.0744	-0.0119	3.1024	11.4532	0.0033
$D_{5.50\ \%,10} - \hat{D}_{5.50\ \%,10}$	0.0002	0.0553	0.2907	-0.2657	0.0699	0.0112	3.1062	12.2205	0.0022
$D_{6.00\ \%,\infty} - \hat{D}_{6.00\ \%,\infty}$	-0.0029	0.0531	0.2871	-0.2715	0.0672	0.0048	3.1208	15.2239	0.0005

Panel B: Maximizing Log Likelihood

	Simulated Error Distribution							Jarque/Bera	
	ME	MAE	MAX	MIN	STD	$SKEW$	$KURT$	JBSTAT	P-Val.
$\hat{\sigma}_\eta - \hat{\sigma}_\eta$	-0.0001	0.0005	0.0014	-0.0022	0.0006	-0.3128	3.3932	11.1065	0.0039
$\mu - \hat{\mu}$	-0.0001	0.0002	0.0006	-0.0009	0.0003	-0.2992	3.3159	9.3247	0.0094
$\eta - \hat{\eta}$	-0.4252	1.2506	4.3028	-5.9902	1.5576	-0.2781	3.2759	401.2455	0.0000
$E - \hat{E}$	-0.0035	0.0383	0.2644	-0.2255	0.0486	0.0086	3.3984	165.4339	0.0000
$D_{4.50\ \%,2} - \hat{D}_{4.50\ \%,2}$	-0.0001	0.0386	0.2062	-0.1837	0.0485	-0.0212	3.0364	3.2293	0.1990
$D_{5.00\ \%,4} - \hat{D}_{5.00\ \%,4}$	-0.0101	0.0635	0.3219	-0.3621	0.0792	-0.0289	3.1010	14.0665	0.0009
$D_{5.50\ \%,10} - \hat{D}_{5.50\ \%,10}$	-0.0134	0.0606	0.3347	-0.3153	0.0747	-0.0040	3.0146	0.2788	0.8699
$D_{6.00\ \%,\infty} - \hat{D}_{6.00\ \%,\infty}$	-0.0126	0.0590	0.2824	-0.3125	0.0732	-0.0326	3.0440	6.4222	0.0403

bankruptcy state and the starting values of EBIT-drift and volatility lie above the true parameter values. A standard search algorithm is then able to approach the true parameter values relatively quickly. Given the first estimates, the log likelihood function of the filter is maximized which is expected to refine the parameter estimates. The Kalman filter is not only able to identify the EBIT-volatility but also the EBIT-risk-neutral drift from the imprecisely observed security prices which no other method was able to before. In previous empirical studies, the drift has been arbitrarily set to the risk-free rate adjusted for an estimate of a constant payout ratio disregarding the theoretical inconsistency that the payout ratio changes together with EBIT in a dynamic way.

Both filters work remarkably well despite our small sample size of only 50 data points per filter optimization. However, the GBM-estimators do not improve in the second estimation step which is attributable to the non-linearity of the state function. The small sample properties of the GBM-model are therefore less favorable. Larger sample time-series are needed for better convergence.

The filter even works under a second setting where security pricing errors are misspecified. Although the estimators are biased, both filters find optima close to the true parameter values. However, for the GBM-filter more observations are needed to get reasonable sample distributions.

Having succeeded in a simulation study, we can take the next step and use time series of bond and equity prices of individual firms to test the model directly. Section 5.3 discusses some of the practical issues. Depending on the individual firm and the information available about the firm, additional assumption might be needed. However, the proposed model gives a sound foundation for reasonable and consistent adjustments.

6

Concluding Remarks

6.1 Summary

With the increased availability of disaggregated, high frequency corporate bond data, structural credit models can be subjected to better and more reliable econometric tests and thus have regained academic interest. Earlier empirical studies reported the inability of firm value models to price corporate bonds correctly. With more accurately and more frequently recorded time series of bond prices, new econometric analysis became feasible. However, recent evidence is not convincing yet.

Exploring the empirical and the theoretical literature in detail, a serious drawback of existing structural credit risk models became visible: its evident simplification of the financing structure of the firm. Moreover, the economic assumptions behind the theory are disguised so that empirical models have itself been applied inadequately and no final results can be drawn from this empirical work.

In this thesis we offer a proposal to overcome some of the theoretical deficiencies of structural credit risk models and to fill the gap between the theory and the empirical literature. Chapter 2 develops a very general economic framework for the pricing of corporate securities. The firm's earnings before interest and taxes, EBIT, are assumed to follow an Itô-process and the Corporate Securities Framework allows a firm to have multiple finite maturity debt issues. The complex capital structure requires a consistent model of bankruptcy proceedings because the capital structure will eventually change in the future. As a result, the recovery for an individual bond issue might change from subperiod to subperiod. By the same argument, the bankruptcy barrier might change. We show that the model can be solved analytically if there exists an explicit solution for the firm value, for all the bankruptcy probabilities, and for the Arrow-Debreu bankruptcy prices. Despite its technical representation, Chapter 2 concentrates on the economic assumptions which are presented in a rigorous mathematical form. The chapter offers two distinct extensions to the current literature: First, the Corporate Securities Framework is developed from economic assumptions and mathematical formalism is used to keep the model consistent. Therefore, important modeling decisions such as the choice of the stochastic process for EBIT can be justified economically. Our approach considerably increases the flexibility of the model. Second, the framework is general enough to be extended for various refinements because any model feature is independent of others. Stochastic interest rates or strategic bankruptcy behavior of equity holders can be introduced conceptually without difficulty. Our model also provides an interesting setting to analyze refinancing strategies thoroughly. In particular, if a firm must decide whether to replace an existing debt issue by reissuing debt or by asking equity holders to infuse money, which strategy is then optimal for equity holders? Which maturity, which notional, and which coupon should be offered to potential debt holders? How does the decision depend on the current state of the firm? Such questions can be consistently addressed within our framework.

Chapter 3 demonstrates that the Corporate Securities Framework is tractable if EBIT follows either an arithmetic or a geometric Brownian motion. We emphasize the fact that the widely used but hardly questioned standard assumption that state variables follow a geometric Brownian motion is no longer needed to solve the model analytically. There is even a strong case in favor of EBIT following an arithmetic Brownian motion because EBIT can then become negative and a much wider class of firms can be covered in an empirical application. By choosing arithmetic Brownian motion, we do not only extend the existing literature of credit risk models to this particular process type but we also show that the geometric Brownian motion assumption is one among many. Our framework offers the possibility of selecting a suitable stochastic process on economic rather than on mathematical considerations. For solving the bankruptcy probabilities with several finite maturity debt issues, new hitting probabilities of a Brownian motion facing a changing barrier are derived. The results should prove useful for other applications in option pricing or real option analysis, as well. The analytical solution speeds up the calculation for the simulation study conducted in Chapter 5.

Additionally, we propose two numerical methods which enable us to price derivatives on corporate securities and to extend the basic analytical model with more advanced decision rules. The standard trinomial lattice approach is capable of incorporating more complicated tax regimes and optimal bankruptcy decision of equity owners. A numerical integration approach can be used for pricing derivatives on corporate securities of the European type, as well. The integration approach is fast, accurate, and can be applied to comparable problems and more complicated derivatives in practice.

In Chapter 4 the analytical solution for corporate securities is illustrated by numerical examples. The Corporate Securities Framework produces a rich set of comparative statics results which are open to an intuitive explanation. Most interestingly, we document that in both of our analytical examples equity value is a concave function of the EBIT-volatility and therefore exhibits room for optimization given the initial EBIT and the capital structure. This property propels further explanations and a more general analysis of a firm's choice of risk.

A distinct focus is laid on the detection of how the Corporate Securities Framework can be applied in a direct empirical study. Several features of the model prove important. The tax structure of EBIT-based models cannot be neglected. As well, future debt issues are important. However, a reasonable refinancing model can ease the computational burden for an empirical analysis because the optimal bankruptcy barrier becomes almost flat which facilitates the calculation of the hitting probabilities and prices.

As a second application in Chapter 4, we investigate option prices on equity if the firm has a complex capital structure. The Corporate Securities Framework offers the flexibility to change the capital structure in the future and to show that the term structure of debt is an important determinant of option prices. Since the density of equity values and equity returns are obtainable directly in our framework, we are able to link the current firm's condition to the shape of equity densities and to option implied volatilities. The level of the smile increases the closer the firm is to bankruptcy. Debt repayments before option maturity tend to shift implied volatility curves down. In contrast to the broad literature on implied volatilities, we recommend to link the shape and level of implied volatilities to the moments of the equity return distribution only if the firm is currently in a good state. The examples in Subsection 4.2 indicate that higher moments of the distribution of equity returns can be misleading for firms close to bankruptcy.

In Chapter 5, the empirical literature on structural credit risk models is critically reviewed. As a result of the drawbacks of proposed estimation procedures, we suggest a Kalman filter to estimate EBIT as a latent state variable and the parameters of the models. In a simulation study, we demonstrate that the Kalman filter allows to identify the true parameters correctly. Convergence for the ABM- and GBM-model is fast and the estimators seem to be unbiased even in small samples. The estimators are robust against misspecified observation errors. However, the GBM-model needs longer time series of security prices to obtain reasonable estimator properties in this case. Most promisingly, we are able to identify not only the EBIT-volatility but also the risk-neutral EBIT-drift in both models. The latter finding is new to the literature and of particular interest because without the risk-neutral EBIT-drift, the structural credit risk models are likely to contain inconsistencies.

The thesis encompasses a broad scope of topics which are vigorously discussed in academia and in practice. Our Corporate Securities Framework allows for intuitive explanations without resorting to ad-hoc mathematical assumptions which are difficult to justify economically. Apart from its theoretical flexibility, the framework can be directly applied for estimates using a large sample of firms. The empirical tests are not only helpful in academic discussion but also offer a tool to practitioners who prefer working with structural models. To the best of our knowledge, financial institutions favor reduced form models which are primarily used for pricing securities. However, economic effects can hardly be forecasted appropriately if one is restricted to purely statistical methods. A structural framework such as ours is able to capture the essential economic patterns. A flexible structural framework which is easy to handle and produces reliable estimates will eventually convince practitioners to rethink their model choice.

6.2 Future Research

Our economic model of the firm provides a framework which should prove useful for related and extended analysis. We want to highlight three possible fields for further studies.

First, our model need not be restricted to simulations but could be used to perform comprehensive empirical studies. The Corporate Securities Framework is applicable to a much larger group of firms. Compared to other models there no longer exist restrictions on the capital structure. It is even possible to introduce additional debt contracts with embedded options. Moreover, the EBIT-process assumption can be investigated directly and makes it possible to evaluate which of the process assumptions performs better. It was shown in Chapter 5 that the GBM-Kalman filter is not unbiased in small samples. Although time series of stock and bond prices are usually long, empirical studies allow to further explore how the Kalman filter behaves if samples are small and how many observations are necessary to get reliably results. ABM-Kalman filters can be studied analogously.

Second, our model is open enough to analyze recommendations for decision making. The interdependence of bond and stock prices of the same firm is not only of special interest for the corporate budgeting decision of which source of funds to choose from but also for financial institutions or investors who want to measure the risk of their portfolios if they are invested in both instruments. At first sight both problems seem to be two different views of the same problem. This is correct remembering that there is only one firm that managers as well as investors would like to research. But if information is taken into account, managers have an advantage in answering their questions compared to investors. This gives rise to an agency theoretic argument. Both views can be analyzed in our EBIT-firm value framework. A Duffie and Lando (2001)-like notion of incomplete information for outsiders which clarifies how information is processed and how this influences the estimation and the perception of risk, might be an interesting starting point. Financial investors are outsiders in the EBIT-model and have only access to incomplete information. Is there a direct implication of the empirical and theoretical results for them? How can they evaluate the risk of portfolios invested in a firm's equity and debt as well as in treasury bonds correctly if they are exposed to market and credit risk at the same time? Our EBIT-model accounts for these interdependencies and therefore allows for an aggregation of credit and market risk. Simulation studies can show how risk measurement errors distort investment strategies.

Third, our model can be used as a device in integrated risk management. In the last few years risk management in financial institutions advanced considerably. Usually, risk management systems of banks are based on two concepts. Market risks are measured with Value at Risk, credit risks with models that are able to capture correlated defaults of counterparties like CreditRisk+, CreditMetrics or Credit Portfolio View. Due to different assumptions and objectives the two concepts cannot be integrated easily. One of the critical theoretical (economic and mathematical) issues is to combine the two approaches in one comprehensive model. In order to develop the comprehensive market and credit risk model, we can take either a mathematical or an economic approach. The first strand of research tries to isolate the mathematical building blocks of market and credit risk models and to use generalized concepts of stochastic comovement like copulas and extreme value theory to arrive at a meaningful and interpretable estimate of total bank risk. The second economic approach, however, seems more promising. In our economic model, we can keep the mathematics simple, resort to classical stochastic concepts, and motivate the assumptions as well as the derived results by economic intuition.

Our framework can be easily extended to multiple firms. We introduce two correlated EBIT-processes which drive the values of two firms. The interaction allows us to analyze correlated defaults and compare those to observed default correlations. However, a more realistic model of market structure might be warranted if both firms compete directly so that the default of one firm actually improves the outlook for the other firm. If stochastic interest rates are introduced, market and credit risk factors are naturally combined in one model. Then, our Corporate Securities Framework can be contrasted to the separate measurement of two risk components, a VaR of market risk and a VaR of credit risk which are combined by some statistical method, as e.g. proposed by Jorion (2000). By the same argument as above, the problem of the holding period assumptions inherent in the market/credit VaR approach can be resolved. Usually, credit VaR is measured for a longer time horizon than market VaR. Our approach models credit events in an economically intuitive and time-consistent way.

We hope that the quality of the EBIT-model developed in this thesis will be able to trigger research efforts in these fields and finally help to design models which become reliable modules in a next generation of internal risk evaluation tools of financial institutions.

A

Notes on the Equity Option Valuation

A.1 A Note on the Change in Variable of Equity Value and its Return Density Plots

The discussion in 4.2.2 relies heavily on the unconditional partial density plots displayed. The density as of equation (3.96) is defined heuristically for a small interval $d\eta_T$ as explained in Section 3.4.2. Equity values at option maturity and the respective return values are functions of the state variable η_T, the densities with respect to the stochastic Variable η have to be translated into densities of the new variable.

If the distribution function of η for the ABM-case is denoted by $F(\eta_T) = P^{\mathbb{Q}}(\eta \leq \eta_T; M_T > \eta_B)$ with a density of $f(\eta_T) = \partial/\partial\eta(\Phi(\cdot))$ and equity values at maturity T are an invertible function of EBIT $E_T = E(\eta_T)$, the probabilities for the two events must therefore be the same.

$$P^{\mathbb{Q}}(\eta \leq \eta_T; M_T > \eta_B) = P^{\mathbb{Q}}(E \leq E_T; M_T > \eta_B)$$

For the density of equity values f^E this requires that

$$\begin{aligned} f^E(E_T) &= \frac{\partial}{\partial\eta} F(\eta_T) \frac{\partial}{\partial\eta} E^{-1}(\eta_T) \\ &= f(\eta_T) \left(\left. \frac{\partial E_T}{\partial\eta} \right|_{\eta=\eta_T} \right)^{-1}, \end{aligned} \tag{A.1}$$

since the invertible function is one dimensional in the stochastic parameter.

Transforming the equity density into a return density requires the same transformation as above. Denote the equity return with respect to the expected value at the option maturity by $r_E(T) = \ln(E_T/E^{\mathbb{Q}}(E_T))$. So equity return density can be calculated by

$$f^r(r_E(T)) = f^E(E_T)E_T, \tag{A.2}$$

since $\partial r_E(T)/\partial E_T = E_T$.

The GBM-case is equivalent to the ABM-case if η is replaced by $\ln(\bar{\eta})$ and η_B by $\ln(\bar{\eta}_B)$. The derivatives in equation A.1 are then taken with respect to $\ln(\eta_T)$. Equation A.2 does not change.

A.2 Distributional Assumptions and Option Prices

To support the understanding of how distributional assumptions affect option prices, it is necessary to decompose price differences of option with different strikes. Writing the call option value as an expectation under the risk-neutral measure $\mathbb{Q}$

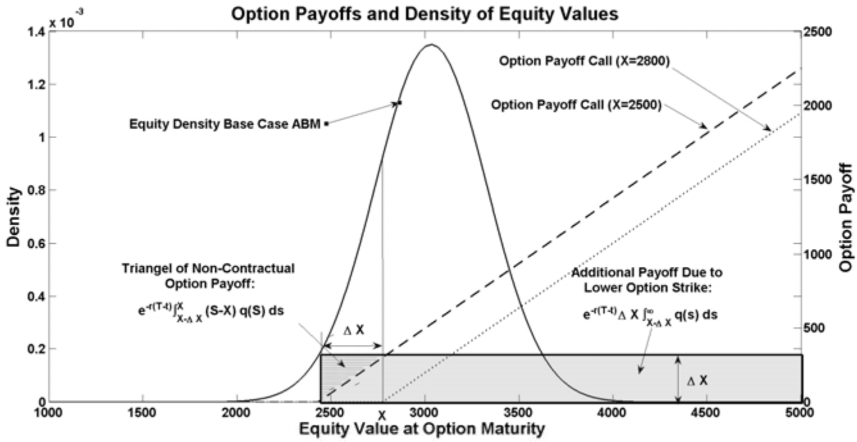
$$\begin{aligned} C(t,T,X) &= E^{\mathbb{Q}}[e^{-r(T-t)}(E_T - X)^+] \\ &= e^{-r(T-t)} \int_0^\infty (E_T - X)^+ q(E_T) dE_T \\ &= e^{-r(T-t)} \int_X^\infty (E_T - X) q(E_T) dE_T, \end{aligned}$$

where E_T denotes the underlying value, X the strike, and T the time of maturity of the call option C at time t, the price difference of a similar call option but with a lower strike $X - \Delta X$, where $\Delta X > 0$ is

$$\begin{aligned} \Delta C(t,T,X,\Delta X) &= e^{-r(T-t)} \left(\int_X^\infty (E_T - X) q(E_T) dE_T \right. \\ &\quad \left. - \int_{X-\Delta X}^\infty (E_T - X - \Delta X) q(E_T) dE_T \right) \\ &= e^{-r(T-t)} \left(\Delta X \int_{X-\Delta X}^\infty q(E_T) dE_T \right. \\ &\quad \left. + \int_{X-\Delta X}^X (E_T - X) q(E_T) dE_T \right). \end{aligned} \tag{A.3}$$

Equation (A.3) can nicely be illustrated by Figure A.1, which shows the payoffs at maturity of two options with strikes X and $X - \Delta X$, respectively. The lower strike option can be replicated as a portfolio of the option with strike X (the area above the gray rectangle and

Fig. A.1. Option price differences due to differences of strike prices



below the dashed line, the area due to the lower strike ΔX (the gray rectangle), and the dotted triangle which the low-strike option holder will not get.

Comparing implicit volatilities is equivalent to comparing the probability mass between $X - \Delta X$ and X. If this probability mass is higher than that of the log-normal distribution, the option price rises more than the Black/Scholes option price leading to an increase in implied volatilities. More generally, the option price difference must exceed

$$\begin{aligned}\Delta C_N(t,T,X,\Delta X) = e^{-r(T-t)} \Bigg[& X \left(N(d_2) - N(d_2 - \frac{\gamma}{\sigma\sqrt{T-t}} \right) \\ & - E^{\mathbb{Q}}(E_T) \left(N(d_1) - N(d_1 - \frac{\gamma}{\sigma\sqrt{T-t}} \right) \\ & + \Delta X N(d_2 - \frac{\gamma}{\sigma\sqrt{T-t}}) \Bigg], \end{aligned} \tag{A.4}$$

where $\gamma = 1 - (\Delta X)/X$ is the proportional decrease of the strike and d_1 and d_2 are defined as in equation (4.1). In equation (A.4), the first two lines represent the dotted triangle of figure A.1 and the third line is equivalent to the gray rectangle.

B

Additional Tables and Figures

Fig. B.1. Equity values after tax as a function of EBIT-volatility and current EBIT value of a GBM-firm.

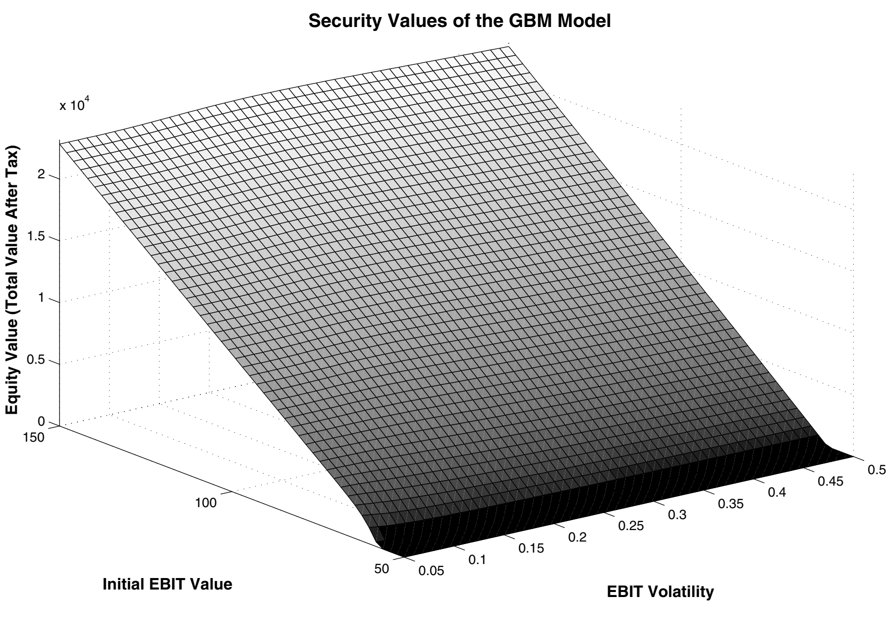


Table B.1. Security values in the ABM-Corporate Securities Framework. Panel A and B show comparative static market values of debt.

Panel A

	Tax Regimes			Optimal Bankruptcy			Future Bond Issues (TR 1)			
	None	1	3	None	TR 1	TR 3	None	1	2	3
4.5 %, 2	593.58	588.21	588.21	588.21	580.26	574.98	580.26	587.59	588.48	588.99
5 %, 4	597.86	586.27	586.27	586.27	578.99	572.69	578.99	568.01	582.60	586.31
5.5 %, 10	617.75	590.02	590.02	590.02	584.31	577.72	584.31	575.67	566.76	582.79
6 %, ∞	1,474.11	1,321.84	1,321.84	1,321.84	1,309.09	1,294.28	1,309.09	1,289.87	1,274.41	1,302.41
6 %, 2-4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-15.61	-1.80	1.45
6 %, 4-10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-29.45	-4.25
6 %, 10-∞	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	64.27
$\sum$ Debt	3,283.31	3,086.33	3,086.33	3,086.33	3,052.65	3,019.68	3,052.65	3,005.53	2,981.00	3,121.97
Equity	2,690.49	1,182.99	1,096.65	1,182.99	1,187.02	1,106.56	1,187.02	1,211.87	1,296.83	1,551.40
BL	26.06	26.06	26.06	26.06	53.35	94.61	53.35	98.03	122.36	48.60
Taxes	0.00	1,704.48	1,790.81	1,704.48	1,698.97	1,771.35	1,698.97	1,675.44	1,594.47	1,273.81
Firm Value	5,999.86	5,999.86	5,999.86	5,999.86	5,992.00	5,992.19	5,992.00	5,990.87	5,994.66	5,995.79

Panel B

	Future Bond Issues (TR 1)				Future Bond Issues (CBB, TR 1)			
	None	1	2	3	None	1	2	3
4.5 %, 2	574.98	584.40	585.64	587.53	588.21	588.21	588.21	588.21
5 %, 4	572.69	560.00	572.77	579.18	582.78	582.14	582.14	582.14
5.5 %, 10	577.72	567.65	556.44	572.17	579.98	579.33	574.86	574.86
6 %, ∞	1,294.28	1,271.61	1,249.68	1,280.42	1,299.90	1,298.55	1,289.22	1,286.35
6 %, 2-4	0.00	-20.59	-8.97	-4.33	0.00	-2.00	-2.00	-2.00
6 %, 4-10	0.00	0.00	-31.17	-11.82	0.00	0.00	-12.19	-12.19
6 %, 10-∞	0.00	0.00	0.00	65.71	0.00	0.00	0.00	65.76
$\sum$ Debt	3,019.68	2,963.09	2,924.39	3,068.86	3,050.88	3,046.22	3,020.24	3,083.13
Equity	1,106.56	1,121.74	1,168.83	1,408.31	1,162.96	1,184.18	1,288.83	1,548.47
BL	94.61	151.40	192.77	101.49	125.27	125.27	125.27	125.27
Taxes	1,771.35	1,754.76	1,709.83	1,417.68	1,696.56	1,679.98	1,601.31	1,278.80
Firm Value	5,992.19	5,990.98	5,995.83	5,996.35	6,035.67	6,035.65	6,035.65	6,035.66

Table B.2. Security values in the GBM-Corporate Securities Framework. Panel A and B show comparative static market values of debt.

Panel A

	Tax Regimes			Optimal Bankruptcy			Future Bond Issues (TR 1)			
	None	1	3	None	TR 1	TR 3	None	1	2	3
4.5 %, 2	593.85	588.56	588.56	588.56	568.98	565.26	568.98	586.53	588.36	588.95
5 %, 4	598.94	587.68	587.68	587.68	567.42	560.85	567.42	546.66	579.10	584.35
5.5 %, 10	620.42	593.43	593.43	593.43	571.17	565.04	571.17	555.08	535.58	565.87
6 %, ∞	1,476.85	1,324.71	1,324.71	1,324.71	1,276.18	1,261.78	1,276.18	1,239.63	1,201.59	1,247.85
6 %, 2-4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-36.08	-5.21	-0.48
6 %, 4-10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-85.90	-34.80
6 %, 10-∞	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	37.45
$\sum$ Debt	3,290.07	3,094.37	3,094.37	3,094.37	2,983.75	2,952.93	2,983.75	2,891.82	2,813.52	2,989.19
Equity	2,692.66	1,182.08	1,095.88	1,182.08	1,197.42	1,119.88	1,197.42	1,237.04	1,349.34	1,579.46
BL	17.25	17.25	17.25	17.25	112.37	150.05	112.37	187.77	236.32	121.66
Taxes	0.00	1,706.28	1,792.48	1,706.28	1,696.11	1,769.10	1,696.11	1,669.92	1,591.95	1,305.74
Firm Value	5,999.98	5,999.98	5,999.98	5,999.98	5,989.66	5,991.96	5,989.66	5,986.56	5,991.13	5,996.06

Panel B

	Future Bond Issues (TR 1)				Future Bond Issues (CBB, TR 1)			
	None	1	2	3	None	1	2	3
4.5 %, 2	565.26	582.79	584.90	587.36	588.56	588.56	588.56	588.56
5 %, 4	560.85	539.93	565.97	574.55	582.52	581.76	581.76	581.76
5.5 %, 10	565.04	548.33	524.80	551.43	572.27	571.50	563.49	563.49
6 %, ∞	1,261.78	1,223.73	1,175.06	1,220.62	1,276.17	1,274.57	1,257.87	1,252.81
6 %, 2-4	0.00	-39.28	-15.13	-8.86	0.00	-2.72	-2.72	-2.72
6 %, 4-10	0.00	0.00	-82.18	-44.98	0.00	0.00	-34.60	-34.60
6 %, 10-∞	0.00	0.00	0.00	43.41	0.00	0.00	0.00	52.20
$\sum$ Debt	2,952.93	2,855.49	2,753.43	2,923.53	3,019.52	3,013.66	2,954.36	3,001.50
Equity	1,119.88	1,143.49	1,191.01	1,401.44	1,161.32	1,183.75	1,320.17	1,577.23
BL	150.05	231.01	306.77	182.06	201.27	201.27	201.27	201.27
Taxes	1,769.10	1,757.15	1,741.58	1,489.31	1,700.97	1,684.38	1,607.25	1,303.08
Firm Value	5,991.96	5,987.14	5,992.78	5,996.33	6,083.08	6,083.06	6,083.04	6,083.07

Fig. B.2. Equity value densities of 6 month equity options in the GBM-Corporate Securities Framework as a function of η_0. Expected equity values are indicated by solid lines. Path probabilities are obtained by differentiating the splined distribution function of EBIT at option maturity.

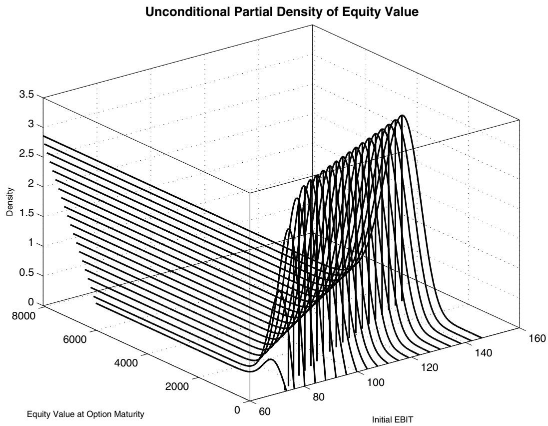


Fig. B.3. Equity value density moments of 6 month equity options in the GBM-Corporate Securities Framework as a function of η_0. The moments are obtained by numerical integration.

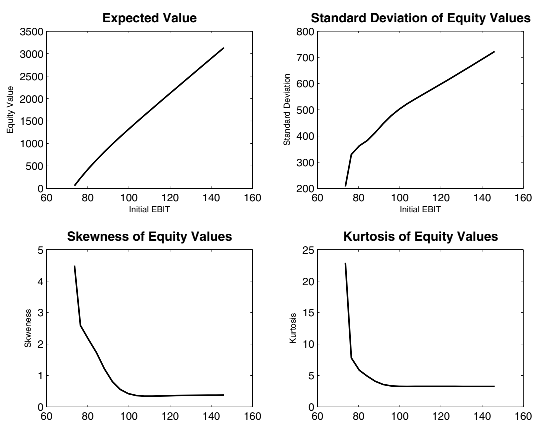


Fig. B.4. Equity return densities of 6 month equity options in the GBM-Corporate Securities Framework as a function of η_0. The 0-returns are indicated by solid lines. Path probabilities are obtained by differentiating the splined distribution function of EBIT at option maturity.

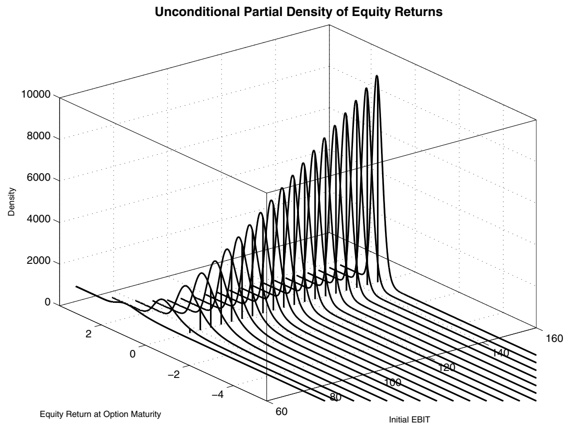


Fig. B.5. Equity return density moments of 6 month equity options in the GBM-Corporate Securities Framework as a function of η_0. The moments are obtained by numerical integration.

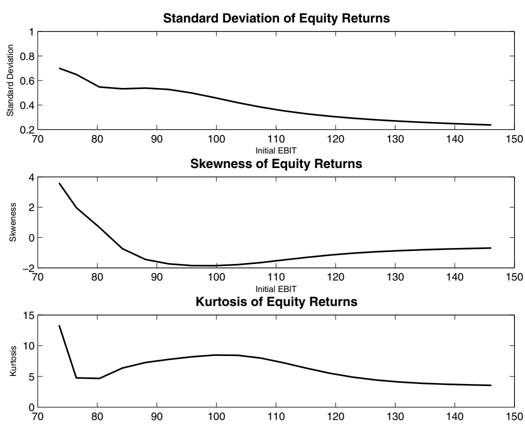


Fig. B.6. Histogram of the estimated ABM-EBIT risk-neutral drift and volatility parameter from 500 simulation runs á 50 periods by minimizing the sum of absolute pricing errors. True parameter values are $\mu = 10$ and $\sigma_\eta = 40$. Observation errors are assumed in the estimation although the observation matrix is free of measurement error.

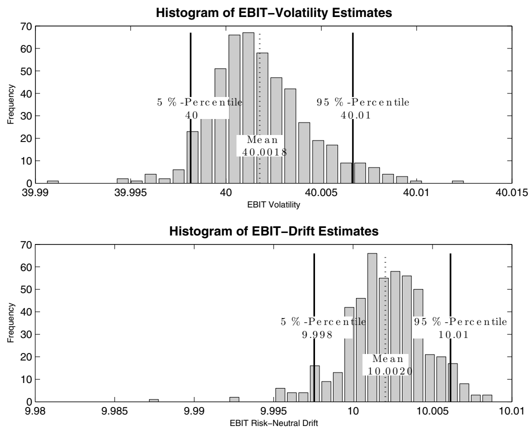


Fig. B.7. Histogram of the estimated ABM-EBIT risk-neutral drift and volatility parameter from 500 simulation runs á 50 periods by maximizing the likelihood function. True parameter values are $\mu = 10$ and $\sigma_\eta = 40$. Observation errors are assumed in the estimation although the observation matrix is free of measurement error.

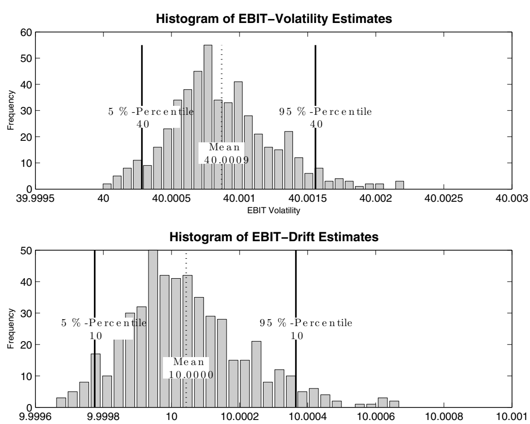


Table B.3. Simulated distribution summary statistics of the estimated ABM-Corporate Securities Framework. The table reports mean errors ME, mean absolute errors MAE, maximum MAX and minimum errors MIN, as well as standard errors STD, skewness $SKEW$ and kurtosis $KURT$ of the distribution. The Jarque-Bera test statistics and its p-value are also provided. The simulation comprised 500 runs á 50 periods. Observation errors are assumed in the estimation although the observation matrix is free of measurement error.

Panel A: Minimizing Absolute Mean Errors

	Simulated Error Distribution							Jarque/Bera	
	ME	MAE	MAX	MIN	STD	$SKEW$	$KURT$	JBSTAT	P-Val.
$\hat{\sigma}_\eta - \hat{\sigma}_\eta$	0.0018	0.0024	0.0125	-0.0094	0.0026	0.4098	4.2998	48.1858	0.0000
$\mu - \hat{\mu}$	0.0020	0.0027	0.0088	-0.0129	0.0026	-0.6440	5.4828	160.5245	0.0000
$\eta - \hat{\eta}$	0.0655	0.0943	0.8325	-0.4949	0.1161	1.0027	6.1845	14749	0.0000
$E - \hat{E}$	0.0015	0.0043	0.0529	-0.0291	0.0063	1.7115	8.0112	38355	0.0000
$D_{4.50\ \%,2} - \hat{D}_{4.50\ \%,2}$	0.0003	0.0007	0.0127	-0.0017	0.0010	2.4963	14.9075	173628	0.0000
$D_{5.00\ \%,4} - \hat{D}_{5.00\ \%,4}$	0.0015	0.0044	0.0533	-0.0126	0.0062	1.7010	7.1668	30135	0.0000
$D_{5.50\ \%,10} - \hat{D}_{5.50\ \%,10}$	0.0017	0.0065	0.0767	-0.0231	0.0091	1.6580	6.8205	26652	0.0000
$D_{6.00\ \%,\infty} - \hat{D}_{6.00\ \%,\infty}$	0.0017	0.0076	0.0882	-0.0275	0.0105	1.6508	6.7713	26163	0.0000

Panel B: Maximizing Log Likelihood

	Simulated Error Distribution							Jarque/Bera	
	ME	MAE	MAX	MIN	STD	$SKEW$	$KURT$	JBSTAT	P-Val.
$\hat{\sigma}_\eta - \hat{\sigma}_\eta$	0.0009	0.0009	0.0022	-0.0000	0.0004	0.5990	3.5747	36.2607	0.0000
$\mu - \hat{\mu}$	0.0000	0.0001	0.0007	-0.0003	0.0002	0.6690	3.5910	43.9954	0.0000
$\eta - \hat{\eta}$	0.0165	0.0728	0.8801	-0.4487	0.1050	1.7976	8.1432	41009	0.0000
$E - \hat{E}$	0.0009	0.0044	0.0527	-0.0283	0.0064	1.7903	8.1469	40939	0.0000
$D_{4.50\ \%,2} - \hat{D}_{4.50\ \%,2}$	0.0002	0.0007	0.0126	-0.0020	0.0010	2.4091	14.3477	158287	0.0000
$D_{5.00\ \%,4} - \hat{D}_{5.00\ \%,4}$	0.0011	0.0045	0.0533	-0.0123	0.0062	1.6944	7.0870	29355	0.0000
$D_{5.50\ \%,10} - \hat{D}_{5.50\ \%,10}$	0.0017	0.0065	0.0768	-0.0233	0.0091	1.6576	6.8297	26721	0.0000
$D_{6.00\ \%,\infty} - \hat{D}_{6.00\ \%,\infty}$	0.0020	0.0075	0.0884	-0.0278	0.0105	1.6563	6.8245	26660	0.0000

Fig. B.8. Histogram of the estimated GBM-EBIT risk-neutral drift and volatility parameter from 500 simulation runs á 50 periods by minimizing the sum of absolute pricing errors. True parameter values are $\bar{\mu} = 3.\dot{3}$ % and $\bar{\sigma}_{\eta} = 18$ %. Observation errors are assumed in the estimation although the observation matrix is free of measurement error.

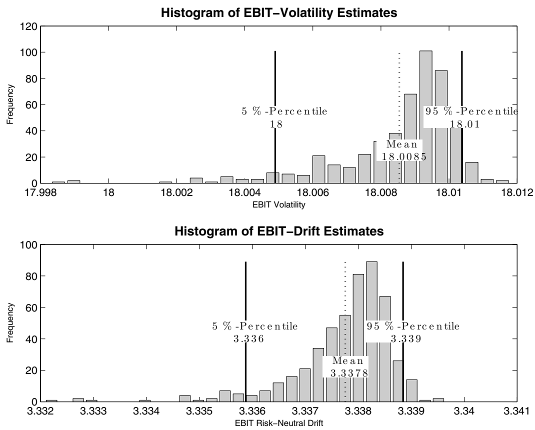


Fig. B.9. Histogram of the estimated GBM-EBIT risk-neutral drift and volatility parameter from 500 simulation runs á 50 periods by maximizing the likelihood function. True parameter values are $\bar{\mu} = 3.\dot{3}$ % and $\bar{\sigma}_{\eta} = 18$ %. Observation errors are assumed in the estimation although the observation matrix is free of measurement error.

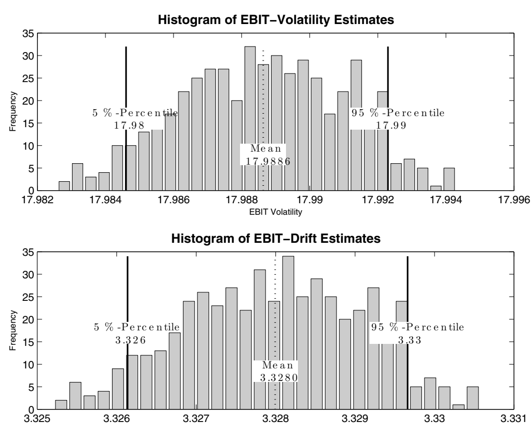


Table B.4. Simulated distribution summary statistics of the estimated GBM-Corporate Securities Framework. The table reports mean errors ME, mean absolute errors MAE, maximum MAX and minimum errors MIN, as well as standard errors STD, skewness $SKEW$ and kurtosis $KURT$ of the distribution. The Jarque-Bera test statistics and its p-value are also provided. The simulation comprised 500 runs á 50 periods. Observation errors are assumed in the estimation although the observation matrix is free of measurement error.

Panel A: Minimizing Absolute Mean Errors

	Simulated Error Distribution							Jarque/Bera	
	ME	MAE	MAX	MIN	STD	$SKEW$	$KURT$	JBSTAT	P-Val.
$\hat{\sigma}_\eta - \hat{\sigma}_\eta$	0.0001	0.0001	0.0001	-0.0000	0.0000	-1.9657	8.4905	940	0.0000
$\mu - \hat{\mu}$	0.0000	0.0000	0.0001	-0.0000	0.0000	-2.0622	9.9654	1351	0.0000
$\eta - \hat{\eta}$	0.2460	0.2469	0.4228	-0.1949	0.0618	-1.6091	7.8251	35031	0.0000
$E - \hat{E}$	-0.0040	0.0052	0.0288	-0.0257	0.0048	0.4695	4.3877	2923	0.0000
$D_{4.50\ \%,2} - \hat{D}_{4.50\ \%,2}$	0.0002	0.0004	0.0114	-0.0074	0.0006	2.4608	41.3900	1560162	0.0000
$D_{5.00\ \%,4} - \hat{D}_{5.00\ \%,4}$	0.0014	0.0036	0.0556	-0.0378	0.0051	0.4453	10.5203	59723	0.0000
$D_{5.50\ \%,10} - \hat{D}_{5.50\ \%,10}$	-0.0005	0.0053	0.0712	-0.0503	0.0075	0.2836	7.7022	23360	0.0000
$D_{6.00\ \%,\infty} - \hat{D}_{6.00\ \%,\infty}$	-0.0027	0.0065	0.0793	-0.0556	0.0085	0.3434	7.4780	21373	0.0000

Panel B: Maximizing Log Likelihood

	Simulated Error Distribution							Jarque/Bera	
	ME	MAE	MAX	MIN	STD	$SKEW$	$KURT$	JBSTAT	P-Val.
$\hat{\sigma}_\eta - \hat{\sigma}_\eta$	-0.0001	0.0001	-0.0001	-0.0002	0.0000	-0.0868	2.3802	8.8748	0.0118
$\mu - \hat{\mu}$	-0.0001	0.0001	-0.0000	-0.0001	0.0000	-0.1040	2.3806	9.1377	0.0104
$\eta - \hat{\eta}$	-0.3467	0.3467	0.0034	-0.6120	0.0886	-0.0285	2.7346	76.8980	0.0000
$E - \hat{E}$	-0.0037	0.0050	0.0298	-0.0249	0.0047	0.4388	4.5383	3266	0.0000
$D_{4.50\ \%,2} - \hat{D}_{4.50\ \%,2}$	-0.0006	0.0007	0.0102	-0.0086	0.0006	0.6483	39.5534	1393332	0.0000
$D_{5.00\ \%,4} - \hat{D}_{5.00\ \%,4}$	-0.0085	0.0089	0.0527	-0.0420	0.0050	0.9109	11.6468	81322	0.0000
$D_{5.50\ \%,10} - \hat{D}_{5.50\ \%,10}$	-0.0118	0.0126	0.0683	-0.0546	0.0076	0.9467	8.2668	32621	0.0000
$D_{6.00\ \%,\infty} - \hat{D}_{6.00\ \%,\infty}$	-0.0113	0.0125	0.0767	-0.0593	0.0085	0.8631	7.9492	28612	0.0000

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