The Open Problems Project

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0.1 Introduction

This is the beginning of a project[1] to record open problems of interest to researchers in computational geometry and related fields. It commenced with the publication of thirty problems in Computational Geometry Column 42 [MO01] (see Problems 1–30), but has grown much beyond that. We encourage correspondence to improve the entries; please send email to TOPP@cs.smith.edu. If you would like to submit a new problem, please fill out this template.

Each problem is assigned a unique number for citation purposes. Problem numbers also indicate the order in which the problems were entered. Each problem is classified as belonging to one or more categories.

The problems are also available as a single Postscript or PDF file.

0.2 Categorized List of All Problems

Below, each category lists the problems that are classified under that category. Note that each problem may be classified under several categories.

arrangements:

- 3-Colorability of Arrangements of Great Circles (Problem 44)

art galleries:

- Vertex π-Floodlights (Problem 23)

coloring:

- 3-Colorability of Arrangements of Great Circles (Problem 44)

combinatorial geometry:

- k-sets (Problem 7)
- Binary Space Partition Size (Problem 14)
- Chromatic Number of the Plane (Problem 57)

[1] Partially supported by NSF grants to the three editors.

stabbing:

- Minimum Stabbing Spanning Tree (Problem 20)

traveling salesman:

- Minimum-Turn Cycle Cover in Planar Grid Graphs (Problem 53)
- Planar Euclidean Maximum TSP (Problem 49)
- Traveling Salesman Problem in Solid Grid Graphs (Problem 54)

triangulations:

- Compatible Triangulations (Problem 38)
- Flip Graph Connectivity in 3D (Problem 28)
- Hamiltonian Tetrahedralizations (Problem 29)
- Minimum Weight Triangulation (Problem 1)
- Pointed Spanning Trees in Triangulations (Problem 50)
- Simple Linear-Time Polygon Triangulation (Problem 10)
- The Number of Pointed Pseudotriangulations (Problem 40)

visibility:

- Trapping Light Rays with Segment Mirrors (Problem 31)
- Vertex π-Floodlights (Problem 23)
- Visibility Graph Recognition (Problem 17)

Voronoi diagrams:

- Dynamic Planar Nearest Neighbors (Problem 63)
- Voronoi Diagram of Lines in 3D (Problem 3)
- Voronoi Diagram of Moving Points (Problem 2)

1 Numerical List of All Problems

The following lists all problems sorted by number. These numbers can be used for citations and correspond to the order in which the problems were entered.

- Problem 1: Minimum Weight Triangulation
- Problem 2: Voronoi Diagram of Moving Points
- Problem 3: Voronoi Diagram of Lines in 3D
- Problem 4: Union of Fat Objects in 3D
- Problem 5: Euclidean Minimum Spanning Tree

- Problem 6: Minimum Euclidean Matching in 2D
- Problem 7: k-sets
- Problem 8: Linear Programming: Strongly Polynomial?
- Problem 9: Edge-Unfolding Convex Polyhedra
- Problem 10: Simple Linear-Time Polygon Triangulation
- Problem 11: 3SUM Hard Problems
- Problem 12: Dynamic Planar Convex Hull
- Problem 13: Point Location in 3D Subdivision
- Problem 14: Binary Space Partition Size
- Problem 15: Output-sensitive Convex Hull in $\mathbb{R}^d$
- Problem 16: Simple Polygonalizations
- Problem 17: Visibility Graph Recognition
- Problem 18: Pushing Disks Together
- Problem 19: Vertical Decompositions in $\mathbb{R}^d$
- Problem 20: Minimum Stabbing Spanning Tree
- Problem 21: Shortest Paths among Obstacles in 2D
- Problem 22: Minimum-Link Path in 2D
- Problem 23: Vertex π-Floodlights
- Problem 24: Polygonal Curve Simplification
- Problem 25: Polyhedral Surface Approximation
- Problem 26: Surface Reconstruction
- Problem 27: Hexahedral Meshing
- Problem 28: Flip Graph Connectivity in 3D
- Problem 29: Hamiltonian Tetrahedralizations
- Problem 30: Thrackles
- Problem 31: Trapping Light Rays with Segment Mirrors
- Problem 32: Bar-Magnet Polyhedra
- Problem 33: Sum of Square Roots

- Problem 34: Extending Pseudosegment Arrangements by Subdivision
- Problem 35: Freeze-Tag: Optimal Strategies for Awakening a Swarm of Robots
- Problem 36: Inplace Convex Hull of a Simple Polygonal Chain
- Problem 37: Counting Polyominoes
- Problem 38: Compatible Triangulations
- Problem 39: Distances among Point Sets in $\mathbb{R}^2$ and $\mathbb{R}^3$
- Problem 40: The Number of Pointed Pseudotriangulations
- Problem 41: Sorting $X + Y$ (Pairwise Sums)
- Problem 42: Vertex-Unfolding Polyhedra
- Problem 43: General Unfoldings of Nonconvex Polyhedra
- Problem 44: 3-Colorability of Arrangements of Great Circles
- Problem 45: Smallest Universal Set of Points for Planar Graphs
- Problem 46: 3D Minimum-Bend Orthogonal Graph Drawings
- Problem 47: Hinged Dissections
- Problem 48: Bounded-Degree Minimum Euclidean Spanning Tree
- Problem 49: Planar Euclidean Maximum TSP
- Problem 50: Pointed Spanning Trees in Triangulations
- Problem 51: Linear-Volume 3D Grid Drawings of Planar Graphs
- Problem 52: Queue-Number of Planar Graphs
- Problem 53: Minimum-Turn Cycle Cover in Planar Grid Graphs
- Problem 54: Traveling Salesman Problem in Solid Grid Graphs
- Problem 55: Pallet Loading
- Problem 56: Packing Unit Squares in a Simple Polygon
- Problem 57: Chromatic Number of the Plane
- Problem 58: Monochromatic Triangles
- Problem 59: Most Circular Partition of a Square
- Problem 60: Transforming Polygons via Vertex-Centroid Moves
- Problem 61: Lines Tangent to Four Unit Balls

Problem 1: Minimum Weight Triangulation

Statement Can a mininimum weight triangulation of a planar point set be found in polynomial time? The *weight* of a triangulation is its total edge length.

Origin Perhaps E. L. Lloyd, 1977: [Llo77], cited in Garey and Johnson [GJ79].

Status/Conjectures Just solved by Wolfgang Mulzer and Günter Rote, January 2006! `http://arxiv.org/abs/cs.CG/0601002`. Entry to be updated later...

This problem is one of the few from Garey and Johnson [GJ79, p. 288] whose complexity status remains unknown.

Partial and Related Results The best approximation algorithms achieve a (large) constant times the optimal length [LK96]; good heuristics are known [DMM95]. If Steiner points are allowed, again constant-factor approximations are known [Epp94, CL98], but it is open to decide even if a minimum-weight Steiner triangulation exists (the minimum might be approached only in the limit).

Appearances [MO01]

Categories triangulations

Entry Revision History J. O'Rourke, 31 Jul. 2001; J. O'Rourke, 3 Jan. 2006.

References

[CL98] Siu-Wing Cheng and Kam-Hing Lee. Quadtree decomposition, Steiner triangulation, and ray shooting. In *ISAAC: 9th Internat. Sympos. Algorithms Computation*, pages 367–376, 1998.

[DMM95] Matthew T. Dickerson, Scott A. McElfresh, and Mark H. Montague. New algorithms and empirical findings on minimum weight triangulation heuristics. In *Proc. 11th Annu. ACM Sympos. Comput. Geom.*, pages 238–247, 1995.

[Epp94] D. Eppstein. Approximating the minimum weight Steiner triangulation. *Discrete Comput. Geom.*, 11:163–191, 1994.

[GJ79] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, New York, NY, 1979.

[Llo77] E. L. Lloyd. On triangulations of a set of points in the plane. In *Proc. 18th Annu. IEEE Sympos. Found. Comput. Sci.*, pages 228–240, 1977.

[LK96] Christos Levcopoulos and Drago Krznaric. Quasi-greedy triangulations approximating the minimum weight triangulation. In *Proc. 7th ACM-SIAM Sympos. Discrete Algorithms*, pages 392–401, 1996.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

Problem 2: Voronoi Diagram of Moving Points

Statement What is the maximum number of combinatorial changes possible in a Euclidean Voronoi diagram of a set of n points each moving along a line at unit speed in two dimensions?

Origin Unknown (to JOR). Perhaps Atallah?

Status/Conjectures Open. Conjectured to be nearly quadratic.

Partial and Related Results The best lower bound known is quadratic, and the best upper bound is cubic [SA95, p. 297]. If the speeds are allowed to differ, the upper bound remains essentially cubic [AGMR98]. The general belief is that the complexity should be close to quadratic; Chew [Che97] showed this to be the case if the underlying metric is L_1 (or L_∞).

Appearances [MO01]

Categories Voronoi diagrams; Delaunay triangulations

Entry Revision History J. O'Rourke, 1 Aug. 2001.

References

[AGMR98] G. Albers, Leonidas J. Guibas, Joseph S. B. Mitchell, and T. Roos. Voronoi diagrams of moving points. *Internat. J. Comput. Geom. Appl.*, 8:365–380, 1998.

[Che97] L. P. Chew. Near-quadratic bounds for the L_1 Voronoi diagram of moving points. *Comput. Geom. Theory Appl.*, 7:73–80, 1997.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[SA95] Micha Sharir and P. K. Agarwal. *Davenport-Schinzel Sequences and Their Geometric Applications*. Cambridge University Press, New York, 1995.

Problem 3: Voronoi Diagram of Lines in 3D

Statement What is the combinatorial complexity of the Voronoi diagram of a set of lines (or line segments) in three dimensions?

Origin Uncertain, pending investigation.

Status/Conjectures Open. Conjectured to be nearly quadradic.

Partial and Related Results There is a gap between a lower bound of $\Omega(n^2)$ and an upper bound that is essentially cubic [Sha94] for the Euclidean case (and yet is quadratic for polyhedral metrics [BSTY98]). A recent advance shows that the "level sets" of the Voronoi diagram of lines, given by the union of a set of cylinders, indeed has near-quadratic complexity [AS00b].

Related Open Problems This problem is closely related to Problem 2, because points moving in the plane with constant velocity yield straight-line trajectories in space-time.

Appearances [MO01]

Categories Voronoi diagrams

Entry Revision History J. O'Rourke, 2 Aug. 2001; 13 Dec. 2001.

References

[AS00b] Pankaj K. Agarwal and Micha Sharir. Pipes, cigars, and kreplach: The union of Minkowski sums in three dimensions. *Discrete Comput. Geom.*, 24(4):645–685, 2000.

[BSTY98] Jean-Daniel Boissonnat, Micha Sharir, Boaz Tagansky, and Mariette Yvinec. Voronoi diagrams in higher dimensions under certain polyhedral distance functions. *Discrete Comput. Geom.*, 19(4):473–484, 1998.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[Sha94] Micha Sharir. Almost tight upper bounds for lower envelopes in higher dimensions. *Discrete Comput. Geom.*, 12:327–345, 1994.

Problem 4: Union of Fat Objects in 3D

Statement What is the complexity of the union of "fat" objects in $\mathbb{R}^3$?

Origin Uncertain, pending investigation.

Status/Conjectures Open. Conjectured to be nearly quadratic.

Partial and Related Results The Minkowski sum of polyhedra of n vertices with the (Euclidean) unit ball has complexity $O(n^{2+\epsilon})$ [AS99], as does the union of n congruent cubes [PSS01]. It is widely believed the same should hold true for *fat* objects, those with a bounded ratio of circumradius to inradius, as it does in $\mathbb{R}^2$ [ES00].

Appearances [MO01]

Categories combinatorial geometry

Entry Revision History J. O'Rourke, 1 Aug. 2001; 1 Jan. 2003 (B. Aronov comment).

References

[AS99] Pankaj K. Agarwal and Micha Sharir. Pipes, cigars, and kreplach: The union of Minkowski sums in three dimensions. In *Proc. 15th Annu. ACM Sympos. Comput. Geom.*, pages 143–153, 1999.

[ES00] A. Efrat and Micha Sharir. On the complexity of the union of fat objects in the plane. *Discrete Comput. Geom.*, 23:171–189, 2000.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[PSS01] Janos Pach, Ido Safruit, and Micha Sharir. The union of congruent cubes in three dimensions. In *Proc. 17th Annu. ACM Sympos. Comput. Geom.*, pages 19–28, 2001.

Problem 5: Euclidean Minimum Spanning Tree

Statement Can the Euclidean *minimum spanning tree* (MST) of n points in $\mathbb{R}^d$ be computed in time close to the lower bound of $\Omega(n \log n)$ [GKFS96]?

Origin Uncertain, pending investigation.

Status/Conjectures Open.

Partial and Related Results Several algorithms have been developed for general graphs with arbitrary edge weights. Chazelle presented an $O(m\alpha(m,n) \log \alpha(m,n))$-time algorithm [Cha97], and then an $O(m\alpha(m,n))$-time algorithm [Cha00b], where $\alpha(m,n)$ is the functional inverse of Ackermann's function, and n and m are the number of vertices and edges respectively in the graph. Pettie and Ramachandran have since given an optimal algorithm for the graph setting [PR02], whose running time is an unknown function between $\Omega(m)$ and $O(m\alpha(m,n))$. In particular, when $m = \Omega(n \log n)$, $\alpha(m,n) = O(1)$ and these time bounds are all linear in the number of edges, m.

But in the geometric setting, the graph is complete, so a time bound linear in the number of edges, m, is quadratic in the number of points, n. And indeed the best upper bounds for the Euclidean MST approach quadratic for large d, e.g., [CK95].

Related Open Problems This problem is intimately related to the bichromatic closest pair problem [AESW91].

Appearances [MO01]

Categories minimum spanning tree; shortest paths

Entry Revision History J. O'Rourke, 2 Aug. 2001; E. Demaine, 7 July 2002.

References

[AESW91] Pankaj K. Agarwal, Herbert Edelsbrunner, O. Schwarzkopf, and Emo Welzl. Euclidean minimum spanning trees and bichromatic closest pairs. *Discrete Comput. Geom.*, 6(5):407–422, 1991.

[Cha97] Bernard Chazelle. A faster deterministic algorithm for minimum spanning trees. In *Proc. 38th Annu. IEEE Sympos. Found. Comput. Sci.*, page To appear, 1997.

[Cha00b] Bernard Chazelle. A minimum spanning tree algorithm with inverse-Ackermann type complexity *J. ACM*, 47(6):1028–1047, November 2000.

[CK95] P. B. Callahan and S. Rao Kosaraju. A decomposition of multidimensional point sets with applications to k-nearest-neighbors and n-body potential fields. *J. Assoc. Comput. Mach.*, 42:67–90, 1995.

[GKFS96] Dima Grigoriev, Marek Karpinski, Friedhelm Meyer auf der Heide, and Roman Smolensky. A lower bound for randomized algebraic decision trees. In *Proc. 28th Annu. ACM Sympos. Theory Comput.*, pages 612–619, 1996.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[PR02] Seth Pettie and Vijaya Ramachandran. An optimal minimum spanning tree algorithm. *J. ACM*, 49(1):16–34, January 2002.

Problem 6: Minimum Euclidean Matching in 2D

Statement What is the complexity of computing a minimum-cost Euclidean matching for $2n$ points in the plane? The *cost* of a matching is the total length of the edges in the matching.

Origin Uncertain, pending investigation.

Status/Conjectures Open.

Partial and Related Results An algorithm that achieves the minimum and runs in nearly $O(n^{2.5})$ time has long been available [Vai89]. This was improved to $O(n^{1.5} \log^5 n)$ in [Var98]. Recently Arora showed how to achieve a $(1+\epsilon)$-approximation in $n(\log n)^{O(1/\epsilon)}$ time [Aro98], and this has been improved to $O((n/\epsilon^3)\log^6 n)$ time [VA99].

A special case of considerable interest is bipartite matching, in which the points are red or blue and the matching connects points of different color. Here $O(n^{2+\epsilon})$ has been achieved [AES99], and a $(1+\epsilon)$-approximation can be found in $O((n/\epsilon)^{1.5} \log^5 n)$ time [VA99].

Appearances [MO01]

Categories shortest paths

Entry Revision History J. O'Rourke, 2 Aug. 2001; 30 Aug. 2001; 13 Dec. 01 (thanks to M. Sharir).

References

[Aro98] Sanjeev Arora. Polynomial time approximation schemes for Euclidean traveling salesman and other geometric problems. *J. Assoc. Comput. Mach.*, 45(5):753–782, 1998.

[AES99] P. K. Agarwal, A. Efrat, and Micha Sharir. Vertical decomposition of shallow levels in 3-dimensional arrangements and its applications. *SIAM J. Comput.*, 29:912–953, 1999.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[Var98] K. Varadarajan. A divide and conquer algorithm for min-cost perfect matching in the plane. In *Proc. 39th Annu. IEEE Sympos. Found. Comput. Sci.*, pages 320–329, 1998.

[Vai89] P. M. Vaidya. Geometry helps in matching. *SIAM J. Comput.*, 18:1201–1225, 1989.

[VA99] K. R. Varadarajan and Pankaj K. Agarwal. Approximation algorithms for bipartite and non-bipartite matching in the plane. In *Proc. 10th ACM-SIAM Sympos. Discrete Algorithms*, pages 805–814, 1999.

Problem 7: k-sets

Statement What is the maximum number of k-sets? (Equivalently, what is the maximum complexity of a k-level in an arrangement of hyperplanes?)

Origin Uncertain, pending investigation.

Status/Conjectures Open.

Partial and Related Results For a given set P of n points, $S \subset P$ is a *k-set* if $|S| = k$ and $S = P \cap H$ for some open halfspace H. Even for points in two dimensions the problem remains open: The maximum number of k-sets as a function of n and k is known to be $O(nk^{1/3})$ by a recent advance of Dey [Dey98], while the best lower bound is only slightly superlinear [Tot00].

Appearances [MO01]

Categories combinatorial geometry; point sets

Entry Revision History J. O'Rourke, 2 Aug. 2001.

References

[Dey98] T. K. Dey. Improved bounds on planar k-sets and related problems. *Discrete Comput. Geom.*, 19:373–382, 1998.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[Tot00] Géza Toth. Point sets with many k-sets. In *Proc. 16th Annu. ACM Sympos. Comput. Geom.*, pages 37–42, 2000.

Problem 8: Linear Programming: Strongly Polynomial?

Statement Is linear programming strongly polynomial?

Origin Nimrod Megiddo [Meg82][Meg83].

Status/Conjectures Open.

Partial and Related Results It is known to be weakly polynomial, exponential in the bit complexity of the input data [Kha80, Kar84]. Subexponential time is achievable via a randomized algorithm [MSW96]. In any fixed dimension, linear programming can be solved in strongly polynomial linear time (linear in the input size), established in dimensions 2 and 3 in [Dye84] and for all dimensions in [Meg84].

Appearances [MO01]

Categories linear programming

Entry Revision History J. O'Rourke, 2 Aug 2001, 16 Jul 2007.

References

[Dye84] M. E. Dyer. Linear time algorithms for two- and three-variable linear programs. *SIAM J. Comput.*, 13:31–45, 1984.

[Kar84] N. Karmarkar. A new polynomial-time algorithm for linear programming. *Combinatorica*, 4:373–395, 1984.

[Kha80] L. G. Khachiyan. Polynomial algorithm in linear programming. *U.S.S.R. Comput. Math. and Math. Phys.*, 20:53–72, 1980.

[Meg82] N. Megiddo. Is binary encoding appropriate for the problem-language relationship? *Theoret. Comput. Sci.*, 19:337–341, 1982.

[Meg84] N. Megiddo. Linear programming in linear time when the dimension is fixed. *J. Assoc. Comput. Mach.*, 31:114–127, 1984.

[Meg83] N. Megiddo. Towards a genuinely polynomial algorithm for linear programming. *SIAM J. Comput.*, 12:347–353, 1983.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[MSW96] J. Matoušek, Micha Sharir, and Emo Welzl. A subexponential bound for linear programming. *Algorithmica*, 16:498–516, 1996.

Problem 9: Edge-Unfolding Convex Polyhedra

Statement Can every convex polyhedron be cut along its edges and unfolded flat to a single, nonoverlapping, simple polygon?

Origin First stated in [She75], but in spirit at least goes back to Albrecht Dürer [Dür25].

Status/Conjectures Open. It seems to be a widespead hunch that the answer is YES.

Partial and Related Results The answer is known to be NO for nonconvex polyhedra even with triangular faces [BDE+03], but has been long open for convex polyhedra [She75, O'R00].

Related Open Problems Problem 42: Vertex-Unfolding Polyhedra.
Problem 43: General Unfolding of Nonconvex Polyhedra.
Problem 64: Edge-Unfolding Polyhedra Built from Cubes.

Appearances [She75, O'R00, MO01]

Categories folding and unfolding; polyhedra

Entry Revision History J. O'Rourke, 2 Aug. 2001.

References

[BDE+03] Marshall Bern, Erik D. Demaine, David Eppstein, Eric Kuo, Andrea Mantler, and Jack Snoeyink. Unfoldable polyhedra with convex faces. *Comput. Geom. Theory Appl.*, 24(2):51–62, 2003.

[Dür25] Albrecht Dürer. *The painter's manual: A manual of measurement of lines, areas, and solids by means of compass and ruler assembled by Albrecht Dürer for the use of all lovers of art with appropriate illustrations arranged to be printed in the year MDXXV.*

New York: Abaris Books, 1977, 1525. English translation by Walter L. Strauss of 'Unterweysung der Messung mit dem Zirkel un Richtscheyt in Linien Ebnen uhnd Gantzen Corporen'.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[O'R00] Joseph O'Rourke. Folding and unfolding in computational geometry. In *Proc. 1998 Japan Conf. Discrete Comput. Geom.*, volume 1763 of *Lecture Notes Comput. Sci.*, pages 258–266. Springer-Verlag, 2000.

[She75] Geoffrey C. Shephard. Convex polytopes with convex nets. *Math. Proc. Camb. Phil. Soc.*, 78:389–403, 1975.

Problem 10: Simple Linear-Time Polygon Triangulation

Statement Is there a deterministic, linear-time polygon triangulation algorithm significantly simpler than that of Chazelle [Cha91]?

Origin Implicit since Chazelle's 1990 linear-time algorithm.

Status/Conjectures Open.

Partial and Related Results Simple randomized algorithms that are close to linear-time are known [Sei91], and a recent randomized linear-time algorithm [AGR00] avoids much of the intricacies of Chazelle's algorithm.

Related Open Problems Relatedly, is there a simple linear-time algorithm for computing a shortest path in a simple polygon, without first applying a more complicated triangulation algorithm?

Appearances [MO01]

Categories triangulations

Entry Revision History J. O'Rourke, 2 Aug. 2001; 28 Aug. 2001.

References

[AGR00] Nancy M. Amato, Michael T. Goodrich, and Edgar A. Ramos. Linear-time triangulation of a simple polygon made easier via randomization. In *Proc. 16th Annu. ACM Sympos. Comput. Geom.*, pages 201–212, 2000.

[Cha91] Bernard Chazelle. Triangulating a simple polygon in linear time. *Discrete Comput. Geom.*, 6(5):485–524, 1991.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[Sei91] R. Seidel. A simple and fast incremental randomized algorithm for computing trapezoidal decompositions and for triangulating polygons. *Comput. Geom. Theory Appl.*, 1(1):51–64, 1991.

Problem 11: 3SUM Hard Problems

Statement Can the class of *3SUM hard* problems be solved in subquadratic time? These problems can be reduced from the problem of determining whether, given three sets of integers, A, B, and C with total size n, there are elements $a \in A$, $b \in B$, and $c \in C$ such that $a + b = c$.

Origin [GO95].

Status/Conjectures Open.

Motivation Many fundamental geometric problems fall in this class, e.g., computing the area of the union of n triangles.

Partial and Related Results $\Omega(n^2)$ lower bounds are known for 3SUM and a few 3SUM-hard problems in restricted decision tree models of computation [ES95, Eri99a, Eri99b].

3SUM and its obvious generalizations (4SUM, 5SUM, etc.) are examples of *linear satisfiability* problems. A generic linear satisfiability problem asks, given an array of n integers, do any r of them satisfy the equation

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_r x_r = \alpha_0$$

where $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_r$ are fixed constants. Erickson [Eri99a] proved an $\Omega(n^{\lceil r/2 \rceil})$ lower bound for any problem of this type, in the restricted linear decision tree model. This lower bound is tight except for a logarithmic factor when r is even.

Baran et al. [BDP05] show that subquadratic algorithms for 3SUM are possible in common models of computation that allow more direct manipulation of the numbers instead of just real arithmetic, such as the word RAM. The improvement they obtain is roughly quadratic in the parallelism offered by the model; for example, with $\lg n$-bit words, they obtain an $O(n^2 \left(\frac{\lg \lg n}{\lg n}\right)^2)$-time algorithm. With this word size, the 3SUM problem becomes whether any improvement beyond polylogarithmic factors (or indeed, beyond $\Theta(lg^2 n)$) is possible.

Appearances [MO01]

Categories lower bounds

Entry Revision History J. O'Rourke, 2 Aug. 2001; Jeff Erickson, 20 June 2002; E. Demaine, 7 July 2005.

References

[BDP05] Ilya Baran, Erik D. Demaine, and Mihai Pătraşcu. Subquadratic algorithms for 3SUM. In *Proceedings of the 9th Workshop on Algorithms and Data Structures*, Waterloo, Ontario, Canada, August 2005. To appear.

[Eri99a] Jeff Erickson. Lower bounds for linear satisfiability problems. *Chicago J. Theoret. Comput. Sci.*, 1999(8), 1999.

[Eri99b] Jeff Erickson. New lower bounds for convex hull problems in odd dimensions. *SIAM J. Comput.*, 28:1198–1214, 1999.

[ES95] Jeff Erickson and R. Seidel. Better lower bounds on detecting affine and spherical degeneracies. *Discrete Comput. Geom.*, 13:41–57, 1995.

[GO95] A. Gajentaan and M. H. Overmars. On a class of $O(n^2)$ problems in computational geometry. *Comput. Geom. Theory Appl.*, 5:165–185, 1995.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

Problem 12: Dynamic Planar Convex Hull

Statement Can a planar convex hull be maintained to support both dynamic updates and queries in logarithmic time? More precisely, is there a data structure supporting insertions and deletions of points and supporting various queries about the convex hull of the current set of n points, all in $O(\log n)$ time per operation? An *extreme-point query* asks to find the vertex of the convex hull that is extreme in a given direction. A *tangent query* asks to determine whether a given point is interior to the convex hull, and if not, to find the two tangent lines of the convex hull that passes through the given point. A *gift-wrapping query* asks to find the two vertices of the convex hull adjacent to a given vertex of the convex hull. A *line-stabbing query* asks to find the two edges of the convex hull (if any) that intersect a given line. (Note that two extreme-point queries suffice to determine whether a line intersects the convex hull, while a line-stabbing query determines where exactly the line intersects the convex hull if it does.)

Origin Uncertain, pending investigation.

Status/Conjectures Solved (in a certain sense) by Gerth Brodal and Riko Jacob in a FOCS 2002 paper [BJ02]. See also Jacob's PhD thesis [Jac02] for further details. Their data structure supports insertions and deletions in $O(\log n)$ amortized time and supports extreme-point, tangent, and gift-wrapping queries in $O(\log n)$ worst-case query bounds. It remains open whether a logarithmic bound can be achieved in the worst case, and whether logarithmic bounds can be achieved (amortized or worst case) for line-stabbing queries.

Partial and Related Results For 17 years, the authority on this problem was Overmars and van Leeuwen's paper [OvL81] which describes a data structure supporting insertions and deletions in $O(\log^2 n)$ worst-case time and all types of queries described above in $O(\log n)$ worst-case time. Various structures achieve faster update times when either insertions or deletions are not supported [Pre79, HS92]. But the $O(\log^2 n)$ barrier remained until Chan's FOCS 1999 paper [Cha99], which improved the insertion and deletion time to $O(\log^{1+\epsilon} n)$ amortized for any $\epsilon > 0$. The update time was further improved to $O(\log n \log\log n)$ amortized by Brodal and Jacob [BJ00] until the problem was finally solved in optimal $O(\log n)$ amortized time by the same authors [BJ02, Jac02]. Both the Chan and the Brodal and Jacob structures support extreme-point, tangent, and gift-wrapping queries.

Related Open Problems Problem 63.

Appearances [MO01]

Categories convex hulls; data structures

Entry Revision History J. O'Rourke, 2 Aug. 2001; E. Demaine, 25 Nov. 2002; 22 Aug. 2005; 24 Jan. 2006.

References

[BJ02] Gerth Stølting Brodal and Riko Jacob. Dynamic planar convex hull. In *Proceedings of the 43rd Annual IEEE Symposium on Foundations of Computer Science*, November 2002.

[BJ00] Gerth Stølting Brodal and Riko Jacob. Dynamic planar convex hull with optimal query time and $o(\log n \cdot \log\log n)$ update time. In *Proc. 7th Scand. Workshop Algorithm Theory*, volume 1851 of *Lecture Notes Comput. Sci.*, pages 57–70. Springer-Verlag, 2000.

[Cha99] Timothy M. Chan. Dynamic planar convex hull operations in near-logarithmic amortized time. In *Proc. 40th Annu. IEEE Sympos. Found. Comput. Sci.*, pages 92–99, 1999.

[HS92] J. Hershberger and S. Suri. Applications of a semi-dynamic convex hull algorithm. *BIT*, 32:249–267, 1992.

[Jac02] Riko Jacob. *Dynamic Planar Convex Hull.* PhD thesis, Department of Computer Science, University of Aarhus, Aarhus, Denmark, February 2002.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[OvL81] M. H. Overmars and J. van Leeuwen. Maintenance of configurations in the plane. *J. Comput. Syst. Sci.*, 23:166–204, 1981.

[Pre79] F. P. Preparata. An optimal real-time algorithm for planar convex hulls. *Commun. ACM*, 22:402–405, 1979.

Problem 13: Point Location in 3D Subdivision

Statement Is there an $O(n)$-space data structure that supports $O(\log n)$-time point-location queries in a three-dimensional subdivision of n faces?

Origin Uncertain, pending investigation.

Status/Conjectures Open.

Partial and Related Results Currently $O(n \log n)$ space and $O(\log^2 n)$ queries are achievable [Sno97].

Appearances [MO01]

Categories data structures

Entry Revision History J. O'Rourke, 2 Aug. 2001.

References

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[Sno97] Jack Snoeyink. Point location. In Jacob E. Goodman and Joseph O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, chapter 30, pages 559–574. CRC Press LLC, Boca Raton, FL, 1997.

Problem 14: Binary Space Partition Size

Statement Is it possible to construct a *binary space partition* (BSP) for n disjoint line segments in the plane of size less than $\Theta(n \log n)$?

Origin Paterson and Yao [PY90].

Status/Conjectures Open.

Partial and Related Results The upper bound of $O(n \log n)$ was established by Paterson and Yao [PY90]. Recently Tóth [T01́] improved the trivial $\Omega(n)$ lower bound to $\Omega(n \log n / \log \log n)$. Can the remaining gap be closed?

Appearances [MO01]

Categories data structures; combinatorial geometry

Entry Revision History J. O'Rourke, 2 Aug. 2001.

References

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[PY90] M. S. Paterson and F. F. Yao. Efficient binary space partitions for hidden-surface removal and solid modeling. *Discrete Comput. Geom.*, 5:485–503, 1990.

[T01́] Csaba David Tóth. A note on binary plane partitions. In *Proc. 17th Annu. ACM Sympos. Comput. Geom.*, pages 151–156, 2001.

Problem 15: Output-sensitive Convex Hull in $\mathbb{R}^d$

Statement What is the best output-sensitive convex hull algorithm for n points in $\mathbb{R}^d$?

Origin Uncertain, pending investigation.

Status/Conjectures Open.

Partial and Related Results The lower bound is $\Omega(n \log f + f)$ for f facets (the output size). The best upper bound to date is $O(n \log f + (nf)^{1-\delta} \log^{O(1)} n)$ with $\delta = 1/(\lfloor d/2 \rfloor + 1)$ [Cha96], which is optimal for sufficiently small f.

Appearances [MO01]

Categories convex hulls

Entry Revision History J. O'Rourke, 2 Aug. 2001.

References

[Cha96] Timothy M. Chan. Output-sensitive results on convex hulls, extreme points, and related problems. *Discrete Comput. Geom.*, 16:369–387, 1996.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

Problem 16: Simple Polygonalizations

Statement Can the number of simple polygonalizations of a set of n points in the plane be computed in polynomial time? A *simple polygonalization* is a simple polygon whose vertices are the points.

Origin Uncertain, pending investigation.

Status/Conjectures Open.

Partial and Related Results Certain special cases are known (e.g., for computing the number of monotone simple polygonalizations [ZSSM96]), but the general problem remains open. The problem is closely related to that of generating a "random" instance of a simple polygon on a given set of vertices, with each instance being generated with probability $1/k$, where k is the total number of simple polygonalizations. Heuristic methods are known and implemented [AH96].

See [CHUZ01] and [HMO$^+$09] for related topics and references to relevant papers.

Appearances [MO01]

Categories polygons; point sets

Entry Revision History J. O'Rourke, 2 Aug. 2001; 1 Jan. 2003.

References

[AH96] Thomas Auer and Martin Held. Heuristics for the generation of random polygons. In *Proc. 8th Canad. Conf. Comput. Geom.*, pages 38–43, 1996.

[CHUZ01] Jurek Czyzowicz, Ferran Hurtado, Jorge Urrutia, and Najib Zaguia. On polygons enclosing point sets. *Geombinatorics*, XI:21–28, 2001.

[HMO+09] Ferran Hurtado, C. Merino, D. Oliveros, T. Sakai, Jorge Urrutia, and I. Ventura. On polygons enclosing point sets II. *Graphs Combinatorics*, 25(3):327–329, 2009.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[ZSSM96] Chong Zhu, Gopalakrishnan Sundaram, Jack Snoeyink, and Joseph S. B. Mitchell. Generating random polygons with given vertices. *Comput. Geom. Theory Appl.*, 6:277–290, 1996.

Problem 17: Visibility Graph Recognition

Statement Given a visibility graph G and a Hamiltonian circuit C, determine in polynomial time whether there is a simple polygon whose vertex visibility graph is G, and whose boundary corresponds to C.

Origin ElGindy(?)

Status/Conjectures Open.

Partial and Related Results The problem is not even known to be in NP [O'R93], although it is for "pseudo-polygon" visibility graphs [OS97].

Appearances [MO01]

Categories visibility

Entry Revision History J. O'Rourke, 2 Aug. 2001.

References

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[O'R93] Joseph O'Rourke. Computational geometry column 18. *Internat. J. Comput. Geom. Appl.*, 3(1):107–113, 1993. Also in *SIGACT News* 24:1 (1993), 20–25.

[OS97] Joseph O'Rourke and Ileana Streinu. Vertex-edge pseudo-visibility graphs: Characterization and recognition. In *Proc. 13th Annu. ACM Sympos. Comput. Geom.*, pages 119–128, 1997.

Problem 18: Pushing Disks Together

Statement When a collection of disks are pushed closer together, so that no distance between two center points increases, can the area of their union increase?

Origin Kneser (1955) and Poulsen (1954).

Status/Conjectures Solved by K. Bezdek and R. Connelly. See their web page[2]. (Update as of 3 Aug. 2000.)

Partial and Related Results Previously only settled in the continuous-motion case [BS98], for both this and the corresponding question for intersection area decrease [Cap96]. But now both solved; see above.

Appearances [MO01]

Categories combinatorial geometry

Entry Revision History J. O'Rourke, 2 Aug. 2001; 3 Aug. 2003.

References

[BS98] Marshall Bern and Amit Sahai. Pushing disks together – The continuous-motion case. *Discrete Comput. Geom.*, 20:499–514, 1998.

[Cap96] V. Capoyleas. On the area of the intersection of disks in the plane. *Comput. Geom. Theory Appl.*, 6:393–396, 1996.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

Problem 19: Vertical Decompositions in $\mathbb{R}^d$

Statement What is the complexity of the *vertical decomposition* of n surfaces in $\mathbb{R}^d$, $d \geq 5$?

Origin Uncertain, pending investigation.

Status/Conjectures Open.

Partial and Related Results The lower bound of $\Omega(n^d)$ was nearly achieved up to $d = 3$ [AS00a, p. 271], but a gap remained for $d \geq 4$. A recent result [Kol01] covers $d = 4$ and achieves $O(n^{2d-4+\epsilon})$ for general d, leaving a gap for $d \geq 5$.

[2] http://www.math.cornell.edu/ connelly/kneser.html

Appearances [MO01]

Categories combinatorial geometry

Entry Revision History J. O'Rourke, 2 Aug. 2001.

References

[AS00a] Pankaj K. Agarwal and Micha Sharir. Davenport-Schinzel sequences and their geometric applications. In Jörg-Rüdiger Sack and Jorge Urrutia, editors, *Handbook of Computational Geometry*, pages 1–47. Elsevier Publishers B.V. North-Holland, Amsterdam, 2000.

[Kol01] Vladlen Koltun. Almost tight upper bounds for vertical decompositions in four dimensions. In *Proc. 42nd Annu. IEEE Sympos. Found. Comput. Sci.*, 2001.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

Problem 20: Minimum Stabbing Spanning Tree

Statement What is the complexity of computing a spanning tree of a planar point set P having minimum stabbing number? The *stabbing number* of a tree T is the maximum number of edges of T intersected by a line.

Origin Uncertain, pending investigation.

Status/Conjectures Solved, October 2003: the problem is NP-complete. Significant advance on approximability in 2009.

Partial and Related Results Fekete, Lübbecke, and Meijer [FLM04] proved strong NP-completeness of minimizing the stabbing number or axis-parallel stabbing number or crossing number or axis-parallel crossing number in a perfect matching or spanning tree. They also establish inapproximability by less than a 6/5 factor of minimizing the axis-parallel stabbing number in a perfect matching. They also prove strong NP-completeness of minimizing the axis-parallel crossing number in a triangulation.

The complexity of minimizing the stabbing number or crossing number in a triangulation remains open. Furthermore, it remains open whether any of these problems have constant-factor approximations. See [FLM04] for some ideas.

In the worst case, any set of n points in the plane has a spanning tree of stabbing number $O(\sqrt{n})$ [Aga92, Cha88, Wel93] and this bound is tight. An $O(\sqrt{n})$-approximation follows from this result.

There has been an advance on approximations [HP09]: Har-Peled designed an algorithm that computes a spanning tree of n points P in $\mathbb{R}^d$ whose crossing number is $O(\min(t \log n, n^{1-1/d}))$, where t the minimum crossing number of any spanning tree of P.

Appearances [MO01]

Categories stabbing

Entry Revision History J. O'Rourke, 2 Aug. 2001; E. Demaine, 16 Jan. 2004; J. O'Rourke, 26 Aug 2009.

References

[Aga92] Pankaj K. Agarwal. Ray shooting and other applications of spanning trees with low stabbing number. *SIAM J. Comput.*, 21:540–570, 1992.

[Cha88] Bernard Chazelle. Tight bounds on the stabbing number of spanning trees in Euclidean space. Report CS-TR-155-88, Dept. Comput. Sci., Princeton Univ., Princeton, NJ, 1988.

[FLM04] Sándor P. Fekete, Marco E. Lübbecke, and Henk Meijer. Minimizing the stabbing number of matchings, trees, and triangulations. In *Proc. 15th Ann. ACM-SIAM Sympos. Discrete Algorithms*, pages 430–439, January 2004.

[HP09] Sariel Har-Peled. Approximating spanning trees with low crossing numbers. `http://valis.cs.uiuc.edu/~sariel/papers/09/crossing/`, June 2009.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[Wel93] Emo Welzl. Geometric graphs with small stabbing numbers: Combinatorics and applications. In *Proc. 9th Internat. Conf. Fund. Comput. Theory*, Lecture Notes Comput. Sci., page ?? Springer-Verlag, 1993.

Problem 21: Shortest Paths among Obstacles in 2D

Statement Can shortest paths among h obstacles in the plane, with a total of n vertices, be found in optimal $O(n + h \log h)$ time using $O(n)$ space?

Origin Uncertain, pending investigation.

Status/Conjectures Open.

Partial and Related Results The only algorithm that is linear in n in time and space is quadratic in h [KMM97]; $O(n \log n)$ time, using $O(n \log n)$ space, is known [HS99]. In three dimensions, the Euclidean shortest path problem among general obstacles is NP-hard, but its complexity remains open for some special cases, such as when the obstacles are disjoint unit spheres or axis-aligned boxes; see [Mit00] for a survey.

Appearances [MO01]

Categories shortest paths

Entry Revision History J. O'Rourke, 2 Aug. 2001.

References

[HS99] John Hershberger and Subhash Suri. An optimal algorithm for Euclidean shortest paths in the plane. *SIAM J. Comput.*, 28(6):2215–2256, 1999.

[KMM97] S. Kapoor, S. N. Maheshwari, and Joseph S. B. Mitchell. An efficient algorithm for Euclidean shortest paths among polygonal obstacles in the plane. *Discrete Comput. Geom.*, 18:377–383, 1997.

[Mit00] Joseph S. B. Mitchell. Geometric shortest paths and network optimization. In Jörg-Rüdiger Sack and Jorge Urrutia, editors, *Handbook of Computational Geometry*, pages 633–701. Elsevier Publishers B.V. North-Holland, Amsterdam, 2000.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

Problem 22: Minimum-Link Path in 2D

Statement Can a minimum-link path among polygonal obstacles be found in subquadratic time?

Origin Mitchell [?].

Status/Conjectures Open.

Partial and Related Results The best algorithm known requires essentially quadratic time in the worst case [MRW92].

Related Open Problems What is the complexity of computing minimum-link paths in three dimensions?

Appearances [MO01]

Categories shortest paths

Entry Revision History J. O'Rourke, 2 Aug. 2001.

References

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[MRW92] Joseph S. B. Mitchell, Günter Rote, and G. Woeginger. Minimum-link paths among obstacles in the plane. *Algorithmica*, 8:431–459, 1992.

Problem 23: Vertex π-Floodlights

Statement How many π-floodlights are always sufficient to illuminate any polygon of n vertices, with at most one floodlight placed at each vertex? An *α-floodlight* is a light of aperture α. (We consider here "inward-facing" floodlights, whose defining halfspace lies inside the polygon, locally in the neighborhood of the vertex. Other models of the problem allow general orientations of floodlights or restricted orientations (e.g., "edge-aligned").)

Origin Jorge Urrutia, perhaps first published in [Urr00].

Status/Conjectures Open. Now known that the fraction of n that always suffices lies between 5/8 and 2/3.

Partial and Related Results It was established in [ECOUX95] that for any $\alpha < \pi$, there is a polygon that cannot be illuminated with an α-floodlight at each vertex. When $\alpha = \pi$, $n-2$ lights (trivially) suffice. So it is of interest (as noted in [Urr00]) to learn whether a fraction of n lights suffice for π-floodlights. A (very) special case is that $\lceil n/2 \rceil - 1$ is a tight bound for "monotone mountains" [O'R97]. Tóth established [Tót01] that (roughly) $(3/4)n$ suffice, in the case of general orientation floodlights (not necessarily inward-facing). A lower bound of Santos, that $\lfloor 3(n-1)/5 \rfloor$ inward-facing floodlights are necessary (or $\lfloor 2(n-2)/5 \rfloor$ generally oriented floodlights), stood for several years, but just recently (Jan. 2002) Urrutia constructed examples, based on stitching together copies of Fig. 1, that show that $5(k+1)/(8k+9)$ (inward-facing) floodlights are necessary for each k, thus improving the lower bound factor from 0.6 to 0.625. Also, Speckmann and Tóth [ST01b] showed that $\lfloor n/2 \rfloor$ vertex π-floodlights suffice for general orientations, while $\lfloor (2n - c)/3 \rfloor$ suffice for inward-facing, edge-aligned orientations, where c is the number of convex vertices. In particular, this reduced the upper-bound fraction below 1.

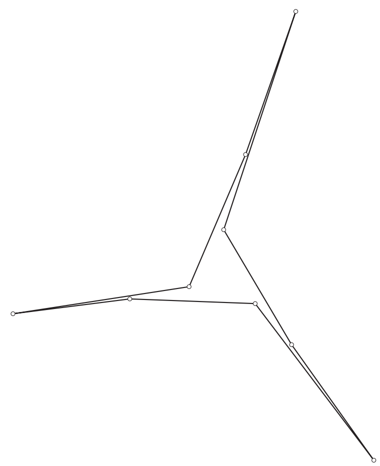

Figure 1: A polygon of 9 vertices that needs 5 vertex π-lights.

Appearances [MO01]

Categories art galleries; visibility

Entry Revision History J. Mitchell, 24 Jan 2001; J. O'Rourke, 2 Aug. 2001; 29 Aug. 2001; 23 Jan. 2002; 30 Sep. 2007.

References

[ECOUX95] V. Estivill-Castro, Joseph O'Rourke, J. Urrutia, and D. Xu. Illumination of polygons with vertex floodlights. *Inform. Process. Lett.*, 56:9–13, 1995.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[O'R97] Joseph O'Rourke. Vertex π-lights for monotone mountains. In *Proc. 9th Canad. Conf. Comput. Geom.*, pages 1–5, 1997.

[ST01b] Bettina Speckmann and Csaba D. Tóth. Vertex π-guards in simple polygons. Technical report, ETH Zürich, December 2001.

[Tót01] Csaba D. Tóth. Illuminating polygons with vertex π-floodlights. In V. N. Alexandrov et al., editors, *Proc. Int. Conf. on Comput. Sci.*, volume 2073 of *Lecture Notes Comput. Sci.*, pages 772–781. Springer-Verlag, 2001.

[Urr00] Jorge Urrutia. Art gallery and illumination problems. In Jörg-Rüdiger Sack and Jorge Urrutia, editors, *Handbook of Computational Geometry*, pages 973–1027. North-Holland, 2000.

Problem 24: Polygonal Curve Simplification

Statement Can an n-vertex polygonal curve be simplified in time nearly linear in n?

Origin Uncertain, pending investigation.

Status/Conjectures Open.

Partial and Related Results The goal is to compute a simplification that uses the fewest vertices of the original curve while approximating it according to some prescribed error criterion. Quadratic-time algorithms have been known for some time [CC96] and a recent algorithm achieves time $O(n^{4/3+\epsilon})$ for a certain error criterion [AV00]. In practice, the Douglas-Peucker algorithm is often used as a heuristic; it can be implemented to run in time $O(n \log n)$ [HS94].

Appearances [MO01]

Categories simplification

Entry Revision History J. O'Rourke, 2 Aug. 2001.

References

[AV00] P. K. Agarwal and Kasturi R. Varadarajan. Efficient algorithms for approximating polygonal chains. *Discrete Comput. Geom.*, 23:273–291, 2000.

[CC96] W. S. Chan and F. Chin. Approximation of polygonal curves with minimum number of line segments or minimum error. *Internat. J. Comput. Geom. Appl.*, 6:59–77, 1996.

[HS94] J. Hershberger and Jack Snoeyink. An $O(n \log n)$ implementation of the Douglas-Peucker algorithm for line simplification. In *Proc. 10th Annu. ACM Sympos. Comput. Geom.*, pages 383–384, 1994.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

Problem 25: Polyhedral Surface Approximation

Statement How efficiently can one compute a polyhedral surface that is an ϵ-approximation of a given triangulated surface in $\mathbb{R}^3$?

Origin Mitchell [?]

Status/Conjectures Open.

Partial and Related Results It is NP-hard to obtain the minimum-facet surface separating two nested convex polytopes [DG97], but polynomial-time approximation algorithms are known ([BG95, MS95, AS98]) for this case, and for separating two terrain surfaces, achieving factors within $O(1)$ or $O(\log n)$ of optimal. However, no polynomial-time approximation results are known for general surfaces.

Appearances [MO01]

Categories simplification

Entry Revision History J. O'Rourke, 2 Aug. 2001.

References

[AS98] P. K. Agarwal and S. Suri. Surface approximation and geometric partitions. *SIAM J. Comput.*, 27:1016–1035, 1998.

[BG95] H. Brönnimann and M. T. Goodrich. Almost optimal set covers in finite VC-dimension. *Discrete Comput. Geom.*, 14:263–279, 1995.

[DG97] G. Das and M. T. Goodrich. On the complexity of optimization problems for 3-dimensional convex polyhedra and decision trees. *Comput. Geom. Theory Appl.*, 8:123–137, 1997.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[MS95] Joseph S. B. Mitchell and Subhash Suri. Separation and approximation of polyhedral objects. *Comput. Geom. Theory Appl.*, 5:95–114, 1995.

Problem 26: Surface Reconstruction

Statement Given a sufficiently dense sample of points on a surface (technically, an ϵ-sample), reconstruct a surface homeomorphic to the original.

Origin Amenta and Bern [?]

Status/Conjectures Open.

Partial and Related Results This has recently been accomplished for smooth surfaces [ACDL00], but remains open for surfaces with sharp edges and corners.

Appearances [MO01]

Categories reconstruction; point sets

Entry Revision History J. O'Rourke, 2 Aug. 2001.

References

[ACDL00] N. Amenta, S. Choi, T. K. Dey, and N. Leekha. A simple algorithm for homeomorphic surface reconstruction. In *Proc. 16th Annu. ACM Sympos. Comput. Geom.*, pages 213–222, 2000.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

Problem 27: Hexahedral Meshing

Statement Can the interior of every simply connected polyhedron whose surface is meshed by an even number of quadrilaterals be partitioned into a hexahedral mesh compatible with the surface meshing? [BEA+99]

Origin Uncertain. Scott Mitchell in [Mit96]?

Status/Conjectures Partially closed, Fall 2006.

Partial and Related Results It was known that a topological hexahedral mesh exists [Mit96, Epp96], with, in general, curved boundaries, but despite the availability of software that extends quadrilateral surface meshes to hexahedral volume meshes, it is not known if a "geometric" hexahedral mesh can be achieved, with all cells having planar faces.

A new result [CS06] settles the practical aspects of the problem, but leaves one question unresolved. This paper provides an explicit algorithm that extends a quadrilateral surface mesh to a hexahedral mesh, where all the hexahedra have straight segment edges. In a sense, these hexahedra are intermediate between the topological and geometric meshes mentioned above. The faces are not necessarily planar, but this is not a crucial aspect in applications, such a fluid dynamics simulations.

The question of whether a hexahedral mesh with planar faces exists remains open.

Related Open Problems See [BE01] for extension of the flipping operation described in Problem 28 to quadrilateral and hexahedral meshes.

Appearances [MO01]

Categories meshing

Entry Revision History J. O'Rourke, 2 Aug. 2001; 18 Feb. 2002 (thanks to D. Eppstein); J. O'Rourke, 24 Oct. 2006.

References

[BEA$^+$99] Marshall Bern, D. Eppstein, P. K. Agarwal, N. Amenta, P. Chew, T. Dey, D. P. Dobkin, Herbert Edelsbrunner, C. Grimm, L. J. Guibas, J. Harer, J. Hass, A. Hicks, C. K. Johnson, G. Lerman, D. Letscher, P. Plassmann, E. Sedgwick, Jack Snoeyink, J. Weeks, C. Yap, and D. Zorin. Emerging challenges in computational topology, 1999. Report on an NSF Workshop Computational Topology, June 11-12, Miami Beach, FL.

[BE01] Marshall Wayne Bern and David Eppstein. Flipping cubical meshes. In *Proc. 10th Int. Meshing Roundtable*, pages 19–29, October 2001. ACM Computing Research Repository, cs.CG/0108020.

[CS06] Carlos D. Carbonera and Jason F. Shepherd. A constructive approach to constrained hexahedral mesh generation. In Phillippe P. Pebay, editor, *Proc. 15th Internat. Meshing Roundtable*. Springer, 2006.

[Epp96] David Eppstein. Linear complexity hexahedral mesh generation. In *Proc. 12th Annu. ACM Sympos. Comput. Geom.*, pages 58–67, 1996.

[Mit96] Scott A. Mitchell. A characterization of the quadrilateral meshes of a surface which admit a compatible hexahedral mesh of the enclosed volume. In *Proc. 13th Sympos. Theoret. Aspects Comput. Sci.*, volume 1046 of *Lecture Notes Comput. Sci.*, pages 465–476. Springer-Verlag, 1996.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

Problem 28: Flip Graph Connectivity in 3D

Statement Is the *flip graph* connected for general-position points in $\mathbb{R}^3$? Given a set of n points in $\mathbb{R}^3$, the flip graph has a node for each tetrahedralization of the set. Two nodes are connected by an arc if there is a 2-to-3 or 3-to-2 "bistellar flip" of tetrahedra between the two simplicial complexes. In the plane, the flips correspond to convex quadrilateral diagonal switches; in $\mathbb{R}^3$, a 5-vertex convex polyhedron is "flipped" between two of its tetrahedralizations.

Origin [EPW90, Joe91]

Status/Conjectures Open.

Partial and Related Results In $\mathbb{R}^2$ the flip graph is connected, providing a basis for algorithms to iterate toward the Delaunay triangulation. A decade ago, several [EPW90, Joe91] asked whether the same holds in $\mathbb{R}^3$ (when no four points are coplanar), but the question remains unresolved. It is not even known whether the flip graph might contain an isolated node. Settled in the negative for points in $\mathbb{R}^5$ by Santos [San00], by constructing polytopes with a space of triangulations not connected via bistellar flips. Settled in the negative for topological tetrahedralizations in 3D, but the constructed tetrahedralization cannot be realized geometrically [DFM04].

Settled in the positive for flip graphs of *regular triangulations* in any dimension in [PL07], based on earlier work of Gelfand, Kapranov and Zelevinsky. The result in [PL07] connects by flips that neither remove nor add vertices (i.e., 2-to-3 or 3-to-2 flips in 3D), whereas the earlier work by Gelfand et al. permits all flips (e.g., 1-to-4 and 4-to-1 flips in 3D).

Related Open Problems Problem 27

Appearances [MO01]

Categories triangulations; Delaunay triangulations

Entry Revision History J. O'Rourke, 2 Aug. 2001; 7 Dec. 2001 (thanks to F. Santos); E. Demaine, 10 Dec. 2001; J. O'Rourke, 18 Feb. 2002 (thanks to D. Eppstein); E. Demaine, 2 Aug. 2004 (thanks to M. Murphy); J. O'Rourke, 20 Aug 2006. J. O'Rourke, 22 Dec 2008 (thanks to L. Pournin).

References

[DFM04] Randall Dougherty, Vance Faber, and Michael Murphy. Unflippable tetrahedral complexes. *Discrete Comput. Geom.*, 32:309–315, 2004.

[EPW90] Herbert Edelsbrunner, F. P. Preparata, and D. B. West. Tetrahedrizing point sets in three dimensions. *J. Symbolic Comput.*, 10(3–4):335–347, 1990.

[Joe91] B. Joe. Construction of three-dimensional Delaunay triangulations using local transformations. *Comput. Aided Geom. Design*, 8(2):123–142, May 1991.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[PL07] Lionel Pournin and Thomas M. Liebling. Constrained paths in the flip-graph of regular triangulations. *Comput. Geom. Theory Appl.*, 37(2):134–140, 2007.

[San00] Francisco Santos. A point configuration whose space of triangulations is disconnected. *J. Amer. Math. Soc.*, 13(3):611–637, 2000.

Problem 29: Hamiltonian Tetrahedralizations

Statement Can every convex polytope in $\mathbb{R}^3$ be partitioned into tetrahedra such that the dual graph has a Hamiltonian path?

Origin [AHMS96].

Status/Conjectures Open.

Partial and Related Results Every convex polygon has such a *Hamiltonian triangulation*, but not every nonconvex polygon does [AHMS96]. The existence of a Hamiltonian path permits faster rendering on a graphics screen via pipelining.

Chin, Ding, and Wang [CDW05] have shown that examples exist for which the *pulling* tetrahedralization of a convex polytope is not Hamiltonian. (A pulling tetrahedralization is obtained by joining one vertex (the apex) to all other vertices of the polytope.) It is open if the shelling tetrahedralization may be always Hamiltonian.

Escalona et al. [EFMU07] prove the conjecture up to $n = 20$: every points set of $n \leq 20$ points admits a Hamiltonian Tetrahedralization. They also detail an algorithm that finds a Hamiltonian Tetrahedralization for n points by adding $O(n)$ Steiner points. The algorithm runs in $O(n^{3/2})$ time.

Appearances [MO01]

Categories triangulations; polyhedra

Entry Revision History J. O'Rourke, 2 Aug. 2001; 13 Dec. 2001; J. Mitchell, 27 Oct. 2005; J. O'Rourke, 30 Sep. 2007.

References

[AHMS96] Esther M. Arkin, Martin Held, Joseph S. B. Mitchell, and Steven S. Skiena. Hamiltonian triangulations for fast rendering. *Visual Comput.*, 12(9):429–444, 1996.

[CDW05] Francis Chin, Qing-Huai Ding, and Cao An Wang. On Hamiltonian tetrahedralizations of convex polyhedra. In Xiang-Sun Zhang, De-Gang Liu, and Ling-Yun Wu, editors, *The Fifth International Symposium (ISORA 05)*, volume 5 of *Operations Research and its Applications, Lecture Notes in Operations Research*, pages 206–216. World Publishing Corporation, August 2005.

[EFMU07] Francisco Escalona, Ruy Fabila-Monroy, and Jorge Urrutia. Hamiltonian tetrahedralizations with steiner points. In *Abstracts 23rd European Worshop on Computational Geometry*, pages 50–53, 2007.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

Problem 30: Thrackles

Statement What is the maximum number of edges in a thrackle? A *thrackle* is a planar drawing of a graph of n vertices by edges which are smooth curves between vertices, with the condition that each pair of edges intersect at exactly one point, and have distinct tangents there. Another phrasing is that nonincident edges cross exactly once, and no incident edges share an interior point.

Origin Conway, late 1960's.

Status/Conjectures Open. Conway's conjecture is that the number edges cannot exceed n.

Partial and Related Results The upper bound was lowered from $O(n^{3/2})$ to $2n-3$ in [LPS95], and further lowered to $(3/2)(n-1)$ in [CN00]. The conjecture has long been known to be true for straight-line thrackles. The conjecture was extended in [CN00] to the claim that a thrackle on n vertices embedded on a surface of genus g has at most $n+2g$ edges. See [BMP05, Sec. 9.5] for a recent discussion and further references.

Reward Conway offers a reward of $1,000 for a resolution.

Appearances [MO01, Weh]

Categories graphs; combinatorial geometry; graph drawing

Entry Revision History J. O'Rourke, 2 Aug. 2001; 13 Dec. 2001; 18 Feb. 2002 (thanks to David Eppstein). E. Demaine, 28 May 2002 (thanks to Stephan Wehner); J. O'Rourke, 22 Sep. 2005.

References

[BMP05] Peter Brass, William Moser, and János Pach. *Research Problems in Discrete Geometry*. Springer, 2005.

[CN00] G. Cairns and Y. Nikolayevsky. Bounds for generalized thrackles. *Discrete Comput. Geom.*, 23(2):191–206, 2000.

[LPS95] László Lovász, János Pach, and Mario Szegedy. On conway's thrackle conjecture. In *Proc. 11th Annu. ACM Sympos. Comput. Geom.*, pages 147–151, 1995.

[MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.

[Weh] Stephan Wehner. On the thrackle problem. `http://www.thrackle.org/thrackle.html`.

Problem 31: Trapping Light Rays with Segment Mirrors

Statement Is it possible to trap all the light from one point source by a finite collection of two-sided disjoint segment mirrors? A light ray is *trapped* if it includes no point strictly exterior to the convex hull of the mirrors. The source point is disjoint from the mirrors. Although several versions of the problem are possible, it seems to make the most sense to treat the mirrors as open segments (i.e., not including their endpoints), but demand that they are disjoint as closed segments.

Origin O'Rourke and Petrovici [OP01]. The question seems natural enough to have been raised earlier, but no other source is known.

Status/Conjectures Conjecture 9 from that paper: "No collection of segment mirrors can trap all the light from one source."

Partial and Related Results In [OP01] several other conjectures are formed that imply a resolution to the posed problem. The strongest—that no collection of mirrors as above can support even a single *nonperiodic* ray, i.e., one that reflects forever (so is trapped) but never rejoins its earlier path—was disproved by Ben Stephens in 2002, who designed a contruction of 8 mirrors that traps a ray reflecting nonperiodically. A similar construction was discovered and described in [MSZ09], which also established that any finite number of rays can be trapped nonperiodically. Milovich [Mil04] proved that if the angles between the lines containing the mirrors are rational multiples of π, then all but a countable number of light rays escape. In his book on billiards, Tabachnikov says, "It is unknown whether one can construct a polygonal trap for a parallel beam of light" [Tab05, p. 116]. This is in contrast to known nonpolygonal traps for such beams.

Related Open Problems Pach's "enchanted forest" of circular mirrors.

Appearances Presented at the Open Problem session of the *13th Canad. Conf. Comput. Geom.*, Waterloo, Ontario, Aug. 2001. Also, Oberwolfach, Jan. 2009.

Categories visibility

Entry Revision History J. O'Rourke, 28 Aug. 2001; 24 Feb. 2003; 5 Oct. 2005; 7 Sep. 2009.

References

[Mil04] David Milovich. Trapping light with mirrors. *MIT Undergrad. J. Math.*, 6:153–180, 2004.

[MSZ09] Zachary Mitchell, Gregory Simon, and Xueying Zhao. Trapping light rays aperiodically with mirrors. Unpublished manuscript, August 2009.

[OP01] Joseph O'Rourke and Octavia Petrovici. Narrowing light rays with mirrors. In *Proc. 13th Canad. Conf. Comput. Geom.*, pages 137–140, 2001.

[Tab05] Serge Tabachnikov. *Geometry and Billiards*, volume 30 of *Mathematics Advanced Study Semesters*. American Mathematical Society, 2005.

Problem 32: Bar-Magnet Polyhedra

Statement Which polyhedra are bar-magnet polyhedra? For reasons detailed below, the problem can be phrased as asking which 3-connected planar graphs may have their edges directed so that the directions "alternate" around each vertex.

Let P be a polyhedron with a set of edges E. For an edge $e \in E$, define a *bar magnet* as a mapping of e to either (N, S) or (S, N), which assigns the endpoints of e opposite poles of a magnet (and corresponds to directing the edge). Call a vertex v of P to be *alternating* under mappings of its edges to bar magnets if the incident edges assigns alternating magnetic poles to v in the cyclic order of those edges on the surface around v: $(N, S, N, S, ...)$. Thus if $\deg(v)$ is even, the poles alternate, and if $\deg(v)$ is odd, at most two like poles are adjacent in the circular sequence. Finally, call a polyhedron a *bar-magnet polyhedron* if there is a bar-magnet assignment of each of its edges so that each of its vertices is alternating.

Origin Joseph O'Rourke, 2001. This problem is inspired by the toy "Roger's Connection," which provides bar magnets and steel balls to construct polyhedra. The structures are most stable when each vertex is alternating.

Status/Conjectures Settled by Bojan Mohar, Apr. 2004.

Partial and Related Results At the presentation of the problem, Therese Biedl proved that the polyhedron formed by gluing together two tetrahedra with congruent bases is not a bar-magnet polyhedron: alternation at the three degree-4 vertices of the common base forces some other edge to be directed both ways. Thus not all polyhedra are bar-magnet polyhedra. Erik Demaine proved that a polyhedron all of whose vertices have even degree is a bar-magnet polyhedron: the graph has a face 2-coloring, and the edges of the faces of color 1 can oriented counterclockwise, which then orients each face of color 2 clockwise. He also observed that if every vertex is of degree 3, Petersen's theorem yields a perfect matching that establishes such "simple" polyhedra are bar-magnet polyhedra.

A clean characterization was provided by Bojan Mohar, who proved [Moh04]:

Theorem 1 *Let G be a planar graph embedded on the surface of a sphere. Define a new graph R whose nodes are the vertices of odd degree in G, with two nodes of R adjacent if they are cofacial in G (lie on a common facial walk). Then G has an NS-orientation (i.e., is a bar-magnet polyhedral skeleton) if and only if R has a perfect matching.*

A *facial walk* is a closed walk along the boundary of a face.

Appearances Posed by J. O'Rourke at the CCCG 2001 open-problem session [DO02].

Categories polyhedra; planar graphs

Entry Revision History J. O'Rourke, 29 Aug. 2001; 11 Oct. 2001; E. Demaine, 31 Aug. 2002; J. O'Rourke, 14 Aug. 2004; 20 Sep. 2004.

References

[DO02] Erik D. Demaine and Joseph O'Rourke. Open problems from CCCG 2001. In *Proceedings of the 14th Canadian Conference on Computational Geometry*, August 2002. `http://www.cs.uleth.ca/~wismath/cccg/papers/open.pdf`.

[Moh04] Bojan Mohar. Bar-magnet polyhedra and NS-orientations of maps. Manuscript, University of Ljubljana, September 2004.

Problem 33: Sum of Square Roots

Statement What is the minimum nonzero difference between two sums of square roots of integers? More precisely, find tight upper and lower bounds on $r(n,k)$, the minimum positive value of

$$\left|\sum_{i=1}^{k}\sqrt{a_i}-\sum_{i=1}^{k}\sqrt{b_i}\right|$$

where a_i and b_i are integers no larger than n. Bounds should be expressed as a function of n and k. Examples:

$$r(20,2)\approx .0002=\sqrt{10}+\sqrt{11}-\sqrt{5}-\sqrt{18}$$

$$r(20,3)\approx .000005=\sqrt{5}+\sqrt{6}+\sqrt{18}-\sqrt{4}-\sqrt{12}-\sqrt{12}$$

Origin Posed in [O'R81]. Perhaps older in other formulations.

Status/Conjectures Open, although some weak bounds are known.

Motivation Of particular importance is whether $\lg 1/r(n,k)$ is bounded above by a polynomial in k and $\lg n$. If this statement is true, then the sign of a sum of square roots of integers can be computed in polynomial time. If this statement is false, however, there still may be a polynomial-time algorithm to compute the sign.

To quote David Eppstein: "A major bottleneck in proving NP-completeness for geometric problems is a mismatch between the real-number and Turing machine models of computation: one is good for geometric algorithms but bad for reductions, and the other vice versa. Specifically, it is not known on Turing machines how to quickly compare a sum of distances (square roots of integers) with an integer or other similar sums, so even (decision versions of) easy problems such as the minimum spanning tree are not known to be in NP."

Partial and Related Results Exponential upper bounds are known through root-separation bounds [BFMS00, MS00]. Specifically, [MS00, BFMS00] proves that $-\lg r(n,k)\le O(2^{2k}\lg n)$. (More generally, [BFMS00, MS00] give finite algorithms to compute the sign of algebraic expressions such as sums of square roots, which are implemented and used in LEDA[3] and CORE[4] for exact geometric computation.)

John A. Anderson (`johnaa333@netzero.net`) has an unpublished proof (Aug 2003) of a similar bound:

$$r(n,k)\ \ge\ [4k^2n]^{1/2-2^{2k-2}}\ .$$

[3]http://www.algorithmic-solutions.com/enleda.htm
[4]http://www.cs.nyu.edu/exact/core/

Cheng et al. [CMSC09] establish an upper bound on $-\lg r(n,k)$ of $2^{O(n/\lg n)}\lg n$, which improves on the above bound $O(2^{2k\lg n})$ whenever $n \le ck\lg k$ for some c.

At the other extreme, Qian and Wang [QW04, QW05] show an upper bound on $r(n,k)$ of $O(n^{-2k+\frac{3}{2}})$. This upper bound on $r(n,k)$ implies a lower bound on $\lg 1/r(n,k)$, that is, on how many bits we need to compute from the square roots to determine the sign of the difference. In particular, it settles (positively) a question posed here by Erik Demaine (Nov. 2001): can the number of bits required to distinguish the difference from zero ever exceed the total number of bits in the input integers?

A slight variation on the problem is to ask (e.g., for $k = 2$), how close can $\sqrt{a}+\sqrt{b}$ be to an integer; Dana Angluin and Sarah Eisenstat [AE04] proved a bound of $\Theta(1/n^{3/2})$ on this integrality gap.

Also, [Blö91] may be relevant.

Appearances [O'R81]; Usenet newsgroup sci.math[5] 25 Dec 90.

Categories numerical computations

Entry Revision History E. Demaine, J. O'Rourke, 19 Nov. 2001; J. O'Rourke, 3 Dec. 2001; 13 Aug. 2003; 18 Aug. 2003; 30 Aug. 2003; 7 Dec. 2003; E. Demaine, 9 Feb. 2004 (thanks to Raimund Seidel); J. O'Rourke, 10 Mar. 2004; J. Mitchell, 30 Sep. 2004; J. Mitchell, 1 Oct. 2004; J. Mitchell, 27 Oct. 2005; J. O'Rourke, 30 Dec. 2005 (thanks to Marc Glisse); J. O'Rourke, 16 May 2006; E. Demaine, 9 Sep. 2009.

References

[AE04] Dana Angluin and Sarah Eisenstat. How close can $\sqrt{a}+\sqrt{b}$ be to an integer?, March 2004.

[Blö91] J. Blömer. Computing sums of radicals in polynomial time. In *Proc. 32nd Annu. IEEE Sympos. Found. Comput. Sci.*, pages 670–677, 1991.

[BFMS00] C. Burnikel, R. Fleischer, K. Mehlhorn, and S. Schirra. A strong and easily computable separation bound for arithmetic expressions involving radicals. *Algorithmica*, 27(1):87–99, 2000.

[CMSC09] Qi Cheng, Xianmeng Meng, Celi Sun, and Jiazhe Chen. Bounding the sum of square roots via lattice reduction. arXiv:0905.4487v1 [cs.CG], 2009.

[MS00] Kurt Mehlhorn and Stefan Schirra. Generalized and improved constructive separation bound for real algebraic expressions.

[5] http://www.ics.uci.edu/ eppstein/junkyard/small-dist.html

Research Report MPI-I-2000-1-004, Max-Planck-Institut für Informatik, Saarbrücken, Germany, November 2000.

[O'R81] Joseph O'Rourke. Advanced problem 6369. *Amer. Math. Monthly*, 88(10):769, 1981.

[QW04] Jianbo Qian and Cao An Wang. New upper bound on difference between two sums of square roots of integers, October 2004.

[QW05] Jianbo Qian and Cao An Wang. How much precision is needed to compare two sums of square roots of integers?. *Information Processing Letters*, 100(5):194–198, December 2005. Also Technical Report, Memorial University of Newfoundland, Oct. 2005.

Problem 34: Extending Pseudosegment Arrangements by Subdivision

Statement How many intersections among an arrangement of pseudosegments in the plane must be added as vertices to allow the pseudosegment arrangment to be extended to a pseudoline arrangement?

An *arrangement of pseudosegments* in the plane is a family of finite-length planar curves such that every two curves intersect in at most one point. An *arrangement of pseudolines* in the plane is a family of planar curves that extend to infinity on both ends such that every two curves intersect in at most one point. Only some pseudosegment arrangements can be *extended* to pseudoline arrangements. However, if we allow turning intersection points into vertices of the arrangement, thereby subdividing the segments, then it is always possible to make a pseudosegment arrangement extendible. The question is how many such vertices must be added in the worst-case in terms of the number n of pseudosegments.

Origin Perhaps [Cha00a], [AACS98], or Boris Aronov?

Status/Conjectures Open.

Partial and Related Results Chan [Cha00a] has proved an upper bound of $O(n \log n)$.

Appearances Posed by Boris Aronov during the open problem session at the Fall Workshop on Computational Geometry, Brooklyn, NY, Nov. 2–3, 2001.

Categories combinatorial geometry

Entry Revision History E. Demaine, 20 Nov. 2001.

References

[AACS98] P. K. Agarwal, Boris Aronov, T. M. Chan, and Micha Sharir. On levels in arrangements of lines, segments, planes, and triangles. *Discrete Comput. Geom.*, 19:315–331, 1998.

[Cha00a] Timothy M. Chan. On levels in arrangements of curves. In *Proc. 41th Annu. IEEE Sympos. Found. Comput. Sci.*, pages 219–227, 2000.

Problem 35: Freeze-Tag: Optimal Strategies for Awakening a Swarm of Robots

Statement An optimization problem that naturally arises in the study of "swarm robotics" is to wake up a set of "asleep" robots, starting with only one "awake" robot. One robot can only awaken another when they are in the same location. As soon as a robot is awake, it may assist in waking up other robots. The goal is to compute an optimal *awakening schedule* such that all robots are awake by time t^*, for the smallest possible value of t^* (the optimal *makespan*). The n robots are initially at n points of a metric space. The problem is equivalent to finding a spanning tree with maximum out-degree two that minimizes the radius from a fixed source.

Is it NP-hard to determine an optimal awakening schedule for robots in the Euclidean (or L_1) plane? In more general metric spaces, can one obtain an approximation algorithm with better than $O(\log n)$ performance ratio?

Origin [ABF$^+$02]

Status/Conjectures [ABF$^+$02] conjecture that the freeze-tag problem is NP-hard in the Euclidean (or L_1) plane. (They show it to be NP-complete in star metrics.)

Motivation What is the most efficient way to "turn on" a large swarm of robots or to distribute to them a secret or a token that requires close proximity in order to pass from one to another?

Partial and Related Results There are a variety of related results given in [ABF$^+$02]. They show that the problem is NP-hard for "star metrics" (each asleep robot is at a leaf of a star graph whose spokes have various lengths). For geometric instances (L_p metrics) in fixed dimension, they give an efficient PTAS. For general metric spaces, they give an $O(\log n)$-approximation algorithm. They also give improved approximation methods for other special cases (star graphs, ultrametrics)

Appearances Posed in [ABF$^+$02], and by Joseph Mitchell during the open problem session at the Fall Workshop on Computational Geometry, Brooklyn, NY, Nov. 2–3, 2001.

Categories optimization; scheduling; robotics

Entry Revision History E. Demaine, 20 Nov. 2001; J. Mitchell, 21 Nov. 2001.

References

[ABF$^+$02] E. M. Arkin, M. A. Bender, S. P. Fekete, J. S. B. Mitchell, and M. Skutella. The freeze-tag problem: How to wake up a swarm of robots. In *Proc. 13th ACM-SIAM Sympos. Discrete Algorithms*, 2002. To appear.

Problem 36: Inplace Convex Hull of a Simple Polygonal Chain

Statement How much extra space is required to compute the convex hull of a simple polygonal chain or simple polygon in linear time?

More precisely, given the n points in order along the chain in an array A, the alogorithm must re-arrange the points inplace in the array and output a number h so that the first h elements in the resulting array are the points on the convex hull in order. The goal is to minimize the extra storage past the array A, say to $O(\log n)$ or ideally $O(1)$.

Origin [BIK$^+$01]

Status/Conjectures Solved [BC04].

Partial and Related Results From the abstract of [BC04]: "we present a simple self-contained solution that uses $O(\log n)$ space, and indicate how to improve it to $O(1)$ space with the same techniques used for stable partition."

Appearances Posed in [BIK$^+$01], and by Hervé Brönnimann during the open problem session at the Fall Workshop on Computational Geometry, Brooklyn, NY, Nov. 2–3, 2001.

Categories convex hulls

Entry Revision History E. Demaine, 21 Nov. 2001; J. O'Rourke, 10 Mar. 2004 (thanks to Ryan Coleman).

References

[BC04] Hervé Brönnimann and Timothy Chan. Space-efficient algorithms for computing the convex hull of a simple polygonal line in linear time. In *Proceedings of the 6th Latin American Symposium on Theoretical Informatics*, volume 2976 of *Lecture Notes in Computer Science*, pages 162–171, 2004.

[BIK+01] Hervé Brönnimann, John Iacono, Jryki Katajainen, Pat Morin, and Jason Morrison. Optimal in-place planar convex hull algorithms. 11th Annu. Fall Workshop Comput. Geom., 2001. `http://geometry.poly.edu/cgwpapers/`.

Problem 37: Counting Polyominoes

Statement How many polyominoes on n squares are there? A *polyomino* is a connected interior-disjoint union of axis-aligned unit squares joined edge-to-edge, in other words, an edge-connected union of cells in the planar square lattice. The *order* of a polyomino is the number of unit squares forming it. The problem asks for the number of polyominoes of order n. The key constraint here is that polyominoes must be edge-connected. There are three variations on the problem, depending on whether two polyominoes are considered equivalent by factoring out just translations (fixed polyominoes), rotations and translations (chiral polyominoes), or reflections, rotations, and translations (free polyominoes).

Origin To quote Klarner [Kla97]: "Polyominoes have a long history, going back to the start of the 20th century, but they were popularized in the present era initially by Solomon Golomb, then by Martin Gardner in his *Scientific American* columns."

Status/Conjectures Open.

Partial and Related Results Asymptotically, results of Klarner et al. [Kla97, Thm. 12.3.1] show that the number of fixed n-ominoes (factoring out just translations), denoted $t(n)$, satisfies

$$\lim_{n\to\infty} [t(n)]^{1/n} = \theta$$

(roughly, $t(n)$ is around n^θ) for a constant θ with $3.9 < \theta < 4.65$, but the precise value of θ remains open.

The exact counts have been computed for small n. See [Slo] for the number of fixed n-ominoes for $n \leq 28$ and for related references. The current record is $n = 46$ by Jensen [Jen01]. See also [Epp] for related links.

Related Open Problems There are many related problems involving polyiamonds (edge-to-edge unions of unit equilateral triangles), polyhexes (edge-to-edge unions of unit regular hexagons), polyabolos (edge-to-edge unions of unit right isosceles triangles), polycubes (face-to-face unions of unit cubes), etc. All of these problems are also open.

Appearances [Kla97]

Categories combinatorial geometry

Entry Revision History E. Demaine & J. O'Rourke, 30 Nov. 2001; E. Demaine, 28 Aug. 2002.

References

[Epp] David Eppstein. Polyominoes and other animals. `http://www1.ics.uci.edu/~eppstein/junkyard/polyomino.html`.

[Jen01] Iwan Jensen. Enumerations of lattice animals and trees. *J. Statistical Physics*, 102:865–881, 2001.

[Kla97] David A. Klarner. Polyominoes. In Jacob E. Goodman and Joseph O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, chapter 12, pages 225–242. CRC Press LLC, Boca Raton, FL, 1997.

[Slo] Neil J. A. Sloane. Sequence A000105. `http://www.research.att.com/cgi-bin/access.cgi/as/njas/sequences/eisA.cgi?Anum=A`

Problem 38: Compatible Triangulations

Statement Is it true that every two sets of n planar points in general position with the same number points on their convex hulls have compatible triangulations? Two triangulations are *compatible* if they have the same combinatorial structure, i.e., if their face lattices are isomorphic. For compatible triangulations T_1 and T_2 of point sets S_1 and S_2, there is a bijection ϕ between the points such that ijk is a triangle of T_1 empty of points of S_1 iff $\phi(i)\phi(j)\phi(k)$ is a triangle of T_2 empty of points of S_2.

Origin [AAK01] and [AAHK02].

Status/Conjectures Open. Conjectured in [AAHK02] to be true.

Motivation Morphing.

Partial and Related Results The answer to the question posed is sometimes NO for points not in general position. If the bijection between the points is given and fixed, then compatible triangulations do not always exist [Saa87]. When the bijection is not given, the conjecture is proven only for point sets with at most three points interior to the hull [AAHK02]. Compatible triangulations can always be achieved by the addition of at most a linear number of Steiner points.

Categories triangulations

Entry Revision History J. O'Rourke, 1 Jan. 2002.

References

[AAHK02] Oswin Aichholzer, Franz Aurenhammer, Ferran Hurtado, and Hannes Krasser. Towards compatible triangulations. *Theoret. Comput. Sci.*, ??:??–??, 2002. `http://www.cis.TUGraz.at/igi/oaich/`.

[AAK01] Oswin Aichholzer, Franz Aurenhammer, and Hannes Krasser. On compatible triangulations of point sets. In *Abstracts 17th European Workshop Comput. Geom.*, pages 23–26. Freie Universität Berlin, 2001.

[Saa87] A. Saalfeld. Joint triangulations and triangulation maps. In *Proc. 3rd Annu. ACM Sympos. Comput. Geom.*, pages 195–204, 1987.

Problem 39: Distances among Point Sets in $\mathbb{R}^2$ and $\mathbb{R}^3$

Statement For a point set P in $\mathbb{R}^d$, let $f_d(P)$ be the number of unit-distance point pairs:

$$f_d(P) = |\{(u,v) \mid u,v \in P, \|u-v\| = 1\}| \ ;$$

and let $f_d(n)$ be the maximum over all sets of n points:

$$f_d(n) = \max_{|P|=n} f_d(P) \ .$$

Further, let $g_d(P)$ denote the number of distinct distances induced by a set of points P:

$$g_d(P) = |\{\|u-v\| \mid u,v \in P\}| \ ;$$

and let $g_d(n)$ be the minimum over all sets of n points:

$$g_d(n) = \min_{|P|=n} g_d(P) \ .$$

Give upper and lower bounds on $f_d(n)$ and $g_d(n)$, particularly for $d = 2$ and $d = 3$.

Origin Paul Erdős [Erd46].

Status/Conjectures Open.

Partial and Related Results $f_2(n) = O(n^{4/3})$ [Szé97, CEG$^+$90, SST84], and $f_2(n) = \Omega(n^{1+c/\log\log n})$ [Erd46]. $f_3(n) = O(n^{5/3})$ and $f_3(n) = \Omega(n^{4/3}\log\log n)$ [Erd60]. For $g_2(n)$, the best result is that $g_2(n) = \Omega(n^{6/7})$ [ST01a]. Erdős conjectured that the correct answer here is $n/\sqrt{\log n}$; this bound is achieved on the grid.

Reward Erdős offered \$500 to settle whether $f_2(n) < cn^{1+\epsilon}$ for some $c > 0$ and for each $\epsilon > 0$, and \$500 to settle whether $g_2(n) = [1 + o(1)]cn/\sqrt{\log n}$.

Appearances [CFG90, pp. 150-1].

Categories combinatorial geometry

Entry Revision History S. Venkatasubramanian, 12 Feb. 2002.

References

[CEG$^+$90] K. Clarkson, Herbert Edelsbrunner, Leonidas J. Guibas, Micha Sharir, and Emo Welzl. Combinatorial complexity bounds for arrangements of curves and spheres. *Discrete Comput. Geom.*, 5:99–160, 1990.

[CFG90] H. P. Croft, K. J. Falconer, and R. K. Guy. *Unsolved Problems in Geometry.* Springer-Verlag, 1990.

[Erd46] P. Erdős. On sets of distances of n points. *Amer. Math. Monthly*, 53:248–250, 1946.

[Erd60] P. Erdős. On sets of distances of n points in Euclidean space. *Publications Mathematical Institute of Hungarian Academy of Sciences*, 5:165–169, 1960.

[Szé97] L. A. Székely. Crossing numbers and hard Erdős problems in discrete geometry. *Combinatorics, Probability and Computing*, 6:353–358, 1997.

[SST84] J. Spencer, E. Szemerédi, and W. T. Trotter. Unit distances in the Euclidean plane. In B. Bollobás, editor, *Graph Theory and Combinatorics*, pages 293–303. Academic Press, New York, NY, 1984.

[ST01a] J. Solymosi and Cs. D. Tóth. Distinct distances in the plane. *Discrete Comput. Geom.*, 25(4):629–634, 2001.

Problem 40: The Number of Pointed Pseudotriangulations

Statement For a planar point set S, is the number of pointed pseudotriangulations always at least the number of triangulations?

A *pseudotriangle* is a planar polygon with exactly three convex vertices. Each pair of convex vertices is connected by a reflex chain, which may be just one segment. (Thus, a triangle is a pseudotriangle.) A *pseudotriangulation* of a set S of n points in the plane is a partition of the convex hull of S into pseudotriangles using S as a vertex set. A minimum pseudotriangulation, or *pointed pseudotriangulation*, has the fewest possible number of edges for a given set S of points.

See [Str00, KKM+01, O'R02a] for examples, explanation of the term "pointed," and further details.

Origin [RRSS01].

Status/Conjectures Open. Conjectured to be true, with equality only when the points of S are in convex position.

Partial and Related Results The conjecture has been established for all sets of at most 10 points: ≤ 9 by [BKPS01], and 10 by Oswin Aichholzer [personal communication, 28 Mar. 2002]. Aichholzer et al. [AAKS02] establish that the number of pointed pseudotriangulations on n points is minimized when the points are in convex position.

Appearances Posed by Jack Snoeyink at the CCCG 2001 open-problem session [DO02].

Categories triangulations; combinatorial geometry

Entry Revision History J. O'Rourke, 20 Mar. 2002; 28 Mar. 2002; E. Demaine, 7 Aug. 2002; 31 Aug. 2002.

References

[AAKS02] Oswin Aichholzer, Franz Aurenhammer, Hannes Krasser, and Bettina Speckmann. Convexity minimizes pseudo-triangulations. In *Proc. 14th Canad. Conf. Comput. Geom.*, pages 158–161, 2002.

[BKPS01] H. Brönnimann, L. Kettner, M. Pocchiola, and J. Snoeyink. Counting and enumerating pseudo-triangulations with the greedy flip algorithm. `http://www.cs.unc.edu/Research/compgeom/pseudoT/`, September 2001.

[DO02] Erik D. Demaine and Joseph O'Rourke. Open problems from CCCG 2001. In *Proceedings of the 14th Canadian Conference on Computational Geometry*, August 2002. `http://www.cs.uleth.ca/~wismath/cccg/papers/open.pdf`.

[KKM+01] Lutz Kettner, David Kirkpatrick, Andrea Mantler, Jack Snoeyink, Bettina Speckmann, and Fumihiko Takeuchi. Tight degree bounds for pseudo-triangulations of points. *Comput. Geom. Th. Appl.*, 2001. To appear. Revision of abstract by L. Kettner, D. Kirkpatrick, and B. Speckmann, in *Proc. 13th Canad. Conf. Comput. Geom.*, pp. 117-120, 2001.

[O'R02a] J. O'Rourke. Computational geometry column 43. *Internat. J. Comput. Geom. Appl.*, 12(3):263–265, 2002. Also in *SIGACT News*, 33(1) Issue 122, Mar. 2002, 58-60.

[RRSS01] D. Randall, G. Rote, F. Santos, and J. Snoeyink. Counting triangulations and pseudotriangulations of wheels. In *Proc. 13th Canad. Conf. Comput. Geom.*, pages 149–152, 2001.

[Str00] Ileana Streinu. A combinatorial approach to planar non-colliding robot arm motion planning. In *Proc. 41st Annu. IEEE Sympos. Found. Comput. Sci.* IEEE, November 2000. 443–453.

Problem 41: Sorting $X + Y$ (Pairwise Sums)

Statement Given two sets of numbers, each of size n, how quickly can the set of all pairwise sums be sorted? In symbols, given two sets X and Y, our goal is to sort the set

$$X + Y = \{x + y \mid x \in X, y \in Y\}.$$

Origin The earliest known reference is Fredman [Fre76], who attributes the problem to Elwyn Berlekamp.

Status/Conjectures Open.

Motivation This is a simple special case of the more general question of *sorting with partial information*: How many comparisons are required to sort if a partial order on the input set is already known? Hernández Barrera [Her96] and Barequet and Har-Peled [BHP01] describe several geometric problems that are "Sorting-$(X + Y)$-hard". Specifically, there is a subquadratic-time transformation from sorting $X + Y$ to each of the following problems: computing the Minkowski sum of two orthogonal-convex polygons, determining whether one monotone polygon can be translated to fit inside another, determining whether one convex polygon can be rotated to fit inside another, sorting the vertices of a line arrangement, or sorting the interpoint distances between n points in $\mathbb{R}^d$. (Although Barequet and Har-Peled [BHP01] claim only that the problems they consider are 3SUM-hard, their proofs immediately imply this stronger result.) Fredman also mentions an immediate application to multiplying sparse polynomials [Fre76].

Partial and Related Results The obvious $O(n^2 \log n)$-time algorithm is also the fastest known. There are $\Omega(n^2)$ lower bounds for this problem in various restrictions of the linear decision tree model of computation [Fre76, Die89, Eri99a]. The main problem is whether the logarithmic factor can be removed.

Fredman [Fre76] proved that if a given partial order on m elements has L linear extensions, then the set can be sorted in at most $\log_2 L + 2m$ comparisons. For the sorting $X+Y$ problem, we have $m = n^2$, the Hasse diagram of the partial order is an $n \times n$ diagonal grid, and simple arguments about hyperplane arrangements imply that $L = O(n^{8n})$. Thus, Fredman's algorithm can sort $X+Y$ using only $8n \log n + 2n^2$ comparisons; unfortunately, the algorithm needs exponential time to choose which comparisons to perform! This exponential overhead was reduced to polynomial time by Kahn and Kim [KK95] and then to $O(n^2 \log n)$ by Lambert [Lam92] and Steiger and Streinu [SS95]. These results imply that no superquadratic lower bound is possible in the full linear decision tree model.

If the input consists of n integers between $-M$ and M, an algorithm of Seidel based on fast Fourier transforms runs in $O(n + M \log M)$ time [Eri99a]. The $\Omega(n^2)$ lower bounds require exponentially large integers.

A closely related problem does have a subquadratic solution: find a minimum element of $X+Y$, the so-called *min-convolution* problem, posed by Jeff Erickson [DO06]. See [BCD$^+$06] for the result and a discussion of connections to the sorting problem.

Related Open Problems The decision version of this problem—does the set $X+Y$ have n^2 unique elements?—is 3SUM-hard [BHP01]; see Problem 11.

Categories lower bounds

Entry Revision History E. Demaine, 6 June 2002; Jeff Erickson, 20 June 2002; J. O'Rourke, 20 Aug. 2006.

References

[Her96] A. Hernández Barrera. Finding an $o(n^2 \log n)$ algorithm is sometimes hard. In *Proc. 8th Canad. Conf. Comput. Geom.*, pages 289–294. Carleton University Press, Ottawa, Canada, 1996.

[BCD+06] David Bremner, Timothy M. Chan, Erik D. Demaine, Jeff Erickson, Ferran Hurtado, John Iacono, Stefan Langerman, and Perouz Taslakian. Necklaces, convolutions, and $X+Y$. In *Proceedings of the 14th Annual European Symposium on Algorithms (ESA 2006)*, page to appear, Zürich, Switzerland, September 11–13 2006.

[BHP01] G. Barequet and S. Har-Peled. Polygon containment and translational min-Hausdorff-distance between segment sets are 3SUM-hard. *Internat. J. Comput. Geom. Appl.*, 11:465–474, 2001.

[Die89] M. Dietzfelbinger. Lower bounds for sorting of sums. *Theoret. Comput. Sci.*, 66:137–155, 1989.

[DO06] Erik D. Demaine and Joseph O'Rourke. Open problems from CCCG 2005. In *Proc. 18th Canad. Conf. Comput. Geom.*, pages 75–80, 2006.

[Eri99a] Jeff Erickson. Lower bounds for linear satisfiability problems. *Chicago J. Theoret. Comput. Sci.*, 1999(8), 1999.

[Fre76] M. L. Fredman. How good is the information theory bound in sorting? *Theoret. Comput. Sci.*, 1:355–361, 1976.

[KK95] Jeff Kahn and Jeong Han Kim. Entropy and sorting. *J. Comput. Sys. Sci.*, 51:390–399, 1995.

[Lam92] Jean-Luc Lambert. Sorting the sums $(x_i + y_j)$ in $O(n^2)$ comparisons. *Theoret. Comput. Sci.*, 103:137–141, 1992.

[SS95] W. Steiger and Ileana Streinu. A pseudo-algorithmic separation of lines from pseudo-lines. *Inform. Process. Lett.*, 53:295–299, 1995.

Problem 42: Vertex-Unfolding Polyhedra

Statement Consider a polyhedron with simply connected facets (no holes on a facet) and without boundary (every edge is incident to exactly two facets). Can the polyhedron be cut along potentially all of its edges, but leaving certain faces connected at vertices, and unfolded into one piece in the plane without overlap? Such an unfolding is called a *vertex-unfolding*, to distinguish from widely studied *edge-unfoldings* (see Problem 9) and *general unfoldings*. An important subproblem here is whether all convex polyhedra have vertex-unfoldings; a negative answer would also resolve Problem 9.

Origin [DEE+02]

Status/Conjectures Open.

Partial and Related Results All simplicial polyhedra have vertex-unfoldings [DEE+02]. These vertex-unfoldings have a special structure called a "facet path" which does not exist in general, even for convex polyhedra [DEE+02].

Related Open Problems Problem 9: Edge-Unfolding Convex Polyhedra. Problem 43: General Unfolding of Nonconvex Polyhedra.

Appearances Originally posed in [DEE+02]. Posed by E. Demaine at the CCCG 2001 open-problem session [DO02].

Categories folding and unfolding; polyhedra

Entry Revision History E. Demaine, 7 Aug. 2002; 31 Aug. 2002.

References

[DEE+02] Erik D. Demaine, David Eppstein, Jeff Erickson, George W. Hart, and Joseph O'Rourke. Vertex-unfolding of simplicial manifolds. In *Proceedings of the 18th Annual ACM Symposium on Computational Geometry*, pages 237–243, 2002.

[DO02] Erik D. Demaine and Joseph O'Rourke. Open problems from CCCG 2001. In *Proceedings of the 14th Canadian Conference on Computational Geometry*, August 2002. `http://www.cs.uleth.ca/~wismath/cccg/papers/open.pdf`.

Problem 43: General Unfoldings of Nonconvex Polyhedra

Statement Can every closed polyhedron be cut along its surface and unfolded into one piece in the plane without overlap? Such an unfolding is called a *general unfolding* to distinguish from *edge-unfoldings* (see Problem 9) and *vertex-unfoldings* (see Problem 42).

Origin Perhaps [BDE+03].

Status/Conjectures Open.

Partial and Related Results It is known that every convex polyhedron has a general unfolding. In fact, there are three general methods for unfolding convex polyhedra: the star unfolding [AO91, AAOS97], the source unfolding [MMP87], and unfolding via quasigeodesics [IOV07].

On the nonconvex side, Bern et al. [BDE+03] show a general unfolding for a nonconvex simplicial polyhedron (whose faces are all triangles) that has no edge unfolding, establishing that general unfoldings are more powerful than edge unfoldings. (This was known earlier [BDD+98] but with an example using nonconvex faces.)

It is now known that all orthogonal polyhedra (those with all edges parallel to coordinate axes) have a general unfolding [DFO07], although the resulting single piece can be exponentially thin and long. See [O'R08] for a survey of progress on orthogonal polyhedra.

Related Open Problems Problem 9: Edge-Unfolding Convex Polyhedra. Problem 42: Vertex-Unfolding Polyhedra.

Appearances [BDE$^+$03], [DO07a, Open Prob. 22.3].

Categories folding and unfolding; polyhedra

Entry Revision History E. Demaine, 7 Aug. 2002; J. O'Rourke, 24 Jul 2008.

References

[AAOS97] Pankaj K. Agarwal, Boris Aronov, Joseph O'Rourke, and Catherine A. Schevon. Star unfolding of a polytope with applications. *SIAM J. Comput.*, 26:1689–1713, 1997.

[AO91] Boris Aronov and Joseph O'Rourke. Nonoverlap of the star unfolding. In *Proc. 7th Annu. ACM Sympos. Comput. Geom.*, pages 105–114, 1991.

[BDD$^+$98] Therese Biedl, Erik D. Demaine, Martin L. Demaine, Anna Lubiw, Joseph O'Rourke, Mark Overmars, Steve Robbins, and Sue Whitesides. Unfolding some classes of orthogonal polyhedra. In *Proc. 10th Canad. Conf. Comput. Geom.*, pages 70–71, 1998. Full version in *Elec. Proc.*: `http://cgm.cs.mcgill.ca/cccg98/proceedings/cccg98-biedl-unfolding.ps.gz%`.

[BDE$^+$03] Marshall Bern, Erik D. Demaine, David Eppstein, Eric Kuo, Andrea Mantler, and Jack Snoeyink. Ununfoldable polyhedra with convex faces. *Comput. Geom. Theory Appl.*, 24(2):51–62, 2003.

[DFO07] Mirela Damian, Robin Flatland, and Joseph O'Rourke. Epsilon-unfolding orthogonal polyhedra. *Graphs and Combinatorics*, 23[Suppl]:179–194, 2007. Akiyama-Chvátal Festschrift.

[DO07a] Erik D. Demaine and Joseph O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra*. Cambridge University Press, July 2007. `http://www.gfalop.org`.

[IOV07] Jin-ichi Itoh, Joseph O'Rourke, and Costin Vîlcu. Unfolding convex polyhedra via quasigeodesics. Technical Report 085, Smith College, July 2007. arXiv:0707.4258v2 [cs.CG].

[MMP87] Joseph S. B. Mitchell, David M. Mount, and Christos H. Papadimitriou. The discrete geodesic problem. *SIAM J. Comput.*, 16:647–668, 1987.

[O'R08] Joseph O'Rourke. Unfolding orthogonal polyhedra. In J.E. Goodman, J. Pach, and R. Pollack, editors, *Proc. Snowbird Conference Discrete and Computational Geometry: Twenty Years Later*, pages 307–317. American Mathematical Society, 2008.

Problem 44: 3-Colorability of Arrangements of Great Circles

Statement Is every zonohedron face 3-colorable when viewed as a planar map? An equivalent question, under a different guise, is the following: is the arrangement graph of great circles on the sphere always vertex 3-colorable? (The *arrangement graph* has a vertex for each intersection point, and an edge for each arc directly connecting two intersection points.) Assume that no three circles meet at a point, so that this arrangement graph is 4-regular.

Origin The zonohedron-face version is due to Stan Wagon, deriving from the work in [SW00]. The origin of the arrangement guise of the problem is [FHNS00].

Status/Conjectures Open.

Partial and Related Results Arrangement graphs of circles in the plane, or general circles on the sphere, can require four colors [Koe90]. The key property in this problem is that the circles must be great. All arrangement graphs of up to 11 great circles have been verified to be 3-colorable by Oswin Aichholzer (August, 2002). See [Wag02] for more details.

Appearances Posed by Stan Wagon at the CCCG 2002 open-problem session.

Categories arrangements; coloring; polyhedra

Entry Revision History E. Demaine & J. O'Rourke, 28 Aug. 2002.

References

[FHNS00] Stefan Felsner, Ferrán Hurtado, Marc Noy, and Ileana Streinu. Hamiltonicity and colorings of arrangement graphs. In *11th Annu. ACM-SIAM Symp. Discrete Algorithms (SODA)*, pages 155–164, January 2000.

[Koe90] G. Koester. 4-critical, 4-valent planar graphs constructed with crowns. *Math. Scand.*, 67:15–22, 1990.

[SW00] Thomas Sibley and Stan Wagon. Rhombic Penrose tilings can be 3-colored. *American Mathematics Monthly*, 106:251–253, 2000.

[Wag02] Stan Wagon. A machine resolution of a four-color hoax. In *Proc. 14th Canad. Conf. Comput. Geom.*, pages 181–193, August 2002.

Problem 45: Smallest Universal Set of Points for Planar Graphs

Statement How many points must be placed in the plane to support planar drawing of all planar graphs on n vertices? More precisely, call a set of points *universal* if every planar graph on n vertices can be drawn with straight-line edges and without crossings by placing the vertices on a subset of the points. What is the smallest universal set of points as a function of n? In particular, is it $O(n)$?

Origin Attributed to Mohar by János Pach (23 Nov. 2002). See also [CH89] for some of the history.

Status/Conjectures Open. Between $\Theta(n)$ and $\Theta(n^2)$.

Partial and Related Results By definition, a universal set of points must have size at least n. Chrobak and Karloff [CH89] proved the stronger result that any universal set of points must have at least $1.098n$ points.

On the other side, it is well-known that there are universal sets of points of size $O(n^2)$. In particular, every planar graph can be drawn on the $O(n) \times O(n)$ square grid [dFPP90, Sch90]. However, any universal set of points forming a grid must have size at least $n/3 \times n/3$ [CH89].

Stephen Kobourov asks for the smallest value of n for which a universal point set of size n does not exist. He has checked by exhaustive search that there is a universal point set of size n for all $n \leq 14$.

Appearances Posed by Stephen Koborouv during an open-problem session at the DIMACS Workshop on Computational Geometry (12th Annual Fall Workshop on Computational Geometry), Nov. 2002.

Categories graphs; point sets; graph drawing

Entry Revision History E. Demaine, 23 Nov. 2002; 20 Sep. 2003 (thanks to Sergio Cabello).

References

[CH89] M. Chrobak and H.Karloff. A lower bound on the size of universal sets for planar graphs. *SIGACT News*, 20:83–86, 1989.

[dFPP90] H. de Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10(1):41–51, 1990.

[Sch90] W. Schnyder. Embedding planar graphs on the grid. In *Proc. 1st ACM-SIAM Sympos. Discrete Algorithms*, pages 138–148, 1990.

Problem 46: 3D Minimum-Bend Orthogonal Graph Drawings

Statement Does every simple graph with maximum vertex degree $\Delta \leq 6$ have a 3D orthogonal point-drawing with no more than two bends per edge? A *3D orthogonal point-drawing* of a graph maps each vertex to a unique point of the 3D cubic lattice, and maps each edge to a lattice path between the endpoints; these paths can only intersect at common endpoints. In this problem, each path must have at most two bends, that is, consist of at most three orthogonal line segments (links).

Origin Likely [ESW00].

Status/Conjectures Open.

Partial and Related Results Two bends would be best possible, because any drawing of K_5 uses at least two bends on at least one edge. If $\Delta \leq 5$, two bends per edge suffice [Woo03]. Two bends also suffice for the complete multipartite 6-regular graphs K_7, $K_{2,2,2,2}$, $K_{3,3,3}$, and $K_{6,6}$ [Woo00]. In general, there is a drawing with an average number of bends per edge of at most $2 + \frac{2}{7}$ [Woo03]. Additionally, three bends per edge always suffice, even for multigraphs [ESW00, PT99, Woo01].

Two-dimensional versions of this problem have also been studied. A *2D orthogonal point-drawing* of a graph maps each vertex to a unique point of the 2D square lattice, and maps each edge to a lattice path between the endpoints; the paths are allowed to intersect at common endpoints and at proper crossings (points at which two paths meet but do not bend), but must be edge-disjoint. Every graph with maximum vertex degree $\Delta \leq 4$ has a 2D orthogonal point-drawing with at most two bends per edge, and furthermore within a $2n \times 2n$ rectangle of the grid [Sch95]. On the other hand, as in 3D, any drawing of K_5 uses at least two bends on at least one edge [Sch95], so two bends is again best possible. For planar graphs, we can ask for 2D orthogonal point-drawings that have no (proper) crossings. In this case, again there are drawings with at most two bends per edge, unless the graph has a connected component isomorphic to the icosohedron, in which case three bends per edge is the best possible [BK98, LMS98].

Appearances [ESW00]. Posed by David Wood at the CCCG 2002 open-problem session [DO03b].

Categories graph drawing

Entry Revision History E. Demaine, 21 Dec. 2002; 17 July 2005.

References

[BK98] Therese Biedl and G. Kant. A better heuristic for orthogonal graph drawings. *Comput. Geom. Theory Appl.*, 9:159–180, 1998.

[DO03b] Erik D. Demaine and Joseph O'Rourke. Open problems from CCCG 2002. In *Proceedings of the 15th Canadian Conference on Computational Geometry*, August 2003. To appear; `http://arXiv.org/archive/cs/0212050`.

[ESW00] P. Eades, A. Symvonis, and S. Whitesides. Three dimensional orthogonal graph drawing algorithms. *Discrete Applied Math.*, 103:55–87, 2000.

[LMS98] Yanpei Liu, Aurora Morgana, and Bruno Simeone. A linear algorithm for 2-bend embeddings of planar graphs in the two-dimensional grid. *Discrete Applied Math.*, 81(1–3):69–91, January 1998.

[PT99] Achilleas Papakostas and Ioannis G. Tollis. Algorithms for incremental orthogonal graph drawing in three dimensions. *J. Graph Algorithms Appl.*, 3(4):81–115, 1999.

[Sch95] M. Schäffter. Drawing graphs on rectangular grids. *Discrete Appl. Math.*, 63:75–89, 1995.

[Woo00] David R. Wood. *Three-Dimensional Orthogonal Graph Drawing*. PhD thesis, School of Computer Science and Software Engineering, Monash University, Melbourne, Australia, 2000.

[Woo03] David R. Wood. Optimal three-dimensional orthogonal graph drawing in the general position model. *Theoret. Comput. Sci.*, 2003. To appear.

[Woo01] David R. Wood. Minimising the number of bends and volume in three-dimensional orthogonal graph drawings with a diagonal vertex layout. Technical Report CS-AAG-2001-03, University of Sydney, 2001.

Problem 47: Hinged Dissections

Statement Does every pair of equal-area polygons have a hinged dissection? A *dissection* of one polygon A to another B is a partition of A into a finite number of pieces that may be reassembled to form B. A *hinged dissection* is a dissection where the pieces are hinged at vertices and the reassembling is achieved by rotating the pieces about their hinges in the plane of the polygons.

Origin [DDE$^+$03], [Fre02, p. 3].

Status/Conjectures Now settled: Hinged dissections exist [AAC+08]. Update to this entry soon.

Partial and Related Results There are two main partial results. First, any two *polyominoes* of the same area have a hinged dissection [DDE+03]. A polyomino is a polygon formed by joining unit squares at their edges; see [Kla97] and Problem 37. The polyomino result generalizes to hinged dissections of all edge-to-corresponding-edge gluings of congruent copies of any polygon. Second, any asymmetric polygon has a hinged dissection to its mirror image [Epp01]. Both of these results interpret the problem as ignoring possible intersections between the pieces as they hinge, following what Frederickson calls the "wobbly-hinged" model. This freedom may not be necessary, although this seems not to be established in the literature.

Many specific examples of hinged dissections can be found in [Fre02].

Appearances [O'R02b].

Categories polygons

Entry Revision History J. O'Rourke, 25 Mar 2003; J. O'Rourke, 23 Jan 2009.

References

[AAC+08] Timothy G. Abbott, Zachary Abel, David Charlton, Erik D. Demaine, Martin L. Demaine, and Scott D. Kominers. Hinged dissections exist. In *Proceedings of the 24th Annual ACM Symposium on Computational Geometry (SoCG 2008)*, pages 110–119, College Park, Maryland, June 9–11 2008.

[DDE+03] Erik D. Demaine, Martin L. Demaine, David Eppstein, Greg N. Frederickson, and Erich Friedman. Hinged dissection of polyominoes and polyforms. *Computational Geometry: Theory and Applications*, 2003. To appear. arXiv:cs.CG/9907018, `http://www.arXiv.org/abs/cs.CG/9907018`. Revised version of paper in *Proc. 11th Canad. Conf. Comput. Geom.* 1999, 15–18.

[Epp01] David Eppstein. Hinged kite mirror dissection. ACM Computing Research Repository, June 2001. arXiv:cs.CG/0106032, `http://www.arXiv.org/abs/cs.CG/0106032`.

[Fre02] Greg Frederickson. *Hinged Dissections: Swinging & Twisting.* Cambridge University Press, 2002.

[Kla97] David A. Klarner. Polyominoes. In Jacob E. Goodman and Joseph O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, chapter 12, pages 225–242. CRC Press LLC, Boca Raton, FL, 1997.

[O'R02b] Joseph O'Rourke. Computational geometry column 44. *Internat. J. Comput. Geom. Appl.*, 13(3):273–275, 2002. Also in *SIGACT News*, **34**(2):58–60 (2002), Issue 127.

Problem 48: Bounded-Degree Minimum Euclidean Spanning Tree

Statement What is the complexity of finding a bounded-degree spanning tree for a planar point set, such that the total Euclidean length τ_k is as small as possible, subject to the constraint that no node has more than $k = 4$ edges incident to it?

Origin Papadimitriou and Vazirani [PV84] conjectured the problem to be NP-hard for $k = 4$.

Status/Conjectures Open.

Motivation Natural generalization of finding a shortest geometric Hamiltonian path; arises in network optimization

Partial and Related Results [PV84] proved the problem to be NP-hard for $k = 3$. For $k \geq 5$, the problem is polynomially solvable, as there always is a minimum spanning tree with no point having degree more than 5.

Related Open Problems Various worst-case ratios of minimum weight bounded-degree spanning trees for different degree bounds are still open, in particular comparing τ_k to the weight τ of a minimum spanning tree. [FKK$^+$97] conjecture $\tau_3/\tau \leq 1.103...$, $\tau_4/\tau \leq 1.035...$ for Euclidean distances in the plane, and $\tau_4/\tau \leq 1.25$ for Manhattan distances in the plane, and give matching lower bounds.

[KRY96] show that for Euclidean distances, $\tau_4/\tau \leq 1.25$ and $\tau_3/\tau \leq 1.5$ in the plane, and $\tau_3/\tau \leq 1.66...$ in arbitrary dimensions.

The first two of these bounds were improved to $\tau_4/\tau \leq 1.143$ and $\tau_3/\tau \leq 1.402$ by [Cha03].

Categories minimum spanning tree; optimization; point sets

Entry Revision History S. P. Fekete, 30 July 2003.

References

[Cha03] Timothy M. Chan. Euclidean bounded-degree spanning tree ratios. In *Proc. 19th ACM Sympos. Computational Geometry*, pages 11–19, 2003.

[FKK+97] S. P. Fekete, S. Khuller, M. Klemmstein, B. Raghavachari, and N. Young. A network flow technique for finding low-weight bounded-degree trees. *Journal of Algorithms*, 24:310–324, 1997.

[KRY96] S. Khuller, B. Raghavachari, and N. Young. Low-degree spanning trees of small weight. *SIAM J. Comput.*, 25:355–368, 1996.

[PV84] C. H. Papadimitriou and U. V. Vazirani. On two geometric problems related to the traveling salesman problem. *J. Algorithms*, 5:231–246, 1984.

Problem 49: Planar Euclidean Maximum TSP

Statement What is the complexity of finding a tour of maximum Euclidean length for a planar point set?

Origin [Bar96] showed that there is a PTAS for the problem. No earlier mention is known.

Status/Conjectures Open.

Motivation How does the complexity of a natural problem depend on the geometry of distances?

Partial and Related Results [BJrW98] showed that a maximum length tour can be found in polynomial time for polyhedral metrics in spaces of finite dimension, i.e., for metrics for which the unit ball is a convex body with f facets. The resulting complexity is $O(n^{f-2} \log n)$.

[Fek99] showed that the maximum TSP can be solved in time $O(n)$ for rectilinear distances in the plane, but is NP-hard for Euclidean distances in three-dimensional space, or on the surface of a sphere. Conjectures the case of planar Euclidean distances to be NP-hard.

More recent details and related problems can be found in [BFJ+03].

Related Open Problems The problem is not even known to be in NP. A polynomial algorithm would require some understanding of problem 33 (sum of square roots), at least for classes of instances arising from the computation of tour length.

Also related is the Planar Euclidean maximum scatter TSP: What is the complexity of finding a tour for a planar point set in $\Re^d$, such that the Euclidean length of the shortest edge is maximized? Stated in [ACM+97], shown NP-hard in dimensions $d \geq 3$ in [Fek99], open for $d = 2$. Also, no bounds on approximation are known in a geometric context; the best known aproximation algorithm from [ACM+97] achieves an approximation factor of 2, but does not use geometry.

Appearances [Fek98], [BFJ+03]

Categories traveling salesman; optimization; point sets

Entry Revision History S.P. Fekete, 1 Aug. 2003.

References

[ACM+97] Esther M. Arkin, Y.-J. Chiang, Joseph S. B. Mitchell, Steven S. Skiena, and T. Yang. On the maximum scatter TSP. In *Proc. 8th ACM-SIAM Sympos. Discrete Algorithms*, pages 211–220, 1997.

[BFJ+03] A. Barvinok, S. P. Fekete, D. S. Johnson, A. Tamir, G. J. Woeginger, and R. Wodroofe. The geometric maximum Traveling Salesman problem. *Journal of the ACM*, 50:641–664, 2003.

[Bar96] Alexander I. Barvinok. Two algorithmic results for the TSP. *Math. Oper. Res.*, 21:65–84, 1996.

[BJrW98] Alexander Barvinok, David S. Johnson, Gerhard J. Woeginge r, and Russell Woodroofe. The maximum traveling salesman problem under polyhedral norms. In *Sixth Conference on Integer Programming and Combinatorial Optim ization*, volume 1412 of *Springer LNCS*, pages 195–201, June 1998.

[Fek98] Sándor P. Fekete. Simplicity and hardness of the maximum traveling salesman problem under geometric distances. Manuscript (submitted), Mathematisches Institut, Universität zu Köln, 1998.

[Fek99] S. P. Fekete. Simplicity and hardness of the maximum traveling salesman probl em under geometric distances. In *Proc. 10th ACM-SIAM Sympos. Discrete Algorithms*, pages 337–345, 1999.

Problem 50: Pointed Spanning Trees in Triangulations

Statement Does every triangulation of a set of points in the plane (in general position) contain a pointed spanning tree as a subgraph? A vertex is pointed if one of its incident faces has an angle larger than π at this vertex. A spanning tree is pointed if all of its vertices are pointed.

Origin Oswin Aichholzer, January 2003.

Status/Conjectures Settled negatively, January 2004.

Partial and Related Results Obviously true if a triangulation contains a Hamiltonian path or a pointed pseudotriangulation as a subgraph. For both structures there exist triangulations not containing them. (See, e.g., [O'R02a] for a discussion of pseudotriangulations.) Settled negatively by

Aichholzer et al. [AHK04] with a 124-point counterexample. A consequence is that there are triangulations that require $\Omega(n)$ edge-flips to contain a pointed spanning tree, or to become Hamiltonian.

Related Open Problems Problem 40.

Appearances Posed by Oswin Aichholzer at the CCCG 2003 open-problem session, August 2003. Also posed by Bettina Speckmann as Problem 10 at the First Gremo Workshop on Open Problems in Stels, Switzerland, July 2003.

Categories triangulations; planar graphs

Entry Revision History O. Aichholzer, 13 Aug. 2003; JOR, 15 Jan. 2004.

References

[AHK04] Owin Aichholzer, Clemens Huemer, and Hannes Krasser. Triangulations without pointed spanning trees. In *Abstracts 20th European Workshop Comput. Geom.*, 2004.

[O'R02a] J. O'Rourke. Computational geometry column 43. *Internat. J. Comput. Geom. Appl.*, 12(3):263–265, 2002. Also in *SIGACT News*, 33(1) Issue 122, Mar. 2002, 58-60.

Problem 51: Linear-Volume 3D Grid Drawings of Planar Graphs

Statement Does every n-vertex planar graph have a 3D grid drawing with $O(n)$ volume? A *3D grid drawing* of a graph is a placement of the vertices at distinct points with integer coordinates such that the straight line segments representing the edges are pairwise non-crossing. The volume is of the bounding box.

Origin Felsner, Liotta, and Wismath [FLW02].

Status/Conjectures Open.

Partial and Related Results

1. [FLW02]: Every n-vertex outerplanar graph has a 3D grid drawing with $O(n)$ volume.
2. [DW03b]: Every n-vertex graph with bounded treewidth has a 3D grid drawing with $O(n)$ volume.
3. [DW04]: Every n-vertex planar graph has a 3D grid drawing with $O(n^{3/2})$ volume.
4. [Woo02]: Every n-vertex planar graph has an $O(1) \times O(1) \times O(n)$ grid drawing if and only if planar graphs have $O(1)$ queue-number. (See Problem 52 for a definition of queue-number.)

Related Open Problems Problem 52.

Appearances Above references.

Categories graph drawing

Entry Revision History D. Wood, 6 Dec. 2003; J. O'Rourke, 16 Mar. 2004.

References

[DW04] Vida Dujmović and David R. Wood. Three-dimensional grid drawings with sub-quadratic volume. In János Pach, editor, *Towards a Theory of Geometric Graphs*, volume 342 of *Contemporary Mathematics*, pages 55–66. American Mathematical Society, 2004.

[DW03b] Vida Dujmović and David R. Wood. Tree-partitions of k-trees with applications in graph layout. In Hans Bodlaender, editor, *Proc. 29th Workshop on Graph Theoretic Concepts in Computer Science (WG'03)*, volume 2880 of *Lecture Notes in Comput. Sci.*, pages 205–217. Springer-Verlag, 2003.

[FLW02] Stefan Felsner, Giussepe Liotta, and Stephen Wismath. Straight-line drawings on restricted integer grids in two and three dimensions. In Petra Mutzel, Michael Jünger, and Sebastian Leipert, editors, *Proc. 9th International Symp. on Graph Drawing (GD '01)*, volume 2265 of *Lecture Notes in Comput. Sci.*, pages 328–342. Springer, 2002.

[Woo02] David R. Wood. Queue layouts, tree-width, and three-dimensional graph drawing. In Manindra Agrawal and Anil Seth, editors, *Proc. 22nd Foundations of Software Technology and Theoretical Computer Science (FST TCS '02)*, volume 2556 of *Lecture Notes in Comput. Sci.*, pages 348–359. Springer, 2002.

Problem 52: Queue-Number of Planar Graphs

Statement Does every planar graph have $O(1)$ queue-number? A *queue layout* of a graph consists of a linear order of the vertices and a partition of the edges into non-nested *queues*. Edge xy is *nested* inside edge vw if $v < x < y < w$ in the linear order. The *queue-number* of a graph G is the minimum number of queues in a queue layout of G. This question amounts to asking whether every planar graph has a vertex ordering with a constant number of pairwise nested edges (called a rainbow).

Origin Heath, Leighton, and Rosenberg [HLR92, HR92].

Status/Conjectures Open.

Partial and Related Results

1. [HLR92, HR92]: Every tree has queue-number ≤ 1.
2. [HLR92, HR92]: Every outerplanar graph has queue-number ≤ 2.
3. [DW03b]: Every graph with bounded treewidth has bounded queue-number.
4. [Woo02]: Planar graphs have $O(1)$ queue-number if and only if every n-vertex planar graph has a $O(1) \times O(1) \times O(n)$ grid drawing.
5. [DW03a]: Planar graphs have $O(1)$ queue-number if and only if Hamiltonian bipartite planar graphs have $O(1)$ bipartite thickness. The *bipartite thickness* of a bipartite graph G is the minimum k such that G can be drawn with the vertices on each side of the bipartition along a line, with the two lines parallel, and with each edge assigned to one of k "layers" so that no two edges in the same layer cross (when drawn as straight line segments).

Related Open Problems Problem 51.

Appearances Above references.

Categories graph drawing

Entry Revision History D. Wood, 7 Dec. 2003.

References

[DW03a] Vida Dujmović and David R. Wood. Stacks, queues and tracks: Layouts of graphs subdivisions. Technical Report TR-2003-07, School of Computer Science, Carleton University, Ottawa, Canada, 2003.

[DW03b] Vida Dujmović and David R. Wood. Tree-partitions of k-trees with applications in graph layout. In Hans Bodlaender, editor, *Proc. 29th Workshop on Graph Theoretic Concepts in Computer Science (WG'03)*, volume 2880 of *Lecture Notes in Comput. Sci.*, pages 205–217. Springer-Verlag, 2003.

[HLR92] Lenwood S. Heath, Frank Thomson Leighton, and Arnold L. Rosenberg. Comparing queues and stacks as mechanisms for laying out graphs. *SIAM J. Discrete Math.*, 5(3):398–412, 1992.

[HR92] Lenwood S. Heath and Arnold L. Rosenberg. Laying out graphs using queues. *SIAM J. Comput.*, 21(5):927–958, 1992.

[Woo02] David R. Wood. Queue layouts, tree-width, and three-dimensional graph drawing. In Manindra Agrawal and Anil Seth, editors, *Proc. 22nd Foundations of Software Technology and Theoretical Computer Science (FST TCS '02)*, volume 2556 of *Lecture Notes in Comput. Sci.*, pages 348–359. Springer, 2002.

Problem 53: Minimum-Turn Cycle Cover in Planar Grid Graphs

Statement What is the complexity of finding a cycle-cover of a planar grid graph that has the fewest possible 90° turns? (An 180° U-turn counts as two turns.) A *planar grid graph* is a graph whose vertices are any set of points on the planar integer lattice and whose edges connect every pair of vertices at unit distance.

Origin Aggarwal et al. [ACK+97] show that the more general problem of finding a cycle cover for a planar set of points that minimizes total turn angle is NP-hard. Arkin et al. [ABD+01] consider the problem in grid graphs, but are only able to give approximations.

Status/Conjectures Open.

Motivation Minimizing turns is a natural geometric measure; understanding its algorithmic behavior is of general interest.

Partial and Related Results [ABD+01] show that the problem is polynomially solvable when restricted to *thin* grid graphs, i.e., grid graphs that do not contain an induced 2×2 square. For this special case, the problem behaves somewhat similarly to a Chinese Postman Problem. The problem of finding a minimum-turn tour is known to be NP-complete, even for this special case.

More recent details and related problems can be found in the version [ABD+03].

Related Open Problems Minimum-turn cycle cover in a "solid" (genus-zero) grid graph: What is the complexity of finding a minimum-turn tour for a given planar grid graph without holes?

TSP in a solid grid graph: What is the complexity of finding a minimum-length tour for a given planar grid graph without holes? (Problem 54)

Appearances [ABD+01].

Categories traveling salesman; optimization; point sets; graphs

Entry Revision History S. P. Fekete, 12 Dec. 2003.

References

[ABD+01] E. M. Arkin, M. A. Bender, E. Demaine, S. P. Fekete, J. S. B. Mitchell, and S. Sethia. Optimal covering tours with turn costs. In *Proc. 13th ACM-SIAM Sympos. Discrete Algorithms*, pages 138–147, 2001.

[ABD+03] E. M. Arkin, M. A. Bender, E. Demaine, S. P. Fekete, J. S. B. Mitchell, and S. Sethia. Optimal covering tours with turn costs. Manuscript (submitted), 2003.

[ACK+97] Alok Aggarwal, Don Coppersmith, Sanjeev Khanna, Rajeev Motwani, and Baruch Schieber. The angular-metric traveling salesman problem. In *Proc. 8thh Annual ACM-SIAM Sympos. Discrete Algorithms*, pages 221–229, January 1997.

Problem 54: Traveling Salesman Problem in Solid Grid Graphs

Statement What is the complexity of finding a shortest tour in a solid planar grid graph? A *planar grid graph* is a graph whose vertices are any set of points on the planar integer lattice and whose edges connect every pair of vertices at unit distance. Distances between nodes correspond to induced shortest-path distances in the graph, which corresponds to "Manhattan" distances. A grid graph is *solid* if it does not have any holes, i.e., its complement in the planar integer lattice is connected.

Origin [IPS82] show that the problem is NP-complete in general planar grid graphs.

Status/Conjectures Open.

Motivation

Partial and Related Results [UL97] show that Hamiltonicity of a solid grid graph can be decided in polynomial time. Thus we can decide whether there is a tour of length equal to the number of vertices. In contrast, deciding Hamiltonicity is NP-hard in general planar grid graphs [IPS82].

[ABD+01] observe that finding the shortest tour is polynomially solvable when restricted to *thin* grid graphs, i.e., grid graphs that do not contain an induced 2×2 square. This problem asks about replacing the thin restriction with the solid restriction.

Related Open Problems Minimum-Turn Cycle Cover in Planar Grid Graphs (Problem 53)

Appearances Mentioned in [ABD+01].

Categories traveling salesman; optimization; point sets; graphs

Entry Revision History S. P. Fekete, 20 Dec. 2003; E. Demaine, 16 May 2004.

References

[ABD⁺01] E. M. Arkin, M. A. Bender, E. Demaine, S. P. Fekete, J. S. B. Mitchell, and S. Sethia. Optimal covering tours with turn costs. In *Proc. 13th ACM-SIAM Sympos. Discrete Algorithms*, pages 138–147, 2001.

[IPS82] A. Itai, C. H. Papadimitriou, and J. L. Szwarcfiter. Hamilton paths in grid graphs. *SIAM J. Comput.*, 11:676–686, 1982.

[UL97] Christopher Umans and William Lenhart. Hamiltonian cycles in solid grid graphs. In *Proc. 38th Annu. IEEE Sympos. Found. Comput. Sci.*, pages 496–507, 1997.

Problem 55: Pallet Loading

Statement What is the complexity of the pallet loading problem? Given two pairs of numbers, (A, B) and (a, b), and a number n, decide whether n small rectangles of size $a \times b$, in either axis-parallel orientation, can be packed into a large rectangle of size $A \times B$.

This problem is not even known to be in NP, because of the compact input description, and the possibly complicated structure of a packing, if there is one.

Origin Uncertain, pending investigation.

Status/Conjectures Open.

Motivation Natural packing problem; first-rate example of the relevance of coding input and output.

Partial and Related Results Tarnowsky [Tar92] showed that the problem can be solved in time polynomial in the size of the input if we are restricted to "guillotine" patterns, i.e., arrangements of items that can be obtained by a recursive sequence of edge-to-edge cuts. This result uses some nontrivial algebraic methods.

Related Open Problems What is the complexity of packing a maximal number of unit squares in a simple polygon? (Problem 54)

Appearances [Dow87] claims the problem to be NP-hard; [Exe88] claims the problem to be in NP; but both claims are erroneous. The precise nature of the difficulty is stated in [Nel93].

Categories packing; optimization

Entry Revision History S. P. Fekete, 17 Jan. 2004.

References

[Dow87] K. A. Dowsland. An exact algorithm for the pallet loading problem. *European Journal of Operational Research*, 31:78–84, 1987.

[Exe88] H. Exeler. *Das homogene Packproblem in der betriebswirtschaftlichen Praxis*. Physica-Verlag, Heidelberg, 1988.

[Nel93] J. Nelißen. New approaches to the pallet loading problem. Technical report, RWTH Aachen, 1993.

[Tar92] A. G. Tarnowsky. Exact polynomial algorithm for special case of the two-dimensional cutting stock problem: A guillotine pallet loading problem. Technical Report 9205 - DO, Belarusian State University, 1992.

Problem 56: Packing Unit Squares in a Simple Polygon

Statement What is the complexity of deciding whether a given number of axis-parallel unit squares can be packed into a simple polygon (without holes)?

Origin Unknown.

Status/Conjectures Open.

Motivation Natural packing problem.

Partial and Related Results The problem is known to be NP-hard for polygons with holes [FPT81], even if the polygon is an orthogonal polygon with all coordinates being multiples of 1/2. Recently this version of the problem was shown to be in NP [DEKIvO09], making it NP-complete.

The problem is the decision version for two optimization problems of very different behavior. There is a PTAS for packing the maximum number of squares of fixed size [HM85]. Maximizing the size of squares such that a fixed number of squares can be packed has a lower bound on approximation of 14/13, and there is a 3/2-approximation [BF01].

Related Open Problems What is the complexity of pallet loading? (Problem 55)

Appearances [BF01] conjecture the problem to be polynomially solvable.

Categories packing; optimization

Entry Revision History S. P. Fekete, 16 Jan. 2004; E. Demaine, 3 July 2009.

References

[BF01] C. Baur and S. P. Fekete. Approximation of geometric dispersion problems. *Algorithmica*, 30:450–470, 2001.

[DEKIvO09] Muriel Dulieu, Dania El-Khechen, John Iacono, and Nikolaj van Omme. Packing 2×2 unit squares into grid polygons is NP-complete. In *Proceedings of the 21st Canadian Conference on Computational Geometry*, Vancouver, Canada, August 2009.

[FPT81] R. J. Fowler, M. S. Paterson, and S. L. Tanimoto. Optimal packing and covering in the plane are NP-complete. *Inform. Process. Lett.*, 12(3):133–137, 1981.

[HM85] D. S. Hochbaum and W. Maas. Approximation schemes for covering and packing problems in image processing and VLSI. *J. Assoc. Comput. Mach.*, 32:130–136, 1985.

Problem 57: Chromatic Number of the Plane

Statement How many colors are needed to paint the plane so that no two points a unit distance apart are painted the same color? If the same question is asked of the line, the answer is 2: Coloring $[0, 1)$ red, $[1, 2)$ blue, etc., ensures that no two unit-separated points have the same color. One can view the question as asking for the chromatic number $\chi(\mathbb{E}^2)$ of the infinite *unit-distance graph* G, with every point in the plane a vertex, and an edge between two vertices if they are separated by a unit distance.

Origin Hadwider and Edward Nelson, 1944.

Status/Conjectures Open. Erdős and de Bruijn showed [EdB51] that the chromatic number of the plane is attained for some finite subgraph of G. This result led to narrowing the answer to $4 \le \chi(\mathbb{E}^2) \le 7$. For example, the lower bound of 4 is established by the "Moser graph."

The knowledge gap for the chromatic number of (3D) space is even wider than for the plane: it is only known to satisfy $6 \le \chi(\mathbb{E}^3) \le 15$. See [Gra04a, Gra04b] for further results and references.

There is now some evidence that the chromatic number of the plane may depend on the axioms of set theory. This was first seen possible in examples constructed by Saharon Shelah and Alexander Soifer. Now Payne [Pay09] has constructed unit-distance graphs with the same property.

Related Open Problems Problem 58.

Reward Ron Graham offers $1000 for a solution.

Appearances [O'R04]

Categories combinatorial geometry

Entry Revision History J. O'Rourke, 15 Aug. 2004; 6 July 2009.

References

[EdB51] P. Erdős and N. G. de Bruijn. A colour problem for infinite graphs and a problem in the theory of relations. *Indag. Math.*, 13:371–373, 1951.

[Gra04b] R. L. Graham. Open problems in Euclidean Ramsey theory. *Geocombinatorics*, XIII(4):165–177, April 2004.

[Gra04a] R. L. Graham. Euclidean Ramsey theory. In Jacob E. Goodman and Joseph O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, chapter 11, pages 239–254. CRC Press LLC, Boca Raton, FL, 2nd edition, 2004.

[O'R04] Joseph O'Rourke. Computational geometry column 46. *Internat. J. Comput. Geom. Appl.*, 14(6):475–478, 2004. Also in *SIGACT News*, **35**(3):42–45 (2004), Issue 132.

[Pay09] M. S. Payne. Unit distance graphs with ambiguous chromatic number. arXiv:0707.1177v2 [math.CO], 2009.

Problem 58: Monochromatic Triangles

Statement For any (planar) triangle T, is there is a 3-coloring of the (infinite) plane with no monochromatic copy of T? We imagine congruent copies of T moved around the plane via rigid motions, and seek a spot where T is monochromatic. T is *monochromatic* if its three vertices are painted the same color, by virtue of lying on points of the plane painted that color. Note that the coloring in the question may depend on the given triangle T.

Origin Ron Graham, MSRI, August 2003.

Status/Conjectures Open. Ron Graham conjectures that the answer is YES for all triangles T.

Motivation The question of the chromatic number of the Euclidean plane $\mathbb{E}^2$ has been unresolved for over fifty years (Problem 57). This problem is an interesting, much more restricted variant, posed by Ron Graham as part of his "Geometric Ramsey Theory"" investigation [Gra04a] [Gra04b] at his MSRI lectures[6] in August 2003.

Partial and Related Results See [O'R04] for further explanation.

Related Open Problems Problem 57.

Reward Ron Graham offers $50 for a solution.

Appearances [O'R04]

Categories combinatorial geometry

Entry Revision History J. O'Rourke, 15 Aug. 2004.

References

[Gra04b] R. L. Graham. Open problems in Euclidean Ramsey theory. *Geocombinatorics*, XIII(4):165–177, April 2004.

[Gra04a] R. L. Graham. Euclidean Ramsey theory. In Jacob E. Goodman and Joseph O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, chapter 11, pages 239–254. CRC Press LLC, Boca Raton, FL, 2nd edition, 2004.

[O'R04] Joseph O'Rourke. Computational geometry column 46. *Internat. J. Comput. Geom. Appl.*, 14(6):475–478, 2004. Also in *SIGACT News*, **35**(3):42–45 (2004), Issue 132.

Problem 59: Most Circular Partition of a Square

Statement What is the optimal partition of a square into convex pieces such that the circularity of the pieces is optimized? The *circularity* of a polygon is the ratio of the radius of its smallest circumscribing circle to the radius of its largest inscribed circle. Thus circular pieces have circularity near 1, and noncircular pieces have circularity greater than 1. An optimal partition minimizes the maximum ratio over all pieces in the partition.

Origin [DO03a]

Status/Conjectures Open.

Motivation This is a type of "fat" partition.

[6] http://www.msri.org/publications/video/index07.html

Partial and Related Results It is known from [DO03a] that the equilateral triangle requires an infinite number of pieces to achieve the optimal circularity of 1.5, and that for all regular k-gons, for $k \geq 5$, the one-piece partition is optimal. The square is a difficult intermediate case. It is known that the optimal ratio lies in the narrow interval $[1.28868, 1.29950]$. The upper bound is established by the 92-piece partition shown in Figure 2. It is conjectured in [DO03a] that, as with the equilateral triangle

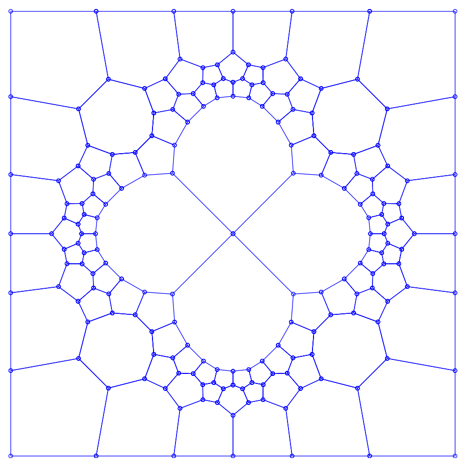

Figure 2: 92-piece partition achieving 1.29950

case, no finite partition achieves the optimal ratio, but rather optimality can be approached as closely as desired as the number of pieces goes to infinity.

Categories packing; meshing

Entry Revision History J. O'Rourke, 16 Aug. 2004.

References

[DO03a] Mirela Damian and Joseph O'Rourke. Partitioning regular polygons into circular pieces I: Convex partitions. In *Proc. 15th Canad. Conf. Comput. Geom.*, pages 43–46, 2003. arXiv:cs.CG/030402.

Problem 60: Transforming Polygons via Vertex-Centroid Moves

Statement Given an arbitrary polygon, transform it by a finite sequence of "vertex-centroid" moves to a regular polygon. A *vertex-centroid move* is a translation of a vertex v along the line vm, where m is the centroid of the vertices of the polygon, i.e., $1/n$-th of the sum of the vertex coordinates. Vertices may move only one at a time, but in any order and any number of times.

Origin Steve Gray, 2003.

Status/Conjectures Open.

Partial and Related Results Let $v(t)$ and $m(t)$ be the positions of the moving vertex and centroid as a function of time t, where t runs from 0 to 1 during the vertex translation. Let L be the line containing $v(0)m(0)$. As $v(t)$ moves on L, $m(t)$ remains on L.

For $n = 3$, a triangle can be made equilateral in two moves. Already for $n = 4$ the situation is less clear.

One could set many other transformational goals besides achieving regularity: scale the polygon by $s > 0$, rotate the polygon, etc. The notion generalizes to arbitrary dimensions.

A more difficult variant would be to use the area centroid rather than the vertex centroid, in which case $m(t)$ does not remain on L, so that a vertex move would have the flavor of pursuit of a moving target.

Appearances

Categories polygons

Entry Revision History J. O'Rourke, 1 Aug. 2005; S. Gray, 15 Aug. 2005.

Problem 61: Lines Tangent to Four Unit Balls

Statement Given a set of n unit-radius balls in $\mathbb{R}^3$, what is the number of lines that are tangent to four of the balls in the set, and miss all the others? (The balls are not necessarily disjoint.)

Origin [AAKS05].

Status/Conjectures Open, conjectured to be $\Omega(n^3)$.

Motivation The number of lines tangent to four unit balls dominates the combinatorial complexity of the space of lines that avoid all the balls. And this complexity is related to questions in visibility and in optimization.

Partial and Related Results In [AAKS05] it is established that the number is $O(n^{3+\epsilon})$ for any $\epsilon > 0$. The best lower bound is $\Omega(n^2)$, which can be achieved, for example, as follows.

Place $n/4$ balls separated along a horizontal line L_1, and another $n/4$ along a parallel line L_2 below, with each of the lower balls directly below an upper ball with their centers 1 unit apart. Thus each pair of balls overlap, their surfaces intersecting in a circle. Arrange a second set of $n/4$ pairs of intersecting balls along lines L_3 and L_4, far from L_1/L_2 and with all four lines parallel, and such that all circles of sphere intersections are coplanar. Now it is easy to see that a line tangent to two circles of intersection, one from the L_1/L_2 group, one from the L_3/L_4 group, is tangent to four balls. And there are $\Omega(n^2)$ such lines. (The same bound can be achieved with disjoint balls with a similar arrangement, but the analysis is slight more complex.)

The problem is also interesting if all balls are disjoint; it is not clear if disjointness affects the answer asymptotically.

Appearances [AAKS05].

Categories combinatorial geometry

Entry Revision History J. O'Rourke, 25 Aug. 2005.

References

[AAKS05] Pankaj K. Agarwal, Boris Aronov, Vladlen Koltun, and Micha Sharir. Lines avoiding unit balls in three dimensions. *Discrete Comput. Geom.*, 34:231–250, 2005.

Problem 62: Volume Maximizing Convex Shape

Statement Let C be a convex piece of paper; its boundary may be a smooth curve, or a polygon. A *perimeter halving folding* is a folding of C obtained by identifying two points x and y on the boundary of C that halve the perimeter, and then folding C by "gluing" xy to yx. This always results in a unique convex shape in 3D, a polyhedron if C is a convex polygon [DO07b]. What unit-area shape C achieves the maximum volume possible via a perimeter-halving folding?

Origin Posed by Joseph Malkevitch in 2002, in a slightly different form: for polygons, and not restricting the folding to perimeter-halving. The modifications above were suggested at CCCG'05 [DO06]. The restriction to perimeter halving eliminates some more complex foldings possible for some convex polygons, and so in that sense simplifies the problem. The extension to smooth shapes is a natural generalization. Smooth shapes only admit perimeter-halving foldings.

Status/Conjectures Open.

Partial and Related Results Even fixing the shape and finding the maximum volume perimeter halving for that shape is difficult. For a circular disk, all perimeter halvings lead to a flat doubly-covered half disk, all of volume zero. The only other shape for which the answer is known, and then only empirically, is the case of C a square [ADO03]. The resulting polyhedron of 6 vertices and 8 faces, shown in Fig. 3, achieves about 60% of the volume of a unit-area sphere.

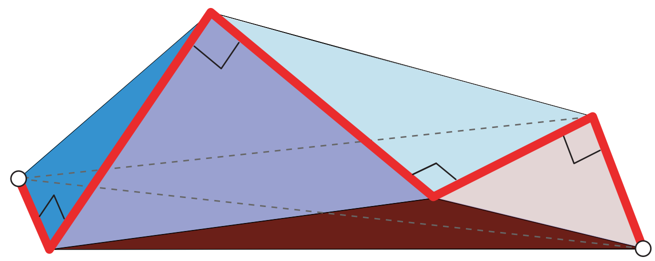

Figure 3: The maximum volume convex polyhedron foldable from a square.

Appearances [DO06].

Categories folding and unfolding

Entry Revision History J. O'Rourke, 26 Aug. 2005.

References

[ADO03] Rebecca Alexander, Heather Dyson, and Joseph O'Rourke. The convex polyhedra foldable from a square. In *Proc. 2002 Japan Conf. Discrete Comput. Geom.*, volume 2866 of *Lecture Notes Comput. Sci.*, pages 38–50. Springer-Verlag, 2003.

[DO07b] Erik D. Demaine and Joseph O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra*. Cambridge University Press, 2007. In press. `http://www.gfalop.org` (formerly `http://www.fucg.org`).

[DO06] Erik D. Demaine and Joseph O'Rourke. Open problems from CCCG 2005. In *Proc. 18th Canad. Conf. Comput. Geom.*, pages 75–80, 2006.

Problem 63: Dynamic Planar Nearest Neighbors

Statement Is there a data structure maintaining a set of n points in the plane subject to insertions, deletions, and nearest-neighbor queries in $O(\log n)$ time? A *nearest-neighbor query* asks to find a point among the set that is nearest (in Euclidean distance) to a given a point in the plane. This problem reduces to maintaining the convex hull of a set of n points in 3D subject to insertions, deletions, and extreme-point queries.

Origin Uncertain, pending investigation.

Status/Conjectures Open.

Motivation This problem is the natural generalization of (1D) search trees to 2D. Standard balanced search tree data structures can maintain n points on the real line subject to insertion, deletion, and predecessor and successor queries (and thus nearest-neighbor queries) in $O(\log n)$ time per operation. (More sophisticated data structures even attain $O(1)$ time per update.)

Partial and Related Results For 14 years, the authority on this problem was Agarwal and Matoušek's FOCS'92 paper [AM95] which describes two data structures: one suports updates in $O(n^\epsilon)$ amortized time and queries in $O(\log n)$ worst-case time, while the other suports updates in $O(\log^2 n)$ amortized time queries in $O(n^\epsilon)$ worst-case time, for any $\epsilon > 0$. The nearest-neighbor problem is a decomposable search problem, so when deletions are forbidden, the general techniques of Bentley and Saxe [BS80] yield an $O(\log^2 n)$ amortized bound for updates and queries. In 2006, Chan [Cha06] obtained the first polylogarithimc data structure for 3D convex hulls and therefore 2D nearest neighbors. His data structure supports insertions in $O(\log^3 n)$ expected amortized time, deletions in $O(\log^6 n)$ expected amortized time, and extreme-point or nearest-neighbor queries in $O(\log^2 n)$ worst-case time.

Related Open Problems Problem 12.

Categories convex hulls; data structures; Voronoi diagrams

Entry Revision History E. Demaine, 24 Jan. 2006.

References

[AM95] P. K. Agarwal and J. Matoušek. Dynamic half-space range reporting and its applications. *Algorithmica*, 13:325–345, 1995.

[BS80] J. L. Bentley and J. B. Saxe. Decomposable searching problems I: Static-to-dynamic transformations. *J. Algorithms*, 1:301–358, 1980.

[Cha06] Timothy Chan. A dynamic data structure for 3-d convex hulls and 2-d nearest neighbor queries. In *Proceedings of the 17th ACM-SIAM Symposium on Discrete Algorithms*, 2006. to appear.

Problem 64: Edge-Unfolding Polycubes

Statement Is there any genus-zero orthogonal polyhedron P built by gluing together cubes face-to-face that cannot be edge-unfolded, where all cube edges on the surface of P are considered edges available for cutting? These orthogonal polyhedra are sometimes known as *polycubes*, 3D versions of 2D *polyominoes*.

Origin George Hart and Joseph O'Rourke, 2004.

Status/Conjectures Open.

Motivation More general problems seem even more difficult.

Partial and Related Results This is a special case of a more general problem, which is equally open. The goal, as in Problem 9, is to cut the surface and unfold without overlap. An *edge unfolding* only permits cutting along edges of the polyhedron. A *grid unfolding* adds extra edges to the surface by intersecting the polyhedron with planes parallel to coordinate planes through every vertex, and so is easier to edge-unfold. Easier still is the posed problem: The orthogonal polyhedron is built from cubes, and all cube edges are available for cutting. Is there any such polyhedron that cannot be edge-unfolded? Such an example would narrow the options, but it may be that every orthogonal polyhedron can be grid-unfolded. (An easy box-on-box example [BDD+98] shows that without some surface refinement [DO05], not all orthogonal polyhedra can be edge-unfolded.) The posed question is among the most specific whose answer would make progress.

Only a few narrow subclasses of orthogonal polyhedra are known to have grid-unfolding algorithms: orthotubes, orthostacks of orthogonally convex slabs, and orthogonal terrains. See [O'R08].

Related Open Problems Problem 9: Edge-Unfolding Convex Polyhedra.
Problem 42: Vertex-Unfolding Polyhedra.
Problem 43: General Unfolding of Nonconvex Polyhedra.

Appearances [DO07b]

Categories folding and unfolding; polyhedra

Entry Revision History J. O'Rourke, 14 Jul 2006, 16 Jul 2007.

References

[BDD+98] Therese Biedl, Erik D. Demaine, Martin L. Demaine, Anna Lubiw, Joseph O'Rourke, Mark Overmars, Steve Robbins, and Sue Whitesides. Unfolding some classes of orthogonal polyhedra. In *Proc. 10th Canad. Conf. Comput. Geom.*, pages 70–71, 1998. Full version in *Elec. Proc.*: `http://cgm.cs.mcgill.ca/cccg98/proceedings/cccg98-biedl-unfolding.ps.gz%`.

[DO07b] Erik D. Demaine and Joseph O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra.* Cambridge University Press, 2007. In press. `http://www.gfalop.org` (formerly `http://www.fucg.org`).

[DO05] Erik D. Demaine and Joseph O'Rourke. Open problems from CCCG 2004. In *Proc. 17th Canad. Conf. Comput. Geom.*, pages 303–306, 2005.

[O'R08] Joseph O'Rourke. Unfolding orthogonal polyhedra. In J.E. Goodman, J. Pach, and R. Pollack, editors, *Proc. Snowbird Conference Discrete and Computational Geometry: Twenty Years Later*, pages 307–317. American Mathematical Society, 2008.

Problem 65: Magic Configurations

Statement Let a finite set of points P in the plane be given, with each point assigned a positive real weight. P is called a *magic configuration* if every line determined by two or more points has the same sum of weights, i.e., the sum of the weights of the points through which each line passes is the same. The problem is to prove or disprove that there are only four essentially distinct magic configurations:

1. Points in general position, with (e.g.) every point assigned weight 1.
2. All points collinear.
3. $n-1$ points collinear with weight (e.g.) 1, and one point not on that line with weight $n-2$.
4. The 7-point configuration shown in Fig. 4, or its projective equivalents.

Origin [Mur71]

Status/Conjectures Settled positively, 2007: [ABK$^+$08]

Motivation The terminology "magic configuration" comes from the notion of *magic squares*, 2D matrices such that every row, column, and (optionally) diagonal sums to the same value.

Partial and Related Results An *ordinary line* is one that passes through exactly two points. Scaling weights of a magic configuration so that the weights on each line sum to 1, the weight of the points on ordinary lines must be $\frac{1}{2}$ in any magic configuration other than the third example above. It is known that, for $n \geq 3$ noncollinear points, at least $\frac{6}{13}n$ lines must be ordinary [CS93].

Settled in [ABK$^+$08], the journal version of a paper that originally appeared in the Proceedings of the 2007 Symposium on Computational Geometry.

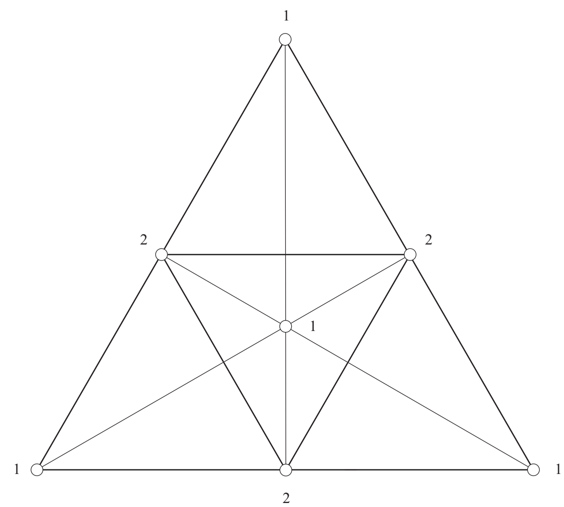

Figure 4: Edge midpoints have weight 2, while all other points have weight 1. All nine lines have sum 4.

Appearances Originally posed by U. S. R. Murty in [Mur71]. Reposed by Murty at a June 2006 celebration of V. Chvátal's 60th birthday. Two people who heard this posing, X. Chen and P. Taslakian, brought the problem to the conference *Discrete and Computational Geometry—Twenty Years Later* in Snowbird, June 2006. In particular, Chen posed the problem at the open-problem session.

Categories point sets

Entry Revision History J. O'Rourke, 14 Jul. 2006; E. Demaine, 15 Jul. 2006; J. O'Rourke, 16 Jul. 2008.

References

[ABK+08] Eyal Ackerman, Kevin Buchin, Christian Knauer, Rom Pinchasi, and Günter Rote. There are not too many magic configurations. *Discrete Comput. Geom.*, 39:3–16, 2008.

[CS93] J. Csima and E. T. Sawyer. There exist $6n/13$ ordinary points. *Discrete & Computational Geometry*, 9(1):187–202, December 1993.

[Mur71] U. S. R. Murty. How many magic configurations are there?. *American Mathematical Monthly*, 78(9):1000–1002, November 1971.

Problem 66: Reflexivity of Point Sets

Statement Let $\rho(S)$ be the fewest number of reflex vertices in a polygonization of a 2D point set S, i.e., the fewest reflexivities of any simple polygon whose vertex set is S. Let $\rho(n)$ be the maximum of $\rho(S)$ over all sets S with n points. What is $\rho(n)$?

Origin [AFH$^+$03]

Status/Conjectures Open.

Partial and Related Results In [AFH$^+$03] the authors prove that $\lfloor n/4 \rfloor \leq \rho(n) \leq \lceil n/2 \rceil$ and conjecture that $\rho(n) = \lfloor n/4 \rfloor$. The upper bound was recently improved to $\frac{5}{12}n + O(1) \approx 0.4167n$ in [AAK08].

Related Open Problems Problem 16: Simple Polygonalizations.

Categories polygons; point sets.

Entry Revision History J. O'Rourke, 3 Aug. 2006; 16 Jul 2008.

References

[AAK08] Eyal Ackerman, Oswin Aichholzer, and Balazs Keszegh. Improved upper bounds on the reflexivity of point sets. *Comput. Geom.: Theory Appl.*, 2008. To appear.

[AFH$^+$03] Esther M. Arkin, Sándor P. Fekete, Ferran Hurtado, Joseph S. B. Mitchell, Marc Noy, Vera Sacristán, and Saurabh Sethia. On the reflexivity of point sets. In B. Aronov, S. Basu, J. Pach, and M. Sharir, editors, *Discrete and Computational Geometry: The Goodman-Pollack Festschrift*, pages 139–156. Springer, 2003.

Problem 67: Fair Partitioning of Convex Polygons

Statement Define a *fair partitioning* of a polygon as a partition of it into a finite number of pieces so that every piece has both the same area and the same perimeter. If all the resulting pieces are convex, call it a *fair convex partitioning*. Given any positive integer n, can any convex polygon be convex fair partitioned into n pieces?

If the answer is "Not always," how does one decide the possibility of such a partitioning for a given polygon and a given n? And if a fair convex partition exists for a specific polygon, how does one find a fair partitioning that minimizes the total length of the cut segments, or minimizes the sum of the perimeters of the pieces?

And finally, what could one say about higher dimensional analogs of this question?

Origin Posed by R. Nandakumar and N. Ramana Rao, June 2007.

Status/Conjectures Open. The originators tend to believe every convex polygon allows a fair convex partition into n pieces for any n. No published work specifically on this problem is known.

Partial and Related Results See [NR08] for an introduction and survey and proof that the conjecture holds for $n = 2$, and a proposed new proof for $n = 4$. This survey cites a new result of Barany, Blagojevic, and Szucs [forthcoming] that establishes the conjecture for $n = 3$. The cited survey also sketches a (different) argument for $n = 3$.

There is work on partitioning convex polygons into equal area convex pieces so that every piece equally shares the boundary of the given target polygon: [ANRCU98] [AKK+98].

A proof of a weaker result—that any polygon allows fair partitioning for any n (where the pieces need not be convex) is proposed at `http://nandacumar.blogspot.com/2006/10/`

Categories polygons; partitioning

Entry Revision History R. Nandakumar and N. Ramana Rao, 14 Jul 2007, 17 Sep 2007; J. O'Rourke, 1 Jan 2009; 23 Jan. 2009.

References

[AKK+98] Jin Akiyama, A. Kaneko, M. Kano, Gisaku Nakamura, Eduardo Rivera-Campo, S. Tokunaga, and Jorge Urrutia. Radial perfect partitions of convex sets in the plane. In *Japan Conf. Discrete Comput. Geom.*, pages 1–13, 1998.

[ANRCU98] Jin Akiyama, Gisaku Nakamura, Eduardo Rivera-Campo, and Jorge Urrutia. Perfect divisions of a cake. In *Proc. Canad. Conf. Comput. Geom.*, pages 114–115, 1998.

[NR08] R. Nandakumar and N. Ramana Rao. 'Fair' partitions of polygons—an introduction. `http://arxiv.org/abs/0812.2241`, 2008.

Problem 68: Rolling a Die over a Labeled Board

Statement Label the faces of a unit cube with numbers 1–6 as in a die. Place the cube to sit on an integer lattice grid, with one corner at the origin and sides aligned with the axes. Completely label every lattice square of a rectangular "board" R, whose corner is at the origin, with numbers in

$\{1, 2, 3, 4, 5, 6\}$. The problem is to roll the cube over its edges so that, for each square $s \in B$ labeled l, the cube lands on s precisely once, and when it does so, the top face of the cube has label l.

What is the computational complexity of solving an instance of this problem?

Origin Version posed by O'Rourke at the 2005 *Canadian Conference on Computational Geometry* [DO06], and subsequently substantially developed and embellished in [BBD$^+$07].

Status/Conjectures Open.

Motivation This problem was inspired by van Deventer's "Rolling block mazes" [vD04]. The paper [BBD$^+$07] uncovered a rich history to rolling cube puzzles going back to the 1960's, which will not be repeated here.

Partial and Related Results The original posed problem labeled an arbitrary connected set S of squares, rather than a rectangular board R; the cells outside of S are *free*, and may be visited any number of times with any number on the die top. That former problem is solved in [BBD$^+$07], which establishes that, as conjectured, the problem is NP-complete.

The posed problem has no free cells, and in fact the labels are all in a rectangular board R. This seems the most interesting specific variant, for it is left possible in [BBD$^+$07] that, if there is a solution for R, it is "uniquely rollable." They establish that there are boards with labeled and *blocked* (i.e., forbidden) cells for which rollable Hamiltonian cycles are not unique, but they leave open fully labeled boards.

Appearances [DO06]; see above.

Categories combinatorial geometry

Entry Revision History J. O'Rourke, 17 Jul 2007.

References

[BBD$^+$07] Kevin Buchin, Maike Buchin, Erik D. Demaine, Martin L. Demaine, Dania El-Khechen, Sandor Fekete, Christian Knauer, André Schulz, and Perouz Taslakian. On rolling cube puzzles. In *Proc. 19th Canad. Conf. Comput. Geom.*, pages 141–148, 2007.

[DO06] Erik D. Demaine and Joseph O'Rourke. Open problems from CCCG 2005. In *Proc. 18th Canad. Conf. Comput. Geom.*, pages 75–80, 2006.

[vD04] M. Oskar van Deventer. Rolling block mazes. In Barry Cipra, Erik D. Demaine, Martin L. Demaine, and Tom Rodgers, editors, *A Tribute to a Mathemagician*, pages 241–250. A K Peters, November 2004.

Problem 69: Isoceles Planar Graph Drawing

Statement Given a planar graph that is interior triangulated (all interior faces are triangles), is there a staight-line drawing of the graph such that each face is an isoceles triangle (i.e., it has two equal-length sides)?

The problem is worth studying both when the drawing must be planar (no crossings allowed) and when it is not.

If such drawings exist, then it is also worth studying what grid-size is needed, and whether it can be done with integer coordinates at all. If such drawings do not always exist, NP-hardness should be investigated.

Origin Joe Malkovitch at Graph Drawing '99.

Status/Conjectures Open.

Partial and Related Results If the graph is a planar 3-tree (i.e., can be obtained by starting from a triangle and repeatedly adding a vertex of degree 3 inside a face, adjacent to all other vertices in the face), then such a drawing can easily be obtained by always placing the vertex at the centroid of the face. However, this drawing will in general be non-planar.

Of particular interest are therefore planar graphs of treewidth 4 and higher.

Categories graph drawing; planar graphs

Entry Revision History T. Biedl, 2 Dec. 2008; J. O'Rourke, 29 Dec. 2008.

Problem 70: Yao-Yao Graph a Spanner?

Statement Is the Yao-Yao Graph a t-spanner for constant t? A geometric graph is a *t-spanner* (or just a *spanner*) if, for every pair of nodes, the shortest distance between the nodes following the edges of the graph is at most t times the Euclidean distance between them. See below for the definition of the Yao-Yao graph.

Origin [WL02](?)

Status/Conjectures Open.

Partial and Related Results The Yao graph YG_k [Yao82] is defined as follows. At each node u, any k equal-separated rays originated at u define k cones. In each cone, choose the shortest edge uv among all edges from u, if there are any, and add a directed edge $\overrightarrow{uv}$ to YG_k. It is known that YG_k, $k > 6$, is a t-spanner, for $t = 1/[1 - 2\sin(\pi/k)]$. The Yao-Yao graph YY_k [WL02] starts with the directed Yao graph, and reduces the maximum degree of nodes as follows. At each node u, all incoming edges from each cone are discarded, except for the shortest one $\overrightarrow{vu}$. And now the result is treated as an undirected graph. Much is known about YY_k, but whether or not it is a t-spanner remains open.

Categories spanners; geometric graphs

Entry Revision History J. O'Rourke, 29 Dec. 2008.

References

[WL02] Yu Wang and Xiang-Yang Li. Distributed spanner with bounded degree for wireless ad hoc networks. In *IPDPS '02: Proc. of the 16th IEEE Int. Parallel and Distributed Processing Symposium*, pages 194–201, 2002.

[Yao82] A. C. Yao. On constructing minimum spanning trees in k-dimensional spaces and related problems. *SIAM J. Comput.*, 11(4):721–736, 1982.

Problem 71: Stretch-Factor for Points in Convex Position

Statement For points S in convex position (i.e., every point is on the hull of S), is the Delaunay triangulation of S a $(\pi/2)$-spanner? A geometric graph is a t-*spanner* (or just a *spanner*) if, for every pair of nodes, the shortest distance between the nodes following the edges of the graph is at most t times the Euclidean distance between them. The constant t is the *stretch factor* or *dilation*.

Origin Prosenjit Bose [DO08].

Status/Conjectures Open.

Partial and Related Results Chew conjectured that the Delaunay triangulation is a t-spanner [Che89] for some constant t. Dobkin et al. [DFS90] established this for $t = \pi(1+\sqrt{5})/2 \approx 5.08$. The value of t was improved to $2\pi/(3\cos(\pi/6)) \approx 2.42$ by Keil and Gutwin [KG92], and further strengthened in [BM04]. Chew showed that t is $\pi/2 \approx 1.57$ for points on a circle, providing a lower bound. "It is widely believed that, for every set of points in $\mathbb{R}^2$, the Delaunay triangulation is a $(\pi/2)$-spanner" [NS07, p. 470].

This history suggests the special case posed above.

There is a new forthcoming result: [CKX09].

Appearances [DO08].

Categories spanners; Delaunay triangulations

Entry Revision History J. O'Rourke, 29 Dec. 2008; 4 July 2009.

References

[BM04] P. Bose and P. Morin. Online routing in triangulations. *SIAM J. Comput.*, 33:937–951, 2004.

[Che89] L. P. Chew. There are planar graphs almost as good as the complete graph. *J. Comput. Syst. Sci.*, 39:205–219, 1989.

[CKX09] Shiliang Cui, Iyad Kanj, and Ge Xia. On the dilation of Delaunay triangulations of points in convex position. In *Proc. Canad. Conf. Comp. Geom.*, 2009. To appear, Aug. 2009.

[DFS90] D. P. Dobkin, S. J. Friedman, and K. J. Supowit. Delaunay graphs are almost as good as complete graphs. *Discrete Comput. Geom.*, 5:399–407, 1990.

[DO08] Erik D. Demaine and Joseph O'Rourke. Open problems from CCCG 2007. In *Proc. 20th Canad. Conf. Comput. Geom.*, 2008.

[KG92] J. M. Keil and C. A. Gutwin. Classes of graphs which approximate the complete Euclidean graph. *Discrete Comput. Geom.*, 7:13–28, 1992.

[NS07] Giri Narasimhan and Michiel Smid. *Geometric Spanner Networks.* Cambridge University Press, 2007.

Problem 72: Polyhedron with Regular Pentagon Faces

Statement Let M be a closed polyhedral surface homeomorphic to S^2 which is entirely composed of equal regular pentagons. If M is immersed in 3-space, is it necessarily the boundary of a union of solid dodecahedra that are glued together at common facets?

Origin Richard Kenyon, first posed in 2006.

Status/Conjectures Open.

Partial and Related Results The corresponding question for equal squares has a positive answer. The question for surfaces embedded in 3-space is also interesting and open. The Kepler-Poinsot great dodecahedron has regular pentagon faces, and is immersed, but is not homeomorphic to S^2 $(V - E + F = -6)$.

Appearances Re-posed at Oberwolfach Workshop, Jan. 2009.

Categories polyhedra

Entry Revision History J. O'Rourke, 23 Jan. 2009.

Problem 73: Congruent Partitions of Polygons

Statement Partition a given polygon P into n mutually congruent pieces so that the area of P not covered by the union of the pieces is as small as possible. A partition which leaves out the least area is an *optimal* congruent partition for that n. If a congruent partition is a perfect cover, leaving no area uncovered, then it is called a *perfect congruent partition*. Two polygons are *congruent* if one can be made to coincide with the other by translation, rotation, or reflection (flipping over).

Origin Posed by R. Nandakumar, May 2009.

Status/Conjectures Please see below.

Partial and Related Results

1. It is known that there exist quadrilaterals with no perfect congruent partition for any n: `http://domino.research.ibm.com/Comm/wwwr_pond`
2. Deciding whether P has a perfect congruent partition appears little explored for $n > 2$. The case of $n = 2$ is solved in [EKFIR08] with an $O(n^3)$ algorithm.
3. If congruence is restricted to translation and rotation only, to what extent does the problem change?
4. Can the left-over area be upper-bounded as a function of P and n?

Related Open Problems Problem 67

Categories polygons; partitioning; dissections

Entry Revision History R. Nandakumar, 13 May 2009; J. O'Rourke, 8 July 2009.

References

[EKFIR08] Dania El-Khechen, Thomas Fevens, John Iacono, and Günter Rote. Partitioning a polygon into two mirror congruent pieces. In *Proc. 20th Canad. Conf. Comput. Geom.*, pages 131–134, August 2008.

Problem 74: Slicing Axes-Parallel Rectangles

Statement Let us say that two rectangles in the place are independent if both their x- and y-axis projections are disjoint. A set of rectangles is then independent if the rectangles are pairwise independent. Suppose that a collection of axes-parallel rectangles contains no independent set of size m or greater. What is the minimal number, $f(m)$, of horizontal and vertical lines needed to slice every rectangle in the collection?

Origin Vincent Vatter, Jun 2009.

Status/Conjectures It is known that $f(m)$ exists and is at most exponential.

Partial and Related Results The problem arises in the study of permutation classes, see [Vat08], where it was proved that $f(m)$ exists and is at most exponential.

Categories combinatorial geometry

Entry Revision History V. Vatter, 24 June 2009

References

[Vat08] Vincent Vatter. Small permutation classes. arXiv:0712.4006v2 [math.CO], 2008.