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Computability of Julia Sets

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Computability of Julia Sets

With 31 Figures

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To our families

Preface

Among all computer-generated mathematical images, Julia sets of rational maps occupy, perhaps, the most prominent position. Their beauty and complexity can be fascinating. They also hold a deep mathematical content, and numerical experiments have become a defining feature of the subject of Complex Dynamics.

Computational hardness of Julia sets is the main subject of this book. By definition, a computable set in the plane can be visualized on a computer screen with an arbitrarily high magnification. In this definition the running time of the visualization algorithm is not limited.

Countless programs to visualize Julia sets have been written. Yet, as we will see,

it is possible to constructively produce examples of quadratic polynomials, whose Julia sets are not computable.

In a way, this result is striking – it says that while a dynamical system can be described numerically with an arbitrary precision, the picture of the dynamics cannot be visualized.

As one indication of how unusual this is, consider the following. Another interesting object for a quadratic polynomial is its filled Julia set. It is obtained by "filling in" all the holes in the Julia set itself. In doing this, the computable properties of the picture can change dramatically:

a filled Julia set is always computable.

The non-computability phenomenon is very subtle, and in describing it we will require a very precise analytic machinery. Many of the techniques we use have only become available in the last few years. Perversely, we are able to construct non-computable examples of Julia sets because we understand Julia sets so well.

Non-computability turns out to be rare. Most Julia sets *are* computable. Their computational hardness, however, may vary. The running time required to produce a high-resolution image of a computable Julia set may be prohibitively high. Already we have seen some further surprises – a class of Julia sets (Julia sets of quadratic polynomials with parabolic orbits) empirically thought of as hard to compute turns out to be easy (and with a practical algorithm).

Our understanding of the computational complexity of Julia sets is in its first stages. Examples of a truly pathological kind (Julia sets of quadratic polynomials with Cremer periodic orbits) turn out to always be computable. No informative pictures of this type have ever been produced, as the running time of all presently existing algorithms renders them impractical. However, it is not known if they are *ever* computationally hard. This is probably the case at least sometimes, but it may also be possible that some of them can be visualized effectively by a clever algorithm. Many interesting problems await further study here.

The goal of the present book is to summarize our present knowledge about the computational properties of Julia sets in a fashion that is as self-contained as possible. While we have found the interplay between theoretical Computer Science and Dynamical Systems extremely fruitful, it makes the presentation more challenging. We have striven to make the book accessible and interesting to experts in both fields. The book assumes no prior knowledge of computability theory, and only a basic familiarity with complex analysis.

We start the book with an introduction to computability theory (Chapter 1) and a survey of the basic principles of dynamics of rational maps (Chapter 2). In Chapter 3 we begin the study of the computability and complexity of Julia sets by looking at some typical examples. We discuss the general positive results in Chapter 4. Non-computability appears in Chapter 5. Chapter 6 serves to understand the topological structure of non-computable examples in more detail.

The material we view as “optional reading” is

 typeset like this.

It is either not directly related to the main storyline, or is too technical, and is directed towards experts in one of the two fields.

Acknowledgments

It is our pleasure to thank our friend and colleague Ilia Binder for the many useful discussions on the computability of Julia sets. Some of the material presented in this book is based on our joint works. We are grateful to John Milnor for his interest in our work, and many insightful comments and questions, which have inspired much of our study. We thank our colleague Michael Shub for useful discussions of our results. Alan Baker's comments were very helpful to us in understanding the connection between number-theoretical properties of parameters and computability of Julia sets. M.B. would like to thank Stephen Cook for his guidance and for sharing his insights on Complexity Theory. Our colleague Boris Khesin was instrumental in getting this book to the publisher.

We thank the Mathematics and Computer Science Departments of the University of Toronto for the wonderful working environment. M.Y. wishes to thank IHÉS, and FIM at ETHZ, and both M.Y. and M.B. thank IPAM at UCLA, for the hospitality extended to us during our visits to these institutes while working on this book. Our research was partially supported by NSERC, through its Canada Graduate Scholarship (M.B.) and a Discovery Grant (M.Y.).

As this work was taking shape, the questions, enthusiasm, and even skepticism of our colleagues have been an invaluable motivation for us. We thank them all.

Contents

List of Notation

$B(y,r)$	the ball with center $y \in \mathbb{R}^n$ and radius r;
$B(Y,r)$	the r-neighborhood of the set Y in $\mathbb{R}^n$;
$\mathbb{U}$	the unit disk in $\mathbb{C}$;
$\mathbb{D}$	the set of dyadic rationals;
$\hat{\mathbb{C}}$	the Riemann sphere;
$\mathbb{T}$	the circle $\mathbb{R}/\mathbb{Z}$;
$\mathscr{M}$	the Mandelbrot set;
$\mathrm{Crit}(R)$	the set of critical points of a rational map R;
$\mathrm{Postcrit}(R)$	the postcritical set of R;
K_n^*	the set of all compact subsets of $\mathbb{R}^n$;
$\mathbb{R}_{\mathscr{C}}$	the set of all computable real numbers;
M^ϕ	an oracle Turing Machine;
$\mathbb{T}$	the circle $\mathbb{R}/\mathbb{Z}$;
S^1	the unit circle $\{\lvert z\rvert = 1\} \subset \mathbb{C}$;
$\mathscr{C}$	the set of finite unions of closed dyadic balls in $\mathbb{R}^k$;
$\mathbb{R}_{\mathscr{C}}$	the field of computable real numbers;
$\mathbb{C}_{\mathscr{C}}$	the field of computable complex numbers;
f^n	unless otherwise specified, the n-th iterate $\underbrace{f \circ f \circ \cdots \circ f}_{n}$;
$J(R)$	the Julia set of the function R;
$K(p)$	the filled Julia set of the polynomial p;
J_c	the Julia set $J(z^2+c)$;
K_c	the filled Julia set $K(z^2+c)$;
$\mathscr{B}$	the set of Brjuno numbers;
$a_n \asymp b_n$, or $a_n = \Theta(b_n)$	$a_n = O(b_n)$ and $b_n = O(a_n)$;
$a_n \lesssim b_n$	$a_n = O(b_n)$.

Chapter 1
Introduction to Computability

One of the main goals of computability theory is to classify problems according to whether or not they can be solved algorithmically. In fact, such questions existed before computers. A famous example is *Hilbert's Tenth Problem*:

"Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: to devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers." [Bulletin of the American Mathematical Society 8 (1902), 437–479.]

In other words:

Is it algorithmically possible to determine if a given Diophantine equation is solvable?

It is fairly clear what an affirmative answer would mean in this case – an explicit method to check if an equation has a solution. Giving a negative answer (which turns out to be the correct one) requires a more formal definition of "methods" that can be used in the solution – as one would need to prove that none of these methods work.

Such formal models of computation precede modern computers. In 1936 two essentially equivalent models were independently proposed by A. Turing [Tur36] and E. Post [Pos36] (and many others have appeared since). Turing's work has been the most influential, and his concept of a *Turing Machine* has become a universally accepted formal model of computation.

1.1 Discrete computability and complexity

1.1.1 Discrete computability and the Turing Machine

A precise definition of a Turing Machine (TM) is somewhat technical and can be found in all texts on computability, e.g. [Pap94, Sip05]. Such a machine consists

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of a *tape*, and a *head* which can read/erase/write the symbols on the tape one at a time, and can shift its position on the tape in either direction. The symbols on the tape come from a finite alphabet, and the TM can be in one of a finitely many *states*. Finally, a simple look-up table tells TM what action to undertake depending on the current state, and the symbol read on the tape. The length of the tape is usually not restricted in either direction, so there is no bound on the "memory" available to the machine.

Despite its apparent simplicity, the computational power of a Turing Machine is equivalent to that of a RAM computer (with an unlimited supply of RAM). In fact, there is a general belief, usually referred to as the Church-Turing Thesis, which states that any computation performed on a physical device can be simulated using a Turing Machine.

Programming a Turing Machine can be a daunting task, and it is much easier to think of an algorithm as a program in one of the many programming languages available on a modern computer. Since the concepts are equivalent, this is the path we will generally follow when describing a particular algorithm.

The formal notion of a Turing Machine gives a natural way of classifying the computability of functions in the *discrete* setting, such as functions acting on the set of naturals $\mathbb{N}$ or the set of finite binary strings $\{0,1\}^*$. Namely:

Definition 1.1.1 *A function $f(x)$ is computable, if there exists a TM which takes x as an input and outputs the value $f(x)$.*

Computable functions are sometimes called *recursive*. They include simple functions such as integer arithmetic operations and lexicographical sorting of strings. They also include problems that appear to be difficult in practice, but can be solved nonetheless if we are willing to wait sufficiently long. These include, for example, computing the prime factorization of an integer and finding the optimal strategy in the game of Go.

On the other hand, there are many functions that are not computable. One way to see this is by a simple counting argument: any TM has a finite description, and hence there are countably many TMs. On the other hand, there are uncountably many functions from $\mathbb{N}$ to $\mathbb{N}$, or even from $\mathbb{N}$ to $\{0,1\}$ – and thus "most" functions are not computable. It is much more interesting to have specific examples of non-computability.

One such example is the *Halting Problem*. The *halting function* H maps a pair (T,w), where T is an encoding of a TM M and w is a binary input, to 1 if the machine M running on input w eventually halts, and 0 otherwise.

Proof (Sketch of proof that H is not computable). We argue by way of a contradiction. Suppose there were a TM M_1 computing the halting function. Let M_2 be the following machine: on an input w, M_2 uses M_1 to compute $H(w,w)$. If $H(w,w) = 0$, then M_2 halts, otherwise it goes into an infinite loop.

Let w_2 be the encoding of M_2. What will be the outcome of running $M_2(w_2)$? If M_2 halts on w_2, then $H(w_2,w_2) = 1$, and thus M_2 cannot halt on w_2 by definition. If M_2 fails to halt on w_2, then $H(w_2,w_2) = 0$, and by its definition M_2 halts on input w_2. In either case we arrive at a contradiction. $\square$

Recall, that a function which takes only boolean values 0 (False) and 1 (True) is called a *predicate*. Consider a predicate $A : \mathbb{N} \times \mathbb{N} \to \{0,1\}$ defined as follows. On an input (x,t), A views x as an encoding of a pair (M,w) of a TM and an input. $A(x,t)$ is 1 if and only if x gives a valid encoding of a pair (M,w) and M halts on w in exactly t steps. It is easy to see that A is a computable predicate. An algorithm to determine the value of $A(x,t)$ is simply to simulate the machine M for exactly t steps with input w, and check whether it terminates at the last step.

On the other hand, computing the predicate

$$B(x) = \exists t\ A(x,t)$$

is as difficult as solving the Halting Problem, and thus B is non-computable. This example will be useful later on.

More generally, a predicate of the form $P(x) = \exists y\ R(x,y)$ for a computable predicate $R(x,y)$ is said to be *recursively enumerable*. Note that every computable predicate is also recursively enumerable. In the case when, as above, for each x there exists at most one y such that $R(x,y)$ holds, we will emphasize it by writing $P(x) = \exists! y\ R(x,y)$.

Another explicit example of a non-computable function is given by the negative solution to Hilbert's Tenth Problem. This famous theorem was proved in 1970 by Yuri Matiyasevich, using earlier results of Julia Robinson, Martin Davis, and Hilary Putnam (see [Mat93] for details and the history of the problem).

Theorem 1.1 *The function that maps an encoding of a diophantine equation E to* 1 *if E is solvable and to* 0 *otherwise, is non-computable.*

1.1.2 Discrete complexity theory

In addition to studying computability properties of problems one is often interested in the amount of computational resources needed to solve a certain problem. The Computational Complexity Theory studies these questions.

Definition 1.1.2 *For a TM M on input w, the* running time *of $M(w)$ is the number of steps $M(w)$ makes before terminating with an output. The running time of M is the function $T_M : \mathbb{N} \to \mathbb{N}$ such that*

$$T_M(n) = \max_{|w|=n} \{\text{the running time of } M(w)\},$$

where $|w|$ denotes the length of w. In other words, $T_M(n)$ is the worst case *running time for inputs of length n.*

For a predicate $P : \{0,1\}^* \to \{0,1\}$ the *time complexity* of P is said to have an upper bound $T_2(n)$ if there exists a Turing Machine M_P with running time bounded by $T_2(n)$ that computes P. The time complexity of P is said to have a lower bound of $T_1(n)$ if, for *any* Turing Machine M computing P with running time T_M, $T_M(n) > T_1(n)$ for infinitely many n's.

In this book we focus on the time complexity of problems. Other complexity notions may include:

- space complexity – the amount of memory used by an algorithm solving the problem,
- randomness complexity – the number of random bits used by an algorithm,
- and communication complexity – the number of bits transferred during an execution of a multi-party computation.

Proving an upper bound typically involves producing an algorithm running in time bounded by $T_2(n)$. Proving lower bounds is generally much more difficult, and most exciting problems such as the "**P** vs. **NP**" (see [Coo06]) fall into this category. For theoretical purposes, a problem is considered "tractable" if its complexity is *polynomial* in n. The class of polynomial-time computable problems is denoted by **P**.

Example 1.1. Consider the satisfiability problem SAT:

Given a boolean formula $\phi(x)$ in k variables, decide whether there is an assignment x' of variables that satisfies ϕ, i.e. such that $\phi(x') = 1$.

Formally, the predicate we would like to compute is

$$SAT(w) = 1 \Longleftrightarrow w \text{ encodes a satisfiable boolean formula } \phi.$$

The naïve way to check whether a formula ϕ in k variables is satisfiable is by trying all possible truth assignments to see whether one of them satisfies ϕ. Checking one assignment takes time bounded by $O(|w|^2)$, and hence SAT can be solved in time $O(|w|^2 \cdot 2^k)$. With a reasonable encoding, $k \leq |w|/2$, and thus the running time is bounded by (recall the notation $n = |w|$):

$$O(|w|^2 \cdot 2^k) \leq O(|w|^2 \cdot 2^{|w|/2}) \leq O(2^{|w|}) = O(2^n).$$

Hence the complexity of SAT is at most *exponential* in n.

On the other hand, it is not hard to see that in order to solve SAT successfully one needs at least to read the entire input w, and hence the complexity of SAT is at least n – *linear* in the size of the input n.

The gap between the upper and lower bound is huge, and the question of the "true" complexity of SAT is of great importance. For example, **P=NP** if and only if SAT has polynomial time complexity. Most researchers believe that this is not the case, but any substantial improvement on the trivial bounds above seems to be remarkably hard to achieve.

There are many "natural" problems that appear to be hard, for which no unconditional super-polynomial lower bounds are known. In addition to the SAT problem mentioned above, these include integer factoring, computing the number of perfect matchings in a graph and finding the optimal strategy in the game of Go. In contrast, using reasoning similar to the proof that the Halting Problem is not computable, one can construct an "artificial" problem of any *reasonable* time complexity. (cf. [Sip05] Section 9.1).

Theorem 1.2 *Suppose that $t : \mathbb{N} \to \mathbb{N}$ is a computable function. Then there is a computable function $f_t : \mathbb{N} \to \{0,1\}$ such that no TM M can compute f_t while running in time $t(n)$ on input n.*

The proof Theorem 1.2 is very similar to the proof that the Halting Problem is not computable. An example of a hard problem is:

Given a machine M and an input w, does M halt on w for $2^{t(|w|)}$ steps?

Note that in order to perform the simulation we need to compute $t(|w|)$ first, and hence the requirement that the function t is computable (the theorem is false without this requirement).

1.2 Computability and complexity of real numbers and functions

1.2.1 Computability and complexity of real numbers

One of Turing's original motivations for introducing the Turing Machine was classifying real numbers into computable and non-computable ones. A number is said to be *computable* if there exists a TM that writes its (infinite) decimal expansion digit by digit. The following definition is equivalent, but slightly less representation dependent:

Definition 1.2.1 *A real number α is said to be computable if there is a computable function $\phi : \mathbb{N} \to \mathbb{Q}$ such that, for all n,*

$$\left| \alpha - \frac{\phi(n)}{2^n} \right| < 2^{-n}.$$

The set of the computable reals is denoted by $\mathbb{R}_{\mathscr{C}}$.

In other words, there exists an algorithm to approximate α with any desired degree of precision. As with discrete functions, "most" numbers are non-computable, while most "nice" numbers, such as π and e are. It is easy to see that $\mathbb{R}_{\mathscr{C}}$ with the usual arithmetic operations forms a field. Moreover, $\mathbb{R}_{\mathscr{C}}$ is a *real closed field*, that is, every real number in the algebraic closure of $\mathbb{R}_{\mathscr{C}}$ belongs to $\mathbb{R}_{\mathscr{C}}$. Computable complex numbers $\mathbb{C}_{\mathscr{C}} = \mathbb{R}_{\mathscr{C}} + i\mathbb{R}_{\mathscr{C}}$ are defined in a similar way, and $\mathbb{C}_{\mathscr{C}}$ is an algebraically closed field.

Below we present a generalization of the definition of a computable number which will be useful later on:

Definition 1.2.2 *A real number α is said to be* right *computable if there is a computable function $\phi : \mathbb{N} \to \mathbb{Q}$ such that*

- *the sequence $\{\phi(n)\}$ is non-increasing: $\phi(1) \geq \phi(2) \geq \ldots$, and*
- *the sequence $\{\phi(n)\}$ converges to α: $\lim_{n\to\infty} \phi(n) = \alpha$.*

Similarly, a number β is left *computable if there is a computable rational sequence $\psi(n)$ that converges to β from below.*

Note that the sets of right-computable and left-computable numbers are countable, because each such number α corresponds to a different TM that computes a sequence converging to α. It is not hard to see that a computable real number is also right- and left-computable. Conversely, if a number is both right-computable and left-computable, then it is computable.

Being just left-computable or right-computable is insufficient for being computable, in general. We will show that, in fact, right-computable numbers that are not computable are dense in $\mathbb{R}$:

Proposition 1.3 *Right computable numbers form a dense subset in $\mathbb{R} \setminus \mathbb{R}_{\mathscr{C}}$.*

Proof. It is sufficient to present a single right computable number which is not computable. If α is a right-computable number in $\mathbb{R} \setminus \mathbb{R}_{\mathscr{C}}$ then $\alpha + q$ is also such a number for any $q \in \mathbb{Q}$, yielding a dense set in $\mathbb{R}$. Let $P(x) = \exists! y\ R(x,y)$ be a non-computable predicate on $\mathbb{N}$ such that $R(x,y)$ is computable, as discussed above. Consider the number

$$\alpha = 1 - \sum_{x=1}^{\infty} P(x) \cdot 4^{-x}.$$

Then α is non-computable, since computing α would also enable us to compute the predicate P. On the other hand, α is right computable, as demonstrated by the following computable function:

$$\phi(n) = 1 - \sum_{x=1}^{n} \sum_{y=1}^{n} R(x,y) \cdot 4^{-x}.$$

$\phi(n)$ is obviously non-increasing, and

$$\lim_{n\to\infty} \phi(n) = 1 - \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} R(x,y) \cdot 4^{-x} = 1 - \sum_{x=1}^{\infty} P(x) \cdot 4^{-x} = \alpha.$$

□

A more detailed discussion on the different extensions of the concept of a computable number can be found in [Wei00].

Similarly to the discussion of the complexity of predicates in the previous section, we can define the time complexity of a real number $\alpha \in \mathbb{R}_{\mathscr{C}}$.

Definition 1.2.3 *We say that functions $T_1(n)$, $T_2(n)$ are lower, upper bounds on the complexity of $\alpha \in \mathbb{R}_{\mathscr{C}}$ if any algorithm which computes the function $\phi(n)$ as in Definition 1.2.2 has running time at least $T_1(n)$ for infinitely many n's, and there is such an algorithm with running time of at most $T_2(n)$ for all sufficiently large values of n. The number α is poly-time computable if $T_2(n)$ can be chosen as a polynomial in n.*

It is not hard to see that the "common" numbers such as 7, π and e are poly-time computable. On the other hand, Theorem 1.2 gives us a way of constructing computable reals of an arbitrarily high computational complexity:

Proposition 1.4 *For any computable function $T : \mathbb{N} \to \mathbb{N}$ there exists a computable real number α of complexity greater than $T(n)$.*

Proof. The same trick as in the construction of a non-computable α in the proof of Proposition 1.3 can be used here. Namely, by Theorem 1.2 there is a computable function

$$f_t : \mathbb{N} \to \{0,1\}$$

such that f_t is computable, but not in time $t(2n)+8n$. Then the number

$$\alpha = \sum_{x=1}^{\infty} f(x) \cdot 4^{-x},$$

is not computable in time $t(n)+n$, otherwise we could use the 4^{-n}-approximation for α to compute $f(n)$ in time $< t(2n)+8n$.

□

The above definitions of computability and complexity using Turing Machines directly apply only to computability questions for discrete objects, such as discrete predicates and finite approximations of real numbers. It has to be extended if we want to discuss the computability of continuous objects such as functions over $\mathbb{R}$ or subsets of $\mathbb{R}^k$.

1.2.2 Oracle computation, computable real functions

The history of defining computability for real objects probably begins with the work of Banach and Mazur [BM37] in 1937, only one year after Turing's paper. This work has founded the tradition of Computable Analysis (sometimes also called Constructive Analysis). Interrupted by war, it was further developed in the book by Mazur [Maz63]. Much research took place in the mid 1950's in the works of Grzegorczyk [Grz55], Lacombe [Lac55], and others. A parallel school of Constructive Analysis was founded by A. A. Markov in Russia in the late 1940's. A modern treatment of the field can be found in [Ko91] and [Wei00].

The definition of computability over the reals presented here falls into this framework.

Consider the simplest case in which we would like to compute a function $f : \mathbb{R} \to \mathbb{R}$. On an input x, we are trying to compute $f(x)$. As in the case with real numbers, the machine M computing f should be able to output $f(x)$ with any given precision 2^{-n}. The machine M, similarly to a practical computer, can only handle a finite amount of information, and thus is not capable of reading or storing an entire input x. Instead, it is allowed to request the input x with an arbitrarily high precision. In other words, it has an external tape and a command *READ(m)* which requests a 2^{-m}-approximation $\phi(m)$ of x to be written on this tape. It can then be read by the machine from the external tape. It is convenient to take all the approximations from the set of *dyadic numbers*

$$\mathbb{D} = \left\{ \frac{k}{2^l} \ : \ k \in \mathbb{Z},\ l \in \mathbb{N} \right\},$$

as they possess a natural finite binary encoding.

To formally define computability of real functions let us first introduce the notion of an *oracle* formalizing the command *READ*:

Definition 1.2.4 *A dyadic-valued function $\phi : \mathbb{N} \to \mathbb{D}$ is called an oracle for a real number x if it satisfies $|\phi(m) - x| < 2^{-m}$ for all m.*

An *oracle Turing Machine* is a TM which can query the value $\phi(m)$ of some oracle ϕ for an arbitrary $m \in \mathbb{N}$. Note that the oracle ϕ itself is not a part of the algorithm, but rather enters as a parameter. We will use a notation M^ϕ to emphasize the dependence of the output of the TM on the values of the oracle.

To get used to the terminology, imagine a trivial algorithm which, given an $n \in \mathbb{N}$ and a good enough approximation of $x \in \mathbb{R}$, outputs a 2^{-n}-approximation of the number $2x$. The algorithm executes the command

$$\textit{READ x WITH PRECISION } 2^{-(n+1)}.$$

At this point the user (playing the role of an oracle in the dictionary sense) enters a dyadic rational d for which $|d - x| < 2^{-(n+1)}$ from the keyboard. The algorithm proceeds to output $2d$ as the answer.

Definition 1.2.5 *Let S be a subset of $\mathbb{R}$, and let $f : S \to \mathbb{R}$ be a real-valued function on S. Then f is said to be computable if there is an oracle Turing Machine $M^\phi(n)$ such that the following holds. If $\phi(m)$ is an oracle for $x \in S$, then for every $n \in \mathbb{N}$, $M^\phi(n)$ returns a dyadic number q such that $|q - f(x)| < 2^{-n}$.*

Note that M^ϕ is supposed to work with *any* valid oracle ϕ for x. The definition generalizes trivially to functions with $k > 1$ variables. While any choice of norm in $\mathbb{R}^k$ would do to measure distance, to fix the ideas we will use the ℓ_2-norm

$$|\bar{x}| = \sqrt{x_1^2 + \cdots + x_k^2} \text{ for } \bar{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k.$$

Examples of computable functions include most common functions such as an integer power, $\exp(x)$, and any trigonometric function. A constant function $f(x) \equiv a$ is computable if and only if a is a computable number.

The oracle terminology allows us to separate the problem of computing the parameter x from the problem of computing the function f on a *given* x. For example, the function $x \mapsto 2x$ is computable. Hence, even if a is a non-computable number, we are still able to compute $2a$, provided we have an oracle access to a. This is despite the fact that $2a$ is a non-computable number in this case.

A fundamental fact about computable functions in this setting is that computable functions are continuous:

Theorem 1.5 *Let $S \subset \mathbb{R}^k$, and suppose that $f : S \to \mathbb{R}$ is computable by an oracle machine M^ϕ. Then f is continuous on S.*

Proof. Let $x \in S$ and $\varepsilon > 0$ be given. Choose an integer m such that $2^{-m} < \varepsilon/2$. Let $\phi(n)$ be an oracle for x satisfying $|\phi(n) - x| < 2^{-n-1}$ for all n (thus "exceeding" the minimum requirement from an oracle). Then $M^\phi(m)$ outputs a number $d \in \mathbb{D}$ such that $|d - f(x)| < 2^{-m}$. The computation of $M^\phi(m)$ terminates after finitely many steps, and hence ϕ is only queried up to some finite precision 2^{-k}. It is now not hard to see that, for any x' such that $|x - x'| < 2^{-k-1}$, there is a valid oracle ϕ' which agrees with ϕ up to precision 2^{-k}. Thus, for any $x' \in S \cap (x - 2^{-k-1}, x + 2^{-k-1})$, $M^{\phi'}(m)$ outputs the same answer d, and we must have $|d - f(x')| < 2^{-m}$. Hence, for every $x' \in S$ such that $|x - x'| < 2^{-k-1}$, we have

$$|f(x) - f(x')| \le |f(x) - d| + |f(x') - d| < 2^{-m} + 2^{-m} < \varepsilon.$$

$\square$

In particular, the theorem shows that discontinuous functions, such as arcsin or $\chi_{\mathbb{Q}}$ cannot be computed by a single machine on the whole domain of definition.

The continuity requirement may seem too restrictive. It is hard to argue, for instance, that the function

$$\operatorname{sign}(x) = \begin{cases} 1 \text{ if } x \ge 0, \\ 0 \text{ if } x < 0 \end{cases}$$

is computationally hard. There exist various natural ways to soften the definition of a computable function, to avoid such counterexamples (see [Bra05]). For most of our negative results, the domain of the function will consist of a single point, and so Theorem 1.5 would not be relevant.

Same considerations can be used to prove a stronger result:

Theorem 1.6 *Under the conditions of Theorem 1.5 there exists an oracle machine $M^{\phi}(k)$ that, for every valid oracle ϕ for $x \in S$, computes a function $\mu_{\phi}(k) : S \times \mathbb{N} \to \mathbb{N}$ such that*

$$|f(y) - f(x)| < 2^{-k} \textit{ whenever } y \in S \textit{ and } |y - x| < 2^{-\mu(x,k)}.$$

We will refer to the this property by saying that f has a *computable local modulus of continuity.*

Note that we cannot say that there is a well defined function $\mu : S \times \mathbb{N} \to \mathbb{N}$, because such a μ would have to be constant (or discontinuous) for a fixed k. In the statement of Theorem 1.6 the value of μ_{ϕ} depends not only on the value of $x \in S$ but also on the particular oracle ϕ for x.

In some cases, for example when $S = [0,1]$, the *global* modulus of continuity (or simply the modulus of continuity) of f on S is also computable. That is, we can compute a function $\mu : \mathbb{N} \to \mathbb{N}$ such that

$$\text{for any } x, y \in S \text{ with } |x - y| < 2^{-\mu(k)} \text{ we have } |f(x) - f(y)| < 2^{-k} \tag{1.2.1}$$

(more generally, this is true whenever S is a compact computable set, as will be defined in the next section).

Conversely, to guarantee that a function $f : [0,1] \to \mathbb{R}$ is computable, it suffices to know that it has a computable modulus of continuity, and have a way to compute its values at all dyadic rational points (cf. Proposition 2.6 in [Ko98]):

Theorem 1.7 *A real function f of the interval $[0,1]$ is computable if and only if it has a computable modulus of continuity and there exists a computable function*

$$g : (\mathbb{D} \cap [0,1]) \times \mathbb{N} \to \mathbb{D}$$

such that

$$|g(r,n) - f(r)| < 2^{-n}.$$

Similarly to the discrete case, the running time $T_M(n)$ of a machine $M^{\phi}(n)$ computing a real function $f : S \to \mathbb{R}$ is the *worst case* number of steps it takes to compute $f(x)$ with precision 2^{-n}. Querying the oracle $\phi(m)$ counts as m time units. Here the "worst case" is taken over all possible $x \in S$ and all valid oracles ϕ for each such x.

Definition 1.2.6 *A function $f : S \to \mathbb{R}$ is said to be poly-time computable if there is a machine M^{ϕ} computing it such that $T_M(n)$ is bounded by a polynomial in n.*

As expected, the common "calculator" functions such as x^2, $\tan x$ and e^x are poly-time computable on suitably chosen domains. For example, $\tan x$ is poly-time computable on any closed sub-interval of $(-\pi/2, \pi/2)$.

1.3 Computability and complexity of subsets of $\mathbb{R}^k$

Let $K \subset \mathbb{R}^k$ be a compact set. We would like to give a definition for K being computable. In the discrete case the distinction between computability of functions and sets is not as important, since a set S is usually said to be computable, or decidable, if and only if its characteristic function χ_S is computable. The same definition would not work over $\mathbb{R}$, since only continuous functions can be computable, and hence χ_K would not be computable unless $K = \emptyset$.

The goal of a machine M computing the set K in our setting is to produce "drawings" of K with any prescribed precision. A "drawing" P of the set K on the computer screen is just a collection of pixels that serve as an accurate description of K (or a portion of K, if the image is zoomed-in). We would expect the following properties from P:

- P should include all pixels that intersect with K (this guarantees that we get a picture of the entire set P); and
- P should not include pixels that are "far" from K, for example pixels that are at least one pixel diameter away from the set K.

By switching from the rectangular computer pixels to the mathematically more convenient round pixels, we see that to "draw" K one should be able to compute a function $f_K : \mathbb{D}^k \times \mathbb{D} \to \{0,1\}$ from the family

$$f_K(d,r) = \begin{cases} 1 & \text{if } B(d,r) \cap K \neq \emptyset \\ 0 & \text{if } B(d,2\cdot r) \cap K = \emptyset \\ 0 \text{ or } 1 & \text{otherwise.} \end{cases} \tag{1.3.1}$$

f_K then can be used to decide whether to include a round pixel with center d and radius r in P. Sample values of a function f_K are illustrated on Figure 1.1.

Definition 1.3.1 *The set K is said to be computable if a Turing Machine M computing a function f_K from the family (1.3.1) exists.*

Definition 1.3.2 *The running time $T_M(n)$ is the worst-case time it could take to compute $f_K(d,r)$ where $r = 2^{-n}$ and d is a dyadic point on the $(\mathbb{Z}/2^{2n})^k$ grid.*

In other words, $T_M(n)$ is the longest time it takes to decide the color of one pixel at resolution $1/2^n$. As before, a set K is said to be poly-time computable if there is a machine M_K computing K such that $T_{M_K}(n)$ is bounded by a polynomial in n.

To see why this is the "right" complexity notion, suppose we are trying to draw a set K on a computer screen which has a 1000×1000 pixel resolution. A 2^{-n}-zoomed-in picture of S has $O(2^{2n})$ pixels of size 2^{-n}, and at this resolution the whole picture would take time $O(T_M(n) \cdot 2^{2n})$ to compute. This quantity is exponential in n, even if $T_M(n)$ is bounded by a polynomial. But we are drawing K on a finite-resolution display, and we will only need to draw $1000 \cdot 1000 = 10^6$ pixels. Deciding these pixels would require $O(10^6 \cdot T_M(n)) = O(T_M(n))$ steps. This running time is polynomial in n if and only if $T_M(n)$ is polynomial. Hence $T_M(n)$ reflects the 'true' cost of zooming in when drawing K.

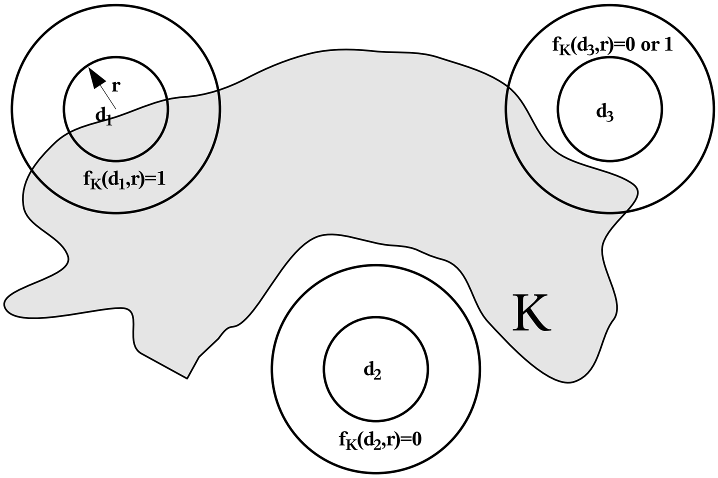

Fig. 1.1 Sample values of the function f_K

Definition 1.3.1 is well-established in the literature (see e.g. [BW99, Wei00, RW03]).

It is not hard to check that simple geometric shapes such as circles and line segments are (poly-time) computable if and only if their parameters are (poly-time) computable. For example, computing a circle (as a set in $\mathbb{R}^2$) is as hard as computing its radius and the coordinates of its center.

The set computability definition above may appear somewhat artificial, but in fact we will see that it is quite robust. For example, it is equivalent to K being approximable in the *metric*. Recall that the Hausdorff metric is a metric on compact subsets of $\mathbb{R}^k$ defined by

$$d_H(X,Y) = \inf\{\varepsilon > 0 | X \subset B(Y,\varepsilon) \text{ and } Y \subset B(X,\varepsilon)\},$$

where $B(X,\varepsilon)$ is the open ε-neighborhood of X:

$$B(X,\varepsilon) := \bigcup_{x\in X} B(x,\varepsilon).$$

We approximate K using a class $\mathscr{C}$ of sets which is dense in the metric d_H among compact sets, and such that elements of $\mathscr{C}$ have a natural binary encoding. Namely $\mathscr{C}$ is the set of finite unions of dyadic balls:

$$\mathscr{C} = \left\{ \bigcup_{i=1}^{n} \overline{B(d_i,r_i)} \;\middle|\; \text{where } d_i \in \mathbb{D}, r_i \in \mathbb{D}^k \right\}.$$

Members of $\mathscr{C}$ can be encoded as binary strings in a natural way. The following theorem connects Definition 1.3.1, approximability in the Hausdorff metric and the computability of functions as per Definition 1.2.5.

Theorem 1.8 *For a compact $K \subset \mathbb{R}^k$ the following are equivalent:*

1. *K is computable as per Definition 1.3.1,*
2. *there is a Turing Machine M that on input n produces a set $C_n \in \mathscr{C}$ that is a 2^{-n}-approximation of K in the Hausdorff metric: $d_H(K, C_n) < 2^{-n}$,*
3. *the* distance function $d_K(x) = \inf\{|x-y| \mid y \in K\}$ *is computable as per Definition 1.2.5.*

Note that the equivalence holds if we are only concerned with *computability* of sets, but it is no longer true if we are concerned with their computational complexity. For example the set $C_n \in \mathscr{C}$ which is a 2^{-n}-approximation of K could typically have exponentially many balls in it, and thus would require an exponential number of steps to compute, even for sets as "simple" as $K_0 = [0,1] \times \{0\} \subset \mathbb{R}^2$.

Remark on the BSS computability model

We note that another approach to the computability of subsets of $\mathbb{R}^k$ has been developed by Blum, Shub, and Smale [BCSS98]. It is based on the concept of decidability in the Blum-Shub-Smale (BSS) model of real computation. The BSS model is very different from the Computable Analysis model we use, and can be very roughly described as based on computation with infinite-precision real arithmetic. Some discussion of the differences between the models may be found in [BC06a] and [Bra05]. In the BSS model, the computer has registers which can hold arbitrary elements of the underlying ring R ($R = \mathbb{R}$ in our case). Computer programs perform exact arithmetic ($+, -, \cdot$, and $\div$) and can branch on conditions based on exact comparisons.

A set S in $\mathbb{R}^2$ is BSS decidable if there is a BSS machine that given a point $(x,y) \in \mathbb{R}^2$ terminates and outputs whether or not $(x,y) \in S$. It is shown in [BCSS98] that the Mandelbrot set (Theorem 2), and most Julia sets (Theorem 3) are not BSS-decidable.

Algebraic in nature, BSS decidability is not well-suited for the study of fractal objects, such as Julia sets. It turns out (see Chapter 2.3 of [BCSS98]) that sets with a fractional Hausdorff dimension, including ones with very simple description, such as the Cantor set and the Koch snowflake (Fig. 1.2), are BSS-undecidable. Moreover, due to the algebraic nature of the model, very simple sets that do not decompose into a countable union of semi-algebraic sets are not decidable. An example of such a set is the graph of the function $f(x) = e^x$ (Fig. 1.2).

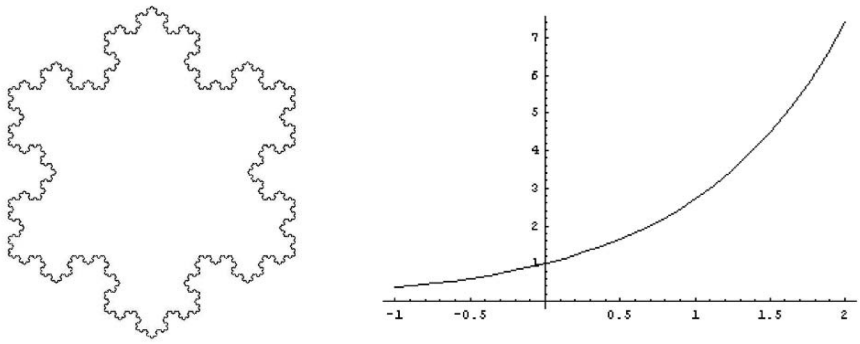

Fig. 1.2 The Koch snowflake and the graph of the function $y = e^x$

1.4 Weakly computable sets

A different definition of set-computability was introduced by Chou and Ko [CK95] under the name of *strong recognizability*. We will refer to it as *weak computability*:

Definition 1.4.1 *We say that a set S is* weakly *computable if there is an oracle Turing Machine $M^{\phi}(n)$ such that, if $\phi = (\phi_1, \phi_2, \ldots, \phi_k)$ represents a point $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$, then the output of $M^{\phi}(n)$ is*

$$M^{\phi}(n) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } B(x, 2^{-(n-1)}) \cap K = \emptyset \\ 0 \text{ or } 1 & \text{otherwise.} \end{cases} \tag{1.4.1}$$

Condition (1.4.1) is similar to condition (1.3.1). The difference is that now we allow x to be any point in $\mathbb{R}^k$ (not just $\mathbb{D}^k$), and we do not require the machine to output 1 if x is not in K but is "close". It is evident from Figure 1.3 that Definition 1.4.1 requires less effort from the algorithm computing K than the original definition. Thus, the new definition appears to be weaker than the definition of set computability from last section, but it turns out that they are equivalent.

Theorem 1.9 *[Bra05] A compact set $K \subset \mathbb{R}^k$ is weakly computable if and only if it is computable as per Definition 1.3.1.*

It is sometimes easier to use weak computability when proving that a certain set is computable. Intuitively, the regular set computability definition requires us to be able to estimate the distance from any point to the set K within a multiplicative factor of 2. This might require a somewhat "global" understanding of the set K. Weak computability, on the other hand, requires us to answer a very local question

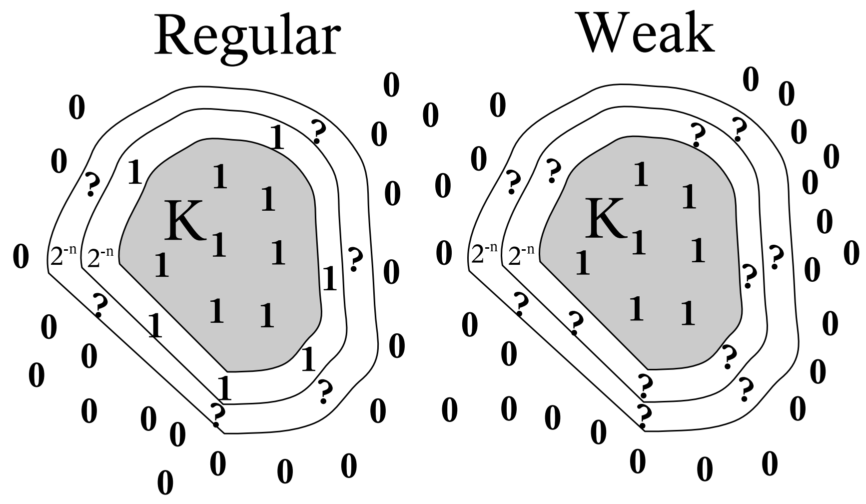

Fig. 1.3 The values of $f_K(\bullet, 2^{-n})$ in the definitions of regular (left) and weak set computability

about the presence of a specific point x in K. It allows us to err if the point x is not in K even if it is very close.

A quantitative version of Theorem 1.9 exists.

Theorem 1.10 *[Bra04] If a set K is weakly computable in time $T(n)$, then it is computable as per Definition 1.3.1 in time $2^{O(T(n)+n)}$.*

Moreover, if we have a poly-time algorithm for computing a set K according to the weak definition, we can derive from it an exponential-time algorithm for computing it in the regular model.

Sketches of proofs of Theorems 1.9 and 1.10 (cf. [Bra05])

The “hard” direction is to show that every weakly computable set is, in fact, computable according to Definition 1.3.1. To simplify matters, suppose that the set S is a one-dimensional set. We assume that S is weakly computable, and want to show that it is computable. The transition to higher dimensions in this case is fairly straightforward. The reduction is “black box” in the sense that we are using repeated simulations of the machine computing S in the weak sense to compute the set S in the regular sense.

We are given a point d in $\mathbb{D}$, and $n > 0$, and want to return 1 if $(d-2^{-n}, d+2^{-n}) \cap S \neq \emptyset$ and 0 if $(d-2\cdot 2^{-n}, d+2\cdot 2^{-n}) \cap S = \emptyset$. Consider the infinite tree T of all the oracles for all the points in $(d-2^{-n}, d+2^{-n})$. In Fig. 1.4, the first three levels of T are presented for $d = \frac{1}{2}$ and $n = 1$ (all the numbers are written in binary). Each infinite path in T represents a real number in the interval we are interested in. There is a path converging to each real x in the interval. In fact, there are usually infinitely many such paths.

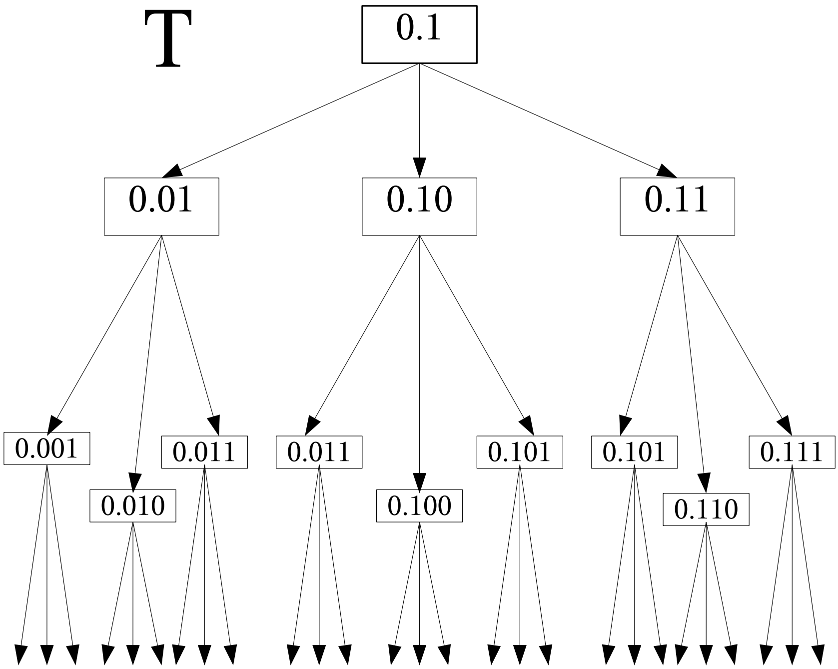

Fig. 1.4 The first three levels of the tree T

We simulate the run of the "weak" machine $M^{\phi}(n)$. The goal is to make the simulation for all the real points in the interval $(d - 2^{-n}, d + 2^{-n})$ simultaneously.
If the machine asks for x with precision 2^{-m}, $m < n$, we respond with d as the approximation. This is a valid oracle value for any x in the interval.
If $m \geq n$, we consider all the possible descendants of d on the level with $m+1$-bit long numbers, and create a separate computation for each of them (thus creating 3^{m-n+1} computations). Consider one of the copies and denote the corresponding value on level $m+1$ by d'. If we are now asked about $\phi(r)$ for some $r < m+1$, we return the value of d' consistent with all the descendants of d'. Otherwise, we again consider all possible descendants of d' on level $r+1$, and split

the computation into 3^{r-m} computations. We continue this process until all computations terminate.
If any of the computations returns 1, we terminate and return 1 for the entire computation; otherwise we return 0. We first show that the computation always terminates.
Suppose that the computation does not terminate. The entire computation can be viewed as a tree where the nodes are the sub-computations described above and a computation C_i is the parent of the 3^s computations it launches. If the entire computation does not terminate, then there are two possibilities: either one of the computations C' fails to terminate without calling to sub-computations, or the tree of all the computations to be performed is infinite.
In the first case the points represented by the oracle leading to C' would cause $M^\phi(n)$ to run forever. In the second case, by König's lemma, there must be an infinite branch in the computations tree. Denote the branch by $C_1, C_2, C_3, \ldots$. That is, C_1 calls C_2, C_2 calls C_3 etc. Note that each C_i works with a node d_i of T and d_{i+1} is a descendant of d_i for each i, and hence the infinite sequence of C_i corresponds to an infinite path p in T. The path converges to a real number $x \in [0,1]$, and p gives rise to an oracle ϕ for x. By the construction, the sequence of $C_1, C_2, C_3, \ldots$ simulates the computation of $M^\phi(n)$. Hence $M^\phi(n)$ does not terminate: a contradiction. This shows that the algorithm terminates. Note that for the proof to work we need the fact that every Cauchy sequence in $\mathbb{D}$ converges to a limit in our domain $\mathbb{R}$.
For the correctness we see that, if there is an $x \in S$ in the interval, then there is an oracle ϕ for x which corresponds to an infinite path in T. The computation branch corresponding to this path will return 1, and the algorithm will output 1. If, on the other hand, x is far from S, then every branch in the computation corresponds to a point which is at least 2^{-n}-far from S, and all of them will return 0, and the algorithm outputs 0 in this case.
Finally, to see that Theorem 1.10 is correct, note that if the weak computation runs in time $T(n)$, it can query the input x with precision of at most $T(n)$, and thus the simulation on the tree T will be limited to the first $T(n)$ levels of the tree, thus running in time exponential in $T(n)$.

1.5 Set-valued functions and uniformity

The problem of computing Julia sets is essentially that of mapping the coefficients of a rational function $R(z)$ to the set J_R. Thus we need a notion of computability of set-valued functions to discuss computability questions about Julia sets.

We can combine the definitions from previous sections to define computability of set-valued functions. Informally, a set-valued function is computable if, given an oracle access to the inputs to the function, we can compute the value of the function according to the definition of set computability. As the focus of the present book is Julia sets, we will consider functions that output two-dimensional sets. The definition extends to sets in $\mathbb{R}^\ell$ for any $\ell \in \mathbb{N}$.

Definition 1.5.1 *Let S be a subset of $\mathbb{R}^k$. Denote by K_2^* the set of all the compact subsets of $\mathbb{R}^2$. Let $F : S \to K_2^*$ be a set-valued function mapping points in S to compact subsets of $\mathbb{R}^2$. The function F is said to be computable on S if there is an oracle TM $M^{\phi_1,\dots,\phi_k}(d,r)$ which, for the oracles representing a point $x = (x_1,x_2,\dots,x_k) \in S$, computes a function $f^{\phi_1,\dots,\phi_k} : \mathbb{D}^2 \times \mathbb{D} \to \{0,1\}$ from the family*

$$f^{\phi_1,\dots,\phi_k}(d,r) = \begin{cases} 1 & \text{if } B(d,r) \cap F(x_1,\dots,x_k) \neq \emptyset \\ 0 & \text{if } B(d,2\cdot r) \cap F(x_1,\dots,x_k) = \emptyset \\ 0 \text{ or } 1 & \text{otherwise.} \end{cases} \tag{1.5.1}$$

The running time $T(n)$ of $M^{\phi_1,\dots,\phi_k}(d,r)$ is the *worst case* time the computation could take when $r = 2^{-n}$ and $d \in (\mathbb{Z}/2^{2n})^2$. We are often interested in the running time $T(x_1,\dots,x_k,n)$ of $M^{\phi_1,\dots,\phi_k}(d,r)$ for a specific value of $(x_1,\dots,x_k)$. In particular, we say that F is poly-time computable on a set S if there is a machine $M^{\phi_1,\dots,\phi_k}(d,r)$ whose running time is bounded by

$$T(x_1,\dots,x_k,n) \le C(x_1,\dots,x_k)\cdot p(n)$$

for all $((x_1,\dots,x_k) \in S$ for some polynomial $p(n)$. Thereby the cost of "zooming in" for any fixed parameter $(x_1,\dots,x_k)$ is always polynomial in n, with a coefficient that may depend on the point $(x_1,\dots,x_k) \in S$.

For computability purposes, the following simple corollary of Theorem 1.8 will be useful.

Theorem 1.11 *For a set-valued function F as above the following are equivalent:*

(I) The function F is computable as per Definition 1.5.1.
(II) There is an oracle TM $M^{\phi_1,\dots,\phi_k}(n)$ that, for oracles representing a point $x = (x_1,x_2,\dots,x_k) \in S$, outputs an encoding of a set $C_n \in \mathscr{C}$ such that the Hausdorff distance $d_H(F(x),C_n) < 2^{-n}$.

In fact, the computability definitions for real functions, sets, and set-valued functions presented above fit in nicely within the much more general framework of Type Two Efficiency (TTE). See [Wei00], and references therein for more details. In particular, Theorem 1.5 stating that computable $\Rightarrow$ continuous holds in a very broad variety of settings. We will only need it in the case of set-valued functions.

Theorem 1.12 *Suppose $F : S \subset \mathbb{R}^k \to K_2^*$ is computable as per Definition 1.5.1. Then F is continuous on S in the Hausdorff metric.*

Proof. The proof is very similar to the proof of Theorem 1.5. It will be convenient for us to use the version of Definition 1.5.1 given by Theorem 1.11(II). Let $M^{\phi}(n)$ be the Turing Machine with an oracle input for $\bar{x} \in S$, whose existence is postulated by this statement. Let $\bar{x}$ be any point in S, and let $\varepsilon = 2^{-k}$ be given. Again, consider an oracle ϕ for $\bar{x}$ such that $|\phi(n) - \bar{x}| < 2^{-(n+1)}$ for all k. We run $M^{\phi}(k+1)$ with this oracle ϕ. By design, it outputs a set L which is a $2^{-(k+1)}$ approximation of $F(\bar{x})$ in the Hausdorff metric.

The computation is performed in a finite amount of time. Hence there is an m such that ϕ is only queried with parameters not exceeding m. Then, for any $\bar{y}$ such that $|\bar{x} - \bar{y}| < 2^{-(m+1)}$, ϕ is a valid oracle for $\bar{y}$ up to parameter value of m. In particular, we can create an oracle ψ for $\bar{y}$ that agrees with ϕ on $1, 2, \ldots, m$. If $\bar{y} \in S$, then the execution of $M^{\psi}(k+1)$ will be identical to the execution of $M^{\phi}(k+1)$, and it will output L which has to be an approximation of $F(\bar{y})$. Thus we have

$$\mathrm{dist}_H(F(\bar{x}), F(\bar{y})) \leq \mathrm{dist}_H(F(\bar{x}), L) + d_H(F(\bar{y}), L) < 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}.$$

This is true for any $\bar{y} \in B(\bar{x}, 2^{-(m+1)}) \cap S$. Hence F is continuous on S. $\square$

Example 1.2. Let the complex plane $\mathbb{C}$ be naturally identified with $\mathbb{R}^2$. Let $d > 1$ be an integer. Consider the multi-valued function $f_d = \sqrt[d]{\ } : \mathbb{C} \to \mathbb{C}$. There is no continuous single-valued branch of f_d on the entire complex plane, and hence there is no computable branch of f_d that is defined on the entire $\mathbb{C}$. There is a computable branch of f_d that is defined everywhere except for a slit connecting 0 to ∞.

On the other hand, if we view the function f_d as a set-valued function that maps a number $z = r \cdot e^{2\pi i \theta}$ to its d roots

$$\{r^{1/d} \cdot e^{2\pi i \theta/d}, r^{1/d} \cdot e^{2\pi i(\theta+1)/d}, \ldots, r^{1/d} \cdot e^{2\pi i(\theta+d-1)/d}\},$$

then it is not hard to see that f_d becomes computable. And indeed, the map $f_d : \mathbb{R}^2 \to K_2^*$ is continuous in the Hausdorff metric.

Note that Definition 1.5.1 makes sense even when $S = \{s\}$ is a singleton. In this case we say that F is *nonuniformly* computable on s. Otherwise, we say that F is *uniformly* computable on the set S. It is generally easier to prove upper bounds and more difficult to prove lower bounds in the non-uniform setting. For example, Theorem 1.12 has no implications in the non-uniform case, as any function on a singleton is continuous.

Theorems 1.9 and 1.10 still apply in the case of set-valued functions, because the reduction from weak to regular computability is a "black box" reduction. Thus, if there is an oracle machine M^{ϕ} that for an oracle ϕ for x *weakly* computes the set $F(x)$ for a set S of parameters x, then there is a machine that computes $F(x)$ on S in the sense of Definition 1.5.1.

Chapter 2
Dynamics of Rational Mappings

2.1 General facts about Riemann surfaces and the hyperbolic metric

A *Riemann surface* is a complex analytic manifold of complex dimension one, which we will also always assume to be connected. As an example, which will be prominently featured in our discussion, consider the *Riemann sphere* $\hat{\mathbb{C}}$. Commonly described as the complex plane $\mathbb{C}$ together with a single point at infinity, it can be specified by an atlas consisting of two charts:

$$\hat{\mathbb{C}} \setminus \{\infty\} \xrightarrow{z \mapsto z} \mathbb{C},$$

$$\hat{\mathbb{C}} \setminus \{0\} \xrightarrow{z \mapsto 1/z} \mathbb{C}.$$

Topologically, $\hat{\mathbb{C}}$ can be identified with the unit sphere $S^2 \subset \mathbb{R}^3$. One such homeomorphism which is particularly convenient, is the stereographic projection of the unit sphere $S^2 = \{x \ : \ |x| = 1\} \subset \mathbb{R}^3$:

$$P : S^2 \setminus \{\text{North Pole}\} \mapsto \mathbb{C}.$$

Its inverse is given by

$$P^{-1} : z \mapsto \left(\frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right); \ P^{-1} : \infty \mapsto (0,0,1).$$

It induces the *spherical metric* on $\hat{\mathbb{C}}$, given by

$$ds = \frac{dz}{1+|z^2|}.$$

Note that for this metric the map $z \mapsto 1/z$ is an isometry.

M. Braverman and M. Yampolsky. *Computability of Julia Sets*.
Algorithms and Computation in Mathematics,

Our definitions of complexity and computability naturally extend to closed subsets of $\hat{\mathbb{C}}$. The easiest way to formalize this is by identifying $\hat{\mathbb{C}}$ with $S^2 \subset \mathbb{R}^3$ via the stereographic projection P, as above:

Definition 2.1.1 *We say that a closed set $K \subset \hat{\mathbb{C}}$ is computable if $P^{-1}(K)$ is a computable subset of $\mathbb{R}^3$. The time complexity of K is again defined to be the time complexity of $P^{-1}(K) \subset \mathbb{R}^3$.*

It is evident that, when $K \subset \mathbb{C} \subset \hat{\mathbb{C}}$, the above definition is equivalent to the computability of K as a subset of $\mathbb{R}^2$. We formulate this as a proposition, together with a corresponding complexity statement, for ease of future reference:

Proposition 2.1 *A closed set $K \subset \hat{\mathbb{C}} \setminus \{\infty\} \cong \mathbb{C}$ is computable as a subset of $\hat{\mathbb{C}}$ if and only if K is computable as a subset of $\mathbb{R}^2$. Similarly, K is poly-time computable as a subset of $\hat{\mathbb{C}}$ if and only if it is poly-time computable as a subset of $\mathbb{C}$.*

The Riemann sphere is an example of a simply-connected Riemann surface. A fundamental Uniformization Theorem, due to Poincaré and Koebe, states that up to isomorphism there are only three such surfaces:

Uniformization Theorem. *Any simply-connected Riemann surface is conformally isomorphic to one of the following:*

- *the complex plane $\mathbb{C}$;*
- *the unit disk $\mathbb{U} = \{|z| < 1\}$;*
- *the Riemann sphere $\hat{\mathbb{C}}$.*

The three possibilities here are really distinct: while there are natural conformal inclusions

$$\mathbb{U} \hookrightarrow \mathbb{C} \hookrightarrow \hat{\mathbb{C}},$$

the Liouville's Theorem and Maximum Modulus Principle of classical complex analysis assert that every holomorphic map $\mathbb{C} \to \mathbb{U}$ or $\hat{\mathbb{C}} \to \mathbb{C}$ must be constant.

Recall that a *covering map* is a continuous map $f : U \to V$ between topological spaces, such that for each $v \in V$ has an open neighborhood W such that $f^{-1}(W)$ is a disjoint union of open neighborhoods X_i and $f : X_i \to W$ is a homeomorphism.

As a corollary to the Uniformization Theorem, we have:

Universal Covering Theorem. *For every Riemann surface S there exists a complex-analytic covering map $U \to S$, where U is one of $\mathbb{U}$, $\mathbb{C}$, or $\hat{\mathbb{C}}$.*

The only Riemann surface whose universal covering is $\hat{\mathbb{C}}$ is $\hat{\mathbb{C}}$ itself. For $\mathbb{C}$ the list is slightly longer: it also includes the cylinder $\mathbb{C}/\mathbb{Z}$ and conformal tori $\mathbb{C}/\Lambda$, where Λ is a two-generator lattice in $\mathbb{C}$. For most of the Riemann surfaces the universal covering is thus the unit disk $\mathbb{U}$, in which case a surface is called *hyperbolic*.

For instance:

Proposition 2.2 *Any domain $W \subset \hat{\mathbb{C}}$ whose complement contains at least three points is a hyperbolic Riemann surface.*

The statement of the Uniformization Theorem thus implies the Riemann Mapping Theorem: every simply-connected bounded domain $W \subset \mathbb{C}$ is conformally isomorphic to $\mathbb{U}$. More precisely:

Riemann Mapping Theorem. *Let $W \subset \mathbb{C}$ be a simply-connected bounded domain, and let w be any point in W. There exists a unique conformal isomorphism*

$$\Psi : W \to \mathbb{U} \text{ with } \Psi(w) = 0, \text{ and } \Psi'(w) > 0.$$

As the Riemann Mapping Theorem is going to play a key role in our investigation, let us consider several illustrations. To visualize Ψ, in Figure 2.1 we draw a polar grid in $\mathbb{U}$, with an increasing density at the boundary, and its image in W for two examples.

For a hyperbolic surface there is a natural choice of a conformally-invariant metric:

Proposition 2.3 *There exists a unique, up to multiplication by a positive constant, Riemannian metric on $\mathbb{U}$ which is invariant under every conformal automorphism of $\mathbb{U}$:*

$$\rho_{\mathbb{U}}(z) = \frac{2|dz|}{1-|z|^2}.$$

With this choice of normalization, $\rho_{\mathbb{U}}$ has a constant Gaussian curvature -1. A covering map $\phi : \mathbb{U} \to S$ transforms $\rho_{\mathbb{U}}$ into a metric on S. By the above proposition, it is independent of the choice of the covering. We will denote this metric ρ_S and call it the *hyperbolic metric* on S. We will further denote by dist_S the distance in the hyperbolic metric, and by $||\cdot||_S$ the norm it induces in the tangent spaces. For a differentiable mapping between two hyperbolic Riemann surfaces

$$f : U \to V,$$

the expression $||f'(z)||_{U,V}$ will stand for the magnitude of the derivative computed with respect to the two hyperbolic norms.

For us, the crucial property of the hyperbolic metric is that it is contracted by a holomorphic map:

Schwarz-Pick Theorem. *If*

$$f : U \to V$$

is a holomoprhic mapping between two hyperbolic Riemann surfaces, then

$$||f'(z)||_{U,V} \leq 1 \text{ for every point } z \in U.$$

Moreover, if equality holds for some $z \in U$, then it holds everywhere, and f is a covering map.

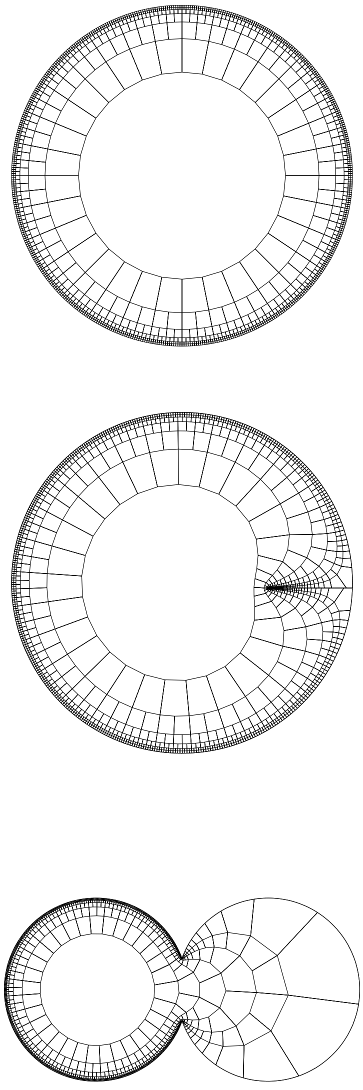

Fig. 2.1 Some illustrations of Riemann mappings: a polar grid in $\mathbb{U}$, and its conformal images for: (a) a disk with a slit $\mathbb{U} \setminus (x,1)$, $x>0$ with marked point $w=0$; (b) the union of two overlapping disks $B(0,1) \cup B(2-\varepsilon,1)$ with a marked point $w=0$. Pictures produced using the numerical package "Zipper" by D. Marshall [Mar].

Passing to the universal coverings for $\phi_U : \mathbb{U} \to U$ and $\phi_V : \mathbb{U} \to V$, with the normalization $\phi_U(0) = z$, $\phi_V(0) = f(z)$, we obtain the Schwarz Lemma of classical complex analysis:

Schwarz Lemma. *If*

$$f : \mathbb{U} \to \mathbb{U}$$

is a holomorphic mapping with $f(0) = 0$, then $|f'(0)| \leq 1$. If the equality holds, then f is a conformal isomorphism of $\mathbb{U}$, in this case a rotation $f(z) = \lambda z$ with $|\lambda| = 1$.

If the equality does not hold, then $|f(z)| < |z|$ for all $z \neq 0$.

Let us quote two more facts of classical complex analysis in a similar vein, which will be useful to us. First we state:

Koebe One-Quarter Theorem. *Let*

$$f : \mathbb{U} \to \mathbb{C}$$

be a one-to-one conformal mapping with $f(0) = 0$ and $f'(0) = 1$. Then the image $f(\mathbb{U})$ covers the disk $B(0, 1/4)$.

Closely related is the following (see e.g. [Pom92]):

Koebe Distortion Theorem. *Let $f : \mathbb{U} \to \mathbb{C}$ be a one-to-one conformal mapping, and fix $r < 1$. For any point z with $|z| < r$, we have:*

$$\frac{1-r}{(1+r)^3} \leq \frac{|f'(z)|}{|f'(0)|} \leq \frac{1+r}{(1-r)^3}.$$

It is not hard to see from the expression for $\rho_{\mathbb{U}}$ that the hyperbolic length of any path in $\mathbb{U}$ which leads to the boundary $\partial\mathbb{U}$ is infinite. On the other hand, on any compact subset of $\mathbb{U}$, the hyperbolic metric is equivalent to the Euclidean one. Let us make a general note for future use:

Proposition 2.4 *Given a hyperbolic domain $W_2 \subset \hat{\mathbb{C}}$, and a subdomain*

$$W_1 \Subset W_2,$$

there exists a constant $C > 1$ such that for all $z \in W_1$

$$C^{-1} < \frac{|dz|}{\rho_{W_2}(z)} < C.$$

Moreover, if W_1 and W_2 are in $\mathscr{C}$, then such a constant can be obtained constructively.

Proof. A crude but quick constructive estimate relies on the Schwarz-Pick Theorem. Apply a Möbius map at first, if necessary, to ensure that $W_2 \Subset \mathbb{C}$. Consider two disks around a point $z \in W_1$:

$$B(z, r) \subset W_2 \subset B(z, R).$$

By the Schwarz-Pick Theorem, the inclusions

$$B(z,r) \underset{\iota_1}{\hookrightarrow} W_2 \underset{\iota_2}{\hookrightarrow} B(z,R)$$

are hyperbolic contractions. Hence

$$\frac{2}{R}|dz| < \rho_{W_2}(z) < \frac{2}{r}|dz|.$$

For a compact subdomain $W_1 \Subset W_2$, the values of r and R can be chosen simultaneously (and constructively) for all points z.

□

A much better way of estimating the hyperbolic metric can be derived from the Koebe One-Quarter Theorem (compare [Mil06]):

Proposition 2.5 *Let $U \subset \mathbb{C}$ be a simply-connected domain, and denote by $r(z)$ the distance from a point z in the plane to the boundary of U. Then for all $z \in U$*

$$\frac{1}{2r(z)} \leq \rho_U(z) \leq \frac{2}{r(z)}.$$

Recall that a mapping $h : X \to Y$ between two topological spaces is *proper* if the preimage of every compact set $K \subset Y$ is compact in X. A non-constant, complex-analytic, and proper mapping

$$f : U \to V$$

between two Riemann surfaces is a *branched covering*. This means that there exists a discrete set of *ramification points* $S \subset U$ such that, restricted to $U \backslash S$, the map becomes a covering. In a neighborhood of each ramification point the map f in appropriate coordinates becomes $z \mapsto z^n$ for some $n \in \mathbb{N}$, with 0 corresponding to the ramification point itself.

As an example of this, consider analytic branched coverings of the Riemann sphere

$$R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}.$$

Every such R is a rational mapping $R(z)$. Its degree $d \in \mathbb{N}$ is the maximum of the degrees of two polynomials $P(z)$, $Q(z)$ without common factors, such that $R = P/Q$. The ramification points of R are its *critical points*

$$R'(z) = 0.$$

There are at most $2d - 2$ of them when counted with multiplicity.

2.2 Julia sets of rational mappings

2.2.1 Basic properties of Julia sets

An excellent general reference for the material in this section is the book of Milnor [Mil06]. For a rational mapping R of degree $\deg R = d \geq 2$ considered as a dynamical system on the Riemann sphere

$$R : \hat{\mathbb{C}} \to \hat{\mathbb{C}},$$

the Julia set is defined as the complement of the set where the dynamics is Lyapunov-stable:

Definition 2.2.1 *Denote by $F(R)$ the set of points $z \in \hat{\mathbb{C}}$ having an open neighborhood $U(z)$ on which the family of iterates $R^n|_{U(z)}$ is equicontinuous. The open set $F(R)$ is called the Fatou set of R and its complement $J(R) = \hat{\mathbb{C}} \setminus F(R)$ is the Julia set.*

It is evident from the definition that

Proposition 2.6 *The Julia set is completely invariant under the action of R, that is, $R^{-1}(J(R)) = J(R)$.*

In the case when the rational mapping is a polynomial

$$P(z) = a_0 + a_1 z + \cdots + a_d z^d : \mathbb{C} \to \mathbb{C}$$

an equivalent way of defining the Julia set is as follows. Obviously, there exists a neighborhood of ∞ on $\hat{\mathbb{C}}$ on which the iterates of P uniformly converge to ∞. Denoting by $A(\infty)$ the maximal such domain of attraction of ∞ we have $A(\infty) \subset F(R)$. We then have

$$J(P) = \partial A(\infty).$$

The closed bounded set $\hat{\mathbb{C}} \setminus A(\infty)$ is called the *filled Julia set*, and denoted $K(P)$; it consists of points whose orbits under P remain bounded:

$$K(P) = \{z \in \hat{\mathbb{C}} | \sup_n |P^n(z)| < \infty\}.$$

The name "filled" stems from the following easy consequence of the Maximum Principle:

Proposition 2.7 *The set $K(P)$ is full, that is, the open set $\hat{\mathbb{C}} \setminus K(P)$ has a single connected component.*

A very simple example

Consider the polynomial mapping $f_0(z) = z^2$ acting on the Riemann sphere $\hat{\mathbb{C}}$. Since the n-th iterate

$$f_0^n(z) = z^{2^n},$$

for all points z with $|z| > 1$ we have $f_0^n(z) \to \infty$, uniformly on compact sets. The closed unit disk $\overline{B(0,1)}$ is invariant under the action of f_0, and therefore coincides with the filled Julia set $K(f)$.

Thus the unit circle $S^1 = J(f_0)$. In the interior $\overset{\circ}{K}(f_0) = B(0,1)$ the iterates $f_0^n(z)$ converge to 0 locally uniformly. Equicontinuity of the family of iterates $\{f_0^n(z)\}$ fails in any open set intersecting with S^1, as arbitrarily near S^1 there are both points whose iterates converge to 0, and to ∞.

Restricted to the unit circle

$$S^1 = \{\exp(2\pi i\theta),\ \theta \in \mathbb{R}\},$$

the dynamics of f_0 becomes the angle-doubling mapping:

$$\theta \mapsto 2\theta \operatorname{mod} \mathbb{Z}.$$

The expansiveness of this linear map implies that for each nonempty open arc $I \subset S^1$ there exists an iterate $f_0^m(I)$ which covers all of S^1. As a consequence, periodic points $\zeta = f_0^j(\zeta)$ are dense in S^1. The reader is invited to verify a more precise version of this statement:

Proposition 2.8 *An angle $\theta \in [0,1)$ is periodic under the doubling map if an only if θ is a rational number, which in its reduced form p/q has an odd denominator q.*

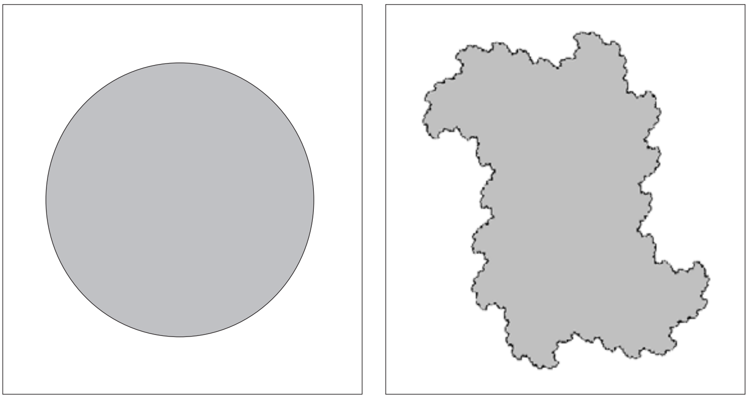

Fig. 2.2 Julia sets of f_0 (left), and $f_\varepsilon = f_0 + \varepsilon$ with $\varepsilon = 0.3 + 0.3i$. The filled Julia set is rendered gray, the Julia set is the black border.

The topological complexity of the dynamics of f_0 on its Julia set is typical, but the geometric simplicity of $J(f_0)$ is an anomaly. To see a more familiar fractal image, consider the picture of the Julia set of $f_\varepsilon(z) = f_0 + \varepsilon$ for a small non-zero value of ε (Figure 2.2). The set $J(f_\varepsilon)$ remains a Jordan curve, and there exists a homeomorphism $\phi : J(f_\varepsilon) \to S^1$ which reduces the dynamics of f_ε to that of f_0:

$$\phi \circ f_\varepsilon \circ \phi^{-1} = f_0.$$

Here is a more interesting example of a Julia set which we have borrowed from a survey of Devaney [Dev04]. A well-known fractal set is obtained as follows. Start with a square of unit size, and divide it into nine sub-squares of equal size. Then remove the interior of the middle square. Now iterate this process, beginning with each of the remaining eight squares. The nowhere dense fractal set S which is obtained as the limit is called the *Sierpinski carpet*.

Among its many interesting properties, is a topological universality in the following sense. A topological dimension of a topological space is the smallest n such that every finite open cover has a refinement in which no point is contained in more than n elements. Any compact connected set whose topological dimension is one can be homeomorphically embedded into S.

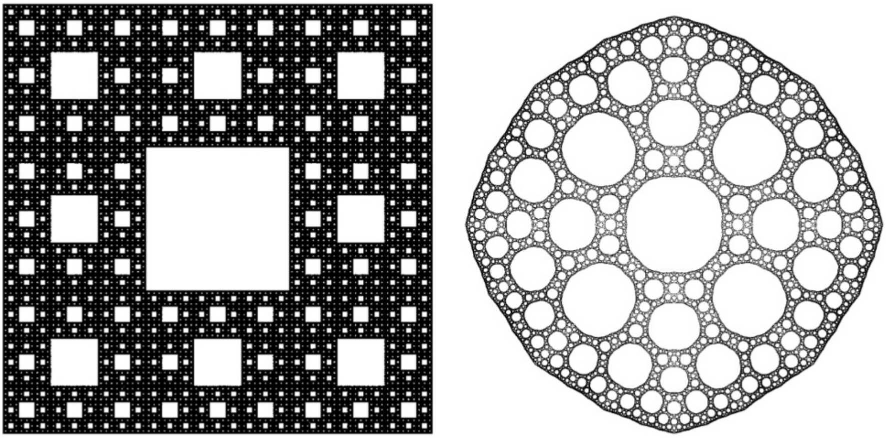

Fig. 2.3 On the left is the Sierpinski carpet. On the right is the Julia set of $h(z) = z^2 - \frac{1}{16z^2}$, which is homeomorphic to the set on the left.

For a "naturally occurring" example of a Sierpinksi carpet, consider for instance, the rational mapping

$$h(z) = z^2 - \frac{1}{16z^2}.$$

The fixed point $\infty \in \hat{\mathbb{C}}$ attracts the orbits of all nearby points. If an orbit of a point converges to infinity, then so do the orbits of all points which are sufficiently near. The open set of all points with this property is the Fatou set $F(h)$. We can describe its structure as follows (without proofs). First, the connected component of $F(h)$ which contains the point ∞ turns out to be a topological disk. Let us denote it by D. The boundary ∂D is a Jordan curve. Since the first iterate of a point in a small neighborhood of the origin will be contained in D, there is a set $D' \Subset \hat{\mathbb{C}} \setminus D$ around the origin which is a component of the first preimage of D. It is not hard to see that it is also simply connected. Removing it from the closed disk $\hat{\mathbb{C}} \setminus D$ has the effect of removing the first middle square in the construction of the Sierpinksi carpet. The next disks to be removed are the four preimages of D' in the annulus $\hat{\mathbb{C}} \setminus (D \cup D')$. The set $F(h)$ is, in fact, homeomorphic to Sierpinski carpet.

For future reference, let us summarize in a proposition below some of the main properties of Julia sets:

Proposition 2.9 *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function. Then the following properties hold:*

- *$J(R)$ is a non-empty compact subset of $\hat{\mathbb{C}}$ which is completely invariant: $R^{-1}(J(R)) = J(R)$;*
- *$J(R) = J(R^n)$ for all $n \in \mathbb{N}$;*
- *$J(R)$ is perfect ($J(R)$ is closed and each of its points is a limit point);*
- *if $J(R)$ has non-empty interior, then it is the whole of $\hat{\mathbb{C}}$;*
- *let $U \subset \hat{\mathbb{C}}$ be any open set with $U \cap J(R) \neq \emptyset$. Then there exists $n \in \mathbb{N}$ such that $R^n(U) \supset J(R)$;*
- *periodic orbits of R are dense in $J(R)$.*

2.2.2 Computability without oracle access to c

To see an example of applying the ideas of computable analysis to Julia sets, let us discuss the following natural question: how easy or how difficult is it to draw a picture of a Julia set of a rational function *without* an oracle access to the values of its coefficients?

As we see below, in such conditions even very simple Julia sets become algorithmically non-computable. As an example, let us consider the family of quadratic polynomials $f_c(z) = z^2 + c$ with one complex coefficient c. Note first the following elementary statement:

Proposition 2.10 *If $c \in (-\infty, -2)$ then $K_c \subset \overline{B(0, \beta_c)}$, where $\beta_c = \sqrt{1/4 - c} + 1/2 > 2$ is a fixed point of f_c.*

Proof. Let $z \in \mathbb{C}$ with $|z| = \beta_c + \delta$, for some $\delta > 0$. By the Triangle Inequality,

$$f_c(z)| = |z^2 + c| \geq |z^2| - |c| = |z|^2 + c = (\beta_c + \delta)^2 + c >$$
$$> \beta_c^2 + c + 2\beta_c\delta = f_c(\beta_c) + 2\beta_c\delta > \beta_c + 4\delta.$$

It follows immediately that $f_c^n(z) \to \infty$, and hence $K_c \subset \overline{B(0, \beta_c)}$. □

As $\beta_c = f_c(\beta_c)$, this point itself lies in K_c. As the above proposition implies,

$$\beta_c \in \partial K_c = J_c.$$

We are now in a position to prove our first negative result on the computability of Julia sets:

Theorem 2.11 *Let $c < -2$ be a non-computable real number. Then the Julia set J_c is non-computable by a Turing Machine without oracle access to c.*

Proof. The fixed point $\beta_c = \sqrt{1/4 - c} + 1/2$ of the mapping f_c is repelling under our assumption on c, and hence lies in the Julia set. By the previous proposition,

$$\beta_c = \sup_{z \in J_c} |z|.$$

Now assume that there exists a Turing Machine $M(n)$ which computes J_c. Use it to determine the largest $j > 0$ such that $j \cdot 2^{-n}$ is at most 2^{-n}-far from all points in J_c. Then

$$0 < (j \cdot 2^{-n} - \beta_c) < 2^{-(n-1)},$$

and hence β_c is computable. But

$$c = \beta_c - \beta_c^2,$$

which contradicts the assumption that c is a non-computable real.

□

In the conditions of the above theorem, the quadratic polynomial f_c is of a particularly simple type (it is hyperbolic, and its Julia set is a Cantor set). In particular, as we will see below, given oracle access to c, its Julia set is computable in poly-time. This example thus confirms what we have expected: the values of the coefficients of a rational map R have to be made available to the algorithm, to have a sensible discussion of the computational hardness of $J(R)$.

2.2.3 Periodic orbits of rational maps and the structure of the Fatou set

For a periodic point $z_0 = R^p(z_0)$ of period p, its *multiplier* is the derivative $\lambda = \lambda(z_0) = DR^p(z_0)$. We may speak of the multiplier of a periodic cycle, as it is the

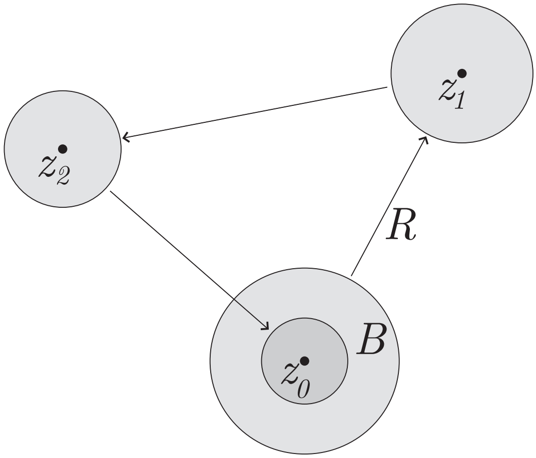

Fig. 2.4 An attracting cycle of a rational map R with period 3. The multiplier $\lambda = DR^3(z_0) = R'(z_0) \cdot R'(z_1) \cdot R'(z_2) \in \mathbb{U}$. In a small enough disk B around z_0, the iterate $R^3(z) \approx z_0 + \lambda(z - z_0)$, and $R^3(B) \Subset B$.

same for all points in the cycle by the Chain Rule. In the case when $|\lambda| \neq 1$, the dynamics in a sufficiently small neighborhood of the cycle is governed by the Mean Value Theorem: when $|\lambda| < 1$, the cycle is *attracting* (*super-attracting* if $\lambda = 0$), and if $|\lambda| > 1$ it is *repelling*.

In both the attracting and repelling cases, the dynamics can be locally linearized:

$$\psi(R^p(z)) = \lambda \cdot \psi(z) \tag{2.2.1}$$

where ψ is a conformal mapping of a small neighborhood of z_0 to a disk around 0. In the case when $|\lambda| = 1$, so that $\lambda = e^{2\pi i\theta}$, $\theta \in \mathbb{R}$, the simplest to study is the *parabolic case* when $\theta = n/m \in \mathbb{Q}$, and so λ is a root of unity. In this case R^p is not locally linearizable; it is not hard to see that $z_0 \in J(R)$. The description of the dynamics in a small neighborhood of a parabolic orbit will be discussed below in some detail.

Irrationally indifferent periodic points

When θ is irrational, the orbit is called *irrationally indifferent*. In this situation, two non-vacuous possibilities are considered: the *Cremer case*, when R^p is not linearizable, and *Siegel case*, when it is. In the latter case, the linearizing map ψ from (2.2.1) conjugates the dynamics of R^p on a neighborhood $U(z_0)$ to the irrational rotation by angle θ (the *rotation angle*) on a disk around the origin. The maximal such neighborhood of z_0 is called a *Siegel disk* (see Figure 2.5).

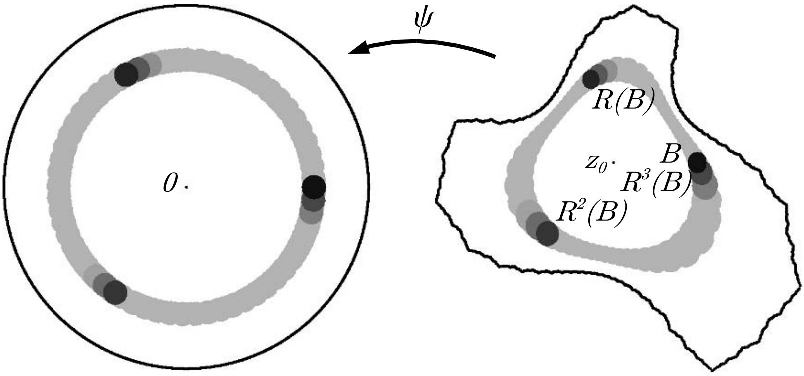

Fig. 2.5 A Siegel disk of a rational mapping R (on the left). The linearizing coordinate ψ is shown. The orbit of a small disk B inside the Siegel disk is indicated, as well as its image on the right.

A different kind of a rotation domain may occur only for a non-polynomial rational mapping R. A *Herman ring A* is a conformal image

$$\nu : \{z \in \mathbb{C} |\, 0 < r < |z| < 1\} \to A,$$

such that

$$R^p \circ \nu(z) = \nu(e^{2\pi i\theta} z),$$

for some $p \in \mathbb{N}$ and $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

The existence of rational maps with Siegel disks was first shown by C. Siegel in 1942 [Sie42]. To formulate his result, several definitions will be needed. For a number $\theta \in [0,1)$ denote by $[r_1, r_2, \dots, r_n, \dots]$, $r_i \in \mathbb{N} \cup \{\infty\}$ its possibly finite continued fraction expansion:

$$[r_1, r_2, \dots, r_n, \dots] \equiv \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cfrac{1}{\cdots + \cfrac{1}{r_n + \cdots}}}} \tag{2.2.2}$$

Such an expansion is defined uniquely if and only if $\theta \notin \mathbb{Q}$. In this case, the *rational convergents* $p_n/q_n = [r_1, \dots, r_n]$ are the closest rational approximants of θ among the numbers with denominators not exceeding q_n. In fact, setting $\lambda = e^{2\pi i\theta}$, we have

$$|\lambda^h - 1| > |\lambda^{q_n} - 1| \text{ for all } 0 < h < q_{n+1},\ h \neq q_n.$$

The difference $|\lambda^{q_n} - 1|$ lies between $2/q_{n+1}$ and $2\pi/q_{n+1}$, and therefore the rate of growth of the denominators q_n describes how well θ may be approximated with rationals.

Definition 2.2.2 *The Diophantine numbers of order k, denoted by $\mathscr{D}(k)$, is the following class of irrationals "badly" approximated by rationals. By definition, $\theta \in \mathscr{D}(k)$ if there exists $c > 0$ such that*

$$q_{n+1} < c q_n^{k-1}$$

The numbers q_n can be calculated from the recurrence relation

$$q_{n+1} = r_{n+1} q_n + q_{n-1}, \text{ with } q_0 = 0,\ q_1 = 1.$$

Therefore $\theta \in \mathscr{D}(2)$ if and only if the sequence $\{r_i\}$ is bounded. Dynamicists call such numbers *bounded type* (number-theorists prefer *constant type*). An extremal example of a number of bounded type is the inverse golden mean

$$\theta_* = \frac{\sqrt{5}-1}{2} = [1,1,1,\ldots].$$

The set

$$\mathscr{D}(2+) \equiv \bigcap_{k>2} \mathscr{D}_k$$

has full measure in the interval $[0,1)$. Siegel showed:

Theorem 2.12 ([Sie42]) *Let R be an analytic map with a periodic point $z_0 \in \hat{\mathbb{C}}$ of period p. Suppose that the multiplier λ of the cycle is*

$$\lambda = e^{2\pi i\theta} \textit{ with } \theta \in \mathscr{D}(2+),$$

then the local linearization equation (2.2.1) holds.

Structure of the Fatou set

The term *basin* in what follows will describe the set of points whose orbits converge to a given periodic orbit under the iteration of R. We will denote $\mathrm{Crit}(R)$ the *critical set of R*, defined as the set of points $z \in \hat{\mathbb{C}}$ where $R'(z) = 0$. Further, we set the *postcritical set of R* to be

$$\mathrm{Postcrit}(R) = \overline{\cup_{i\geq 0} R^i(\mathrm{Crit}(R))}.$$

Fatou made the following observation:

Proposition 2.13 *Let $p_1, \ldots, p_k$ be a periodic orbit of a rational mapping R. If it is either attracting or parabolic, then its basin contains a critical point of R.*

By a perturbative argument, Fatou then concluded that for a rational mapping R with $\deg R = d \geq 2$ at most finitely many periodic orbits are non-repelling. A sharp bound on their number depending on d has been established by Shishikura; it is equal to the number of critical points of R counted with multiplicity:

Fatou-Shishikura Bound. *For a rational mapping of degree d the number of the non-repelling periodic cycles taken together with the number of cycles of Herman rings is at most $2d-2$. For a polynomial of degree d the number of non-repelling periodic cycles in $\mathbb{C}$ is at most $d-1$.*

Therefore, we may refine the last statement of Proposition 2.9:

- *repelling periodic orbits are dense in $J(R)$.*

Classical results of Fatou also imply the following:

Proposition 2.14 *Every Cremer point of a rational mapping R as well as every point of the boundary of a Siegel disk or a Herman ring is contained in* $\mathrm{Postcrit}(R)$.

By definition, the basin of an attracting or a parabolic point, as well as preimages of Siegel disks and Herman rings, belong to the Fatou set. The Fatou-Sullivan Classification Theorem formulated below rules out other possibilities:

Fatou-Sullivan Classification. *For every connected component $W \subset F(R)$ there exists $m \in \mathbb{N}$ such that the image $H = R^m(W)$ is periodic under the dynamics of R. Moreover, each periodic Fatou component H is of one of the following types:*

- *a component of the basin of an attracting or a super-attracting periodic orbit;*
- *a component of the basin of a parabolic periodic orbit;*
- *a Siegel disk;*
- *a Herman ring.*

Quadratic polynomials

To conclude the discussion of the basic properties of Julia sets, let us consider the simplest examples of non-linear rational endomorphisms of the Riemann sphere, the quadratic polynomials. Every affine conjugacy class of quadratic polynomials has a unique representative of the form $f_c(z) = z^2 + c$, the family

$$f_c(z) = z^2 + c,\ c \in \mathbb{C}$$

is often referred to as *the quadratic family*. For a quadratic map the structure of the Julia set is governed by the behavior of the orbit of the only finite critical point 0. In particular, the following dichotomy holds:

Proposition 2.15 *Let $K_c = K(f_c)$ denote the filled Julia set of f_c, and $J_c = J(f_c) = \partial K$. Then:*

- *$0 \in K_c$ implies that K_c is a connected, compact subset of the plane with connected complement;*
- *$0 \notin K_c$ implies that $K_c = J_c$ is a planar Cantor set.*

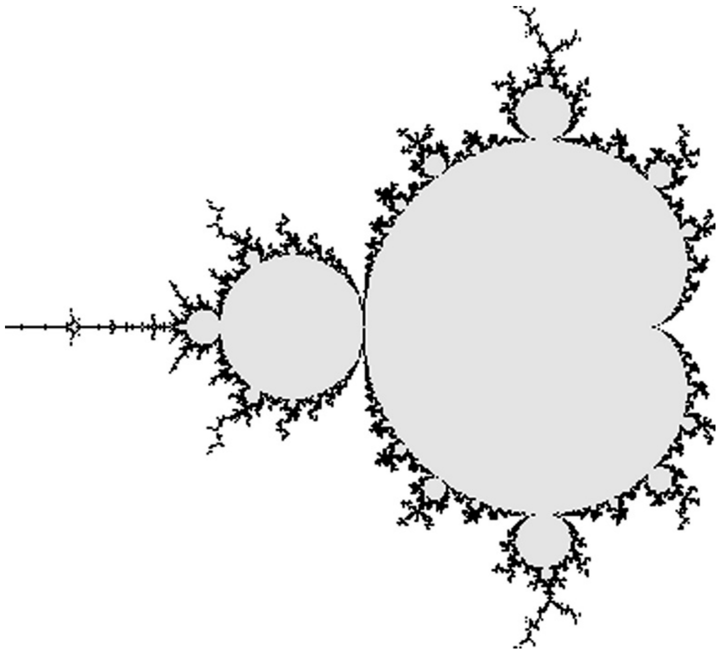

Fig. 2.6 The Mandelbrot set $\mathscr{M}$

The *Mandelbrot set* $\mathscr{M} \subset \mathbb{C}$ is defined as the set of parameter values c for which $J(f_c)$ is connected.

It is easy to verify that for $|c| > 2$ the orbit of the critical point 0 of f_c converges to ∞. Moreover, the only value of c with $|c| = 2$ for which this does not happen is $c = -2$. We thus have:

Proposition 2.16 *The set $\mathscr{M}$ is contained in $B(0,2) \cup \{-2\}$.*

The Fatou-Shishikura Bound implies that a quadratic polynomial has at most one non-repelling cycle in the complex plane. We will therefore call the polynomial f_c (the parameter c, the Julia set J_c) *Siegel, Cremer,* or *parabolic* when it has an orbit of the corresponding type.

Chapter 3
First Examples

3.1 A case study: hyperbolic Julia sets

Unless otherwise specified, in this section "dist" will stand for the distance in the spherical metric in $\hat{\mathbb{C}}$.
A rational mapping $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is called *hyperbolic* if the orbit of every critical point of R is either periodic, or converges to an attracting (or a super-attracting) cycle.
As easily follows from Implicit Function Theorem and considerations of local dynamics of an attracting orbit, hyperbolicity is an open property in the space of coefficients of rational mappings of degree $d \geq 2$. Fatou has conjectured that hyperbolic parameters are also dense in this space. This conjecture, known as the Density of Hyperbolicity Conjecture, forms the central open question in Complex Dynamics.
Considered as a rational mapping of the Riemann sphere, a quadratic polynomial $f_c(z) = z^2 + c$ has two critical points: the origin, and the super-attracting fixed point at ∞. In the case when $c \notin \mathscr{M}$, the orbit of the former converges to the latter, and thus f_c is hyperbolic. Proposition 2.13 implies that whenever f_c has an attracting orbit in $\mathbb{C}$, it is a hyperbolic mapping and $c \in \mathscr{M}$. In the quadratic case, the Density of Hyperbolicity Conjecture thus becomes:

Conjecture (Density of Hyperbolicity in the Quadratic Family). *Hyperbolic parameters are dense in $\mathscr{M}$.*

How accurate is the picture of $\mathscr{M}$ in Figure 2.6? Indeed, our ability to produce accurate images of $\mathscr{M}$ hinges on this set being computable. Peter Hertling [Her05] has shown that Density of Hyperbolicity in the quadratic family implies computability of $\mathscr{M}$.

As an example of a hyperbolic Julia set, consider the quadratic polynomial $f = f_c$ with $c = -0.12 + 0.665i$. This map has a periodic orbit

$$z_0 \approx -0.15+0.19i \mapsto z_1 \approx -0.13+0.61i \mapsto z_2 \approx -0.47+0.5i \mapsto z_0.$$

Its multiplier $\lambda = (f^3)'(z_0)$ satisfies $|\lambda| \approx 0.84 < 1$. We can select a small enough disk $D = B(z_0, \varepsilon)$ so that

$$f^3(z) - z_0 \approx \lambda(z - z_0) \text{ for all } z \in D,$$

and hence $f^3(D) \Subset D$. Using the Fatou-Sullivan Classification together with the Fatou-Shishikura bound, we see that every component of the interior of $K(f)$ belongs to the basin of the attracting orbit. In particular, the orbit of every point in $\overset{\circ}{K}(f)$ eventually passes through D, after which it becomes trapped in $D \cup f(D) \cup f^2(D)$.

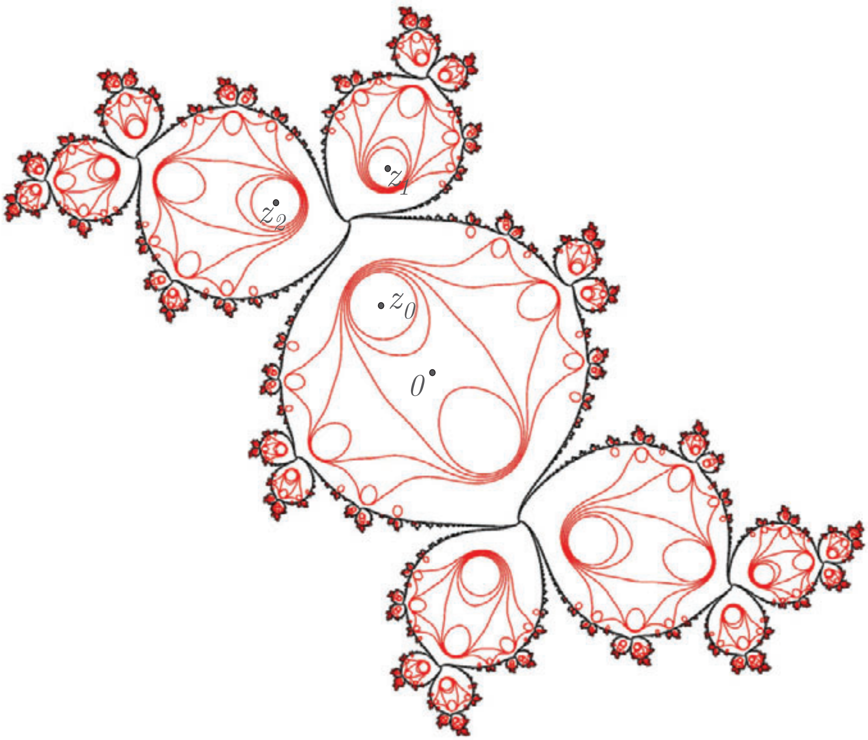

Fig. 3.1 The Julia set of f_c with $c = -0.12+0.665i$. A disk D around an attracting periodic point z_0 is shown together with its first fourteen preimages.

The union of the inverse images

$$\cdots f^{-m}(D) \Supset f^{-(m-1)}(D) \Supset \cdots \Supset f^{-1}(D) \Supset D$$

exhausts the interior of $K(f)$. By Proposition 2.13, one of the sets in this sequence captures the critical point 0 of f. As seen from Figure 3.1 for our choice of D it is the set $f^{-6}(D)$.

Note that the three components of the basin which contain the points of the attracting orbit meet at a "corner" point $p \approx -0.25+0.44i$. The point p itself is fixed under f. The shape of this Julia set is known under the name of a *Douady's rabbit*.

Computability of hyperbolic Julia sets

For a hyperbolic rational map it is easy to account for the points in the complement of the Julia set: they converge to one of the (finitely many) attracting orbits. This fact is key to proving the following.

Theorem 3.1 *Fix $d \geq 2$. There exists a TM M^{ϕ} with oracle access to the coefficients of a rational mapping of degree d which computes the Julia set of every hyperbolic rational map of degree d.*

In preparation to proving the theorem, let us first formulate a general fact:

Proposition 3.2 *Let $Q(z)$ be a complex polynomial. Then there exists a Turing Machine M^{ϕ} with an oracle input for the coefficients of $Q(z)$ such that the following holds. Consider any dyadic ball $B = B(\bar{x}, r) \subset \mathbb{C}$, $\bar{x} \in \mathbb{D}^2$, $r \in \mathbb{D}$, and let $\alpha_1, \dots, \alpha_m$ be the roots of $Q(z)$ contained in B. For any natural number n, the machine M^{ϕ} will take n, r, and $\bar{x}$ as inputs, and will output a finite sequence of complex numbers $\beta_1, \dots, \beta_k$ with dyadic rational real and imaginary parts for which:*

- $\beta_i \in B(\bar{x}, r+2^{-n})$;
- *each β_i lies at a distance not more than 2^{-n} from some root of $Q(z)$;*
- *for every α_j there exists β_i with $|\alpha_j - \beta_i| < 2^{-n}$.*

For a classical reference, see [Wey24]; a review of modern approaches to iterative root-finding algorithms may be found in [BCSS98].

To understand how the Weyl's root-finding algorithm works, consider first how a square of side L in the complex plane can be tested for the presence of zeros of a polynomial Q. Not the most efficient, but a rather straightforward test, is given by the Argument Principle:

$$\frac{1}{2\pi i} \oint_{\partial G} \frac{Q'(z)}{Q(z)} dz = \text{number of zeros inside of } G$$

for any domain G in the plane whose boundary does not contain a zero. One may thus verify whether there is an approximate zero inside the square. If so, one proceeds to subdivide the original square into four congruent squares, and apply the test in each of those. The procedure is then repeated with the squares which tested positive, until the size of a square becomes smaller than the desired precision of the approximation.

Next, we show that if we *know* that a rational map R is hyperbolic, then given enough time we will find all of its attracting periodic orbits. We can actually construct *trapping discs* around the attracting orbits, such as the disc D and its preimages in the example above.

Proposition 3.3 *There is a TM M^ϕ that, given oracle access to the coefficients of R, outputs a finite sequence of discs $B_i = B(c_i, r_i)$ on $\hat{\mathbb{C}}$ with dyadic centers and dyadic radii such that*

- *all the attracting and super-attracting orbits of R belong to $B \equiv \cup B_i$,*
- *all orbits under R originating in B converge to an attracting periodic orbit, and*
- *$R(B) \Subset B$.*

Proof. Let $m \in \mathbb{N}$. By Proposition 3.2, it is possible to constructively approximate all periodic points of R in $\hat{\mathbb{C}}$ whose periods are at most m with precision $2^{-(m+3)}$ in the spherical metric.

For each such a periodic point z_i we will thus obtain its approximate position p_i together with a positive integer k_i such that $R^{k_i}(z_i) = z_i$. We will now approximate the image $R^{k_i}(B(p_i, 2^{-m/2}))$ of a ball around p_i. In other words, we will compute a set $W \in \mathscr{C}$ such that

$$\mathrm{dist}_H(W, R^{k_i}(B(p_i, 2^{-m/2}))) < 2^{-(m+1)},$$

and will attempt to verify that

$$B(W, 2^{-m}) \subset B(p_i, 2^{-m/2}). \tag{3.1.1}$$

This would imply

$$R^{k_i}(B(p_i, 2^{-m/2})) \subset B(W, 2^{-(m+1)}) \subset B(p_i, 2^{-m/2} - 2^{-(m+1)}). \tag{3.1.2}$$

Note that if z_i is an attracting point, then the equation (3.1.1) will hold for any sufficiently large value of m. On the other hand, by the Schwarz Lemma, equation (3.1.2) implies the existence of an attracting orbit in $B(p_i, 2^{-m/2})$, whose basin contains $B(p_i, 2^{-m/2})$. Once m and W satisfying (3.1.1) are found, we can also compute dyadic balls B_j containing each point $R^j(z_i)$ of the cycle

$$z_i \mapsto R(z_i) \mapsto \cdots \mapsto R^{k_i}(z_i) = z_i$$

so that, for each $j = 0, \ldots, k_i - 1$, $R(B_j) \Subset B_{j+1}$, where $j+1$ is taken modulo k_i.

All such B_j's will *eventually* be found, and will satisfy the conditions of the proposition. How will we know when to stop? To this end, we also compute at the m-th step a finite set C_m which is a $2^{-(m+3)}$-approximation of the m-th image $R^m(\mathrm{Crit}(R))$, and attempt to verify that $B(C_m, 2^{-(m+3)})$ is contained in the union

$\cup B_i$ of the balls we have already found. We terminate when this is the case, and output the balls B_i that we have found.

By Fatou's result 2.13, we know that, for each attracting periodic orbit, the orbit of at least one critical point converges to it. Since in our case R is hyperbolic, we know that the orbit of each critical point converges to an attracting periodic orbit, and the algorithm is guaranteed to terminate. □

We are now ready to prove computability of hyperbolic Julia sets.

Proof (Theorem 3.1). As a first step, by looking at one of the balls B_j found in Proposition 3.3, we can compute a dyadic $a \in B_j$ that converges to an attracting orbit of R. Thus $a \notin J(R)$. By conjugating R by the fractional-linear map

$$f_a(z) = \frac{1}{z-a},$$

we obtain a rational map R' with $J(R') = f_a(J(R))$. The effect of the conjugation on the Julia set is to send the point a to ∞. In particular, $\infty \notin J(R')$. Thus, by a simple change of coordinates, we may assume without loss of generality that $\infty \notin J(R)$. By Proposition 2.1 it suffices to prove the computability of $J(R)$ as a subset of $\mathbb{C}$.

Informally, the idea of the argument is to estimate the Julia set from "above" and from "below". On the one hand, we know that the orbit of every point outside $J(R)$ eventually reaches the set B from Proposition 3.3. This allows us to exclude points that are far away from $J(R)$. On the other hand, we know that repelling periodic orbits are dense in $J(R)$, which permits us to eventually identify every point which is close to $J(R)$.

More formally, let $n \in \mathbb{N}$ be the input specifying the required degree of the approximation. The algorithm, which computes a set $J_n \in \mathscr{C}$ with

$$\mathrm{dist}_H(J_n, J(R)) < 2^{-n},$$

works as follows. Denote by U the complement $(\cup B_i)^c$ of the dyadic balls B_i that have been found in Proposition 3.3.

(1) Set $m := 1$.
(2) Compute a set $U_m \in \mathscr{C}$ such that $\mathrm{dist}_H(U_m, R^{-m}(U)) < 2^{-(n+3)}$.
(3) Compute a finite set L_m which approximates with precision $2^{-(n+3)}$ all periodic points of R in U, whose periods are at most m. This is possible by Proposition 3.2.
(4) Check the inclusion $B(L_m, 2^{-(n+1)}) \supset U_m$. If the inclusion holds, output the set $J_n \equiv L_m$ and exit. If not, go to step (5).
(5) Increment $m \leftarrow m+1$ and go to step (2).

Denote by O_m the set of all periodic points of R which are contained in U, and whose periods are at most m. All periodic orbits in U must be repelling. We thus have

$$O_m \subset J(R) = \overline{\cup O_m}. \tag{3.1.3}$$

On the other hand, by the Fatou-Sullivan classification, we have

$$\cap R^{-m}(U) = J(R). \tag{3.1.4}$$

For all m greater than some large enough m_0, the open neighborhood

$$B(O_m, 2^{-(n+3)}) \supset J(R),$$

as seen from the right-hand side of (3.1.3).

On the other hand, for all m greater than some large enough m_1, the open neighborhood

$$B(J(R), 2^{-(n+3)}) \supset R^{-m}(U).$$

Therefore, for each $m \geq \max(m_0, m_1)$ we have

$$B(L_m, 2^{-(n+1)}) \supset B(O_m, 2^{-(n+1)} - 2^{-(n+3)}) \supset B(J(R), 2^{-(n+2)}) \supset \\ B(R^{-m}(U), 2^{-(n+2)} - 2^{-(n+3)}) \supset U_m, \tag{3.1.5}$$

and so the algorithm is guaranteed to terminate. When that happens, by the left-hand side of (3.1.3) we have

$$\operatorname{dist}(z, J(R)) < 2^{-n}$$

for all $z \in L_m$. On the other hand, by (3.1.4) and (3.1.5) we have

$$\operatorname{dist}(z, L_m) < 2^{-n}$$

for all $z \in J(R)$. □

The idea of approximating the Julia set from "above" and "below" which is featured in the above algorithm will be very useful for us in proving positive results. As far as we could tell, its first appearance in the theoretical literature is in the work of Zhong [Zho98]. Its practical applications are, however, rather limited. Of course, one can always attempt to generate images of a Julia set by computing the periodic orbits of periods at most m (or, alternatively, the first m preimages of a single point in $J(R)$). Apart from the fact that the picture may be rather far from the true image of $J(R)$, it will also generally require exponential time to generate.

On the other hand, for a polynomial mapping P, it is easy to determine a domain $U \in \hat{\mathbb{C}}$ whose preimages shrink to the *filled* Julia set $K(P)$. Indeed, any large enough disk around the origin would do. Algorithms approximating $K(P)$ by $P^{-m}(U)$ are perhaps the most widely used. They are known as the *escape-time* algorithms. Their obvious Achilles' heel is the general absence of an estimate on the distance $\operatorname{dist}_H(K(P), P^{-m}(U))$ in terms of m.

Obtaining an estimate on the distance to $J(R)$ in polynomial time requires another idea, which is a key in the proof of the following:

Theorem 3.4 (Cf. [Bra04, Ret05]) *Fix $d \geq 2$. There exists a TM M^{ϕ} with oracle access to the coefficients of a rational mapping of degree d which computes the Julia set of every hyperbolic rational map of degree d with a polynomial complexity bound.*

We begin with the following standard fact:

Proposition 3.5 *A rational mapping R of degree $d \geq 2$ is hyperbolic if and only if there exists a smooth metric μ defined on an open neighborhood of $J(R)$ and a constant $\lambda > 1$ such that the derivative*

$$||DR^n(z)||_\mu > \lambda^n \text{ for every } z \in J(R), n \in \mathbb{N},$$

as long as the image $R^n(z)$ stays in the domain of definition of μ.

Note that the term "hyperbolic" has an established meaning in dynamics. In the context of one dimensional dynamical systems it means "uniformly expanding (or contracting)". Thus Proposition 3.5 justifies the use of the word in our case.

By compactness of $J(R)$, *in the spherical metric*, we will have

$$||DR^n(z)|| > C\lambda^n, \tag{3.1.6}$$

for $C > 0$ independent of n.

The proof of the existence of a metric μ for a hyperbolic mapping R is both instructive and useful for our purposes, and so we outline it below.

Proof (Proposition 3.5.). Let $\{B_i\}$ be the finite collection of dyadic balls around the attracting periodic orbits as in Proposition 3.3. Consider the union $B = \cup B_i$. By Proposition 3.3, the sequence of preimages of B grows successively larger:

$$B \subset R^{-1}(B) \subset R^{-2}(B) \subset \cdots, \text{ and } J(R) \subset \hat{\mathbb{C}} \setminus R^{-k}(B) \text{ for all } k \in \mathbb{N}.$$

By Fatou's result 2.13, there exists $k \in \mathbb{N}$ such that $R^{-k}(B)$ contains the entire post-critical set of R. Setting $V = \hat{\mathbb{C}} \setminus R^{-k}(B)$ and $U = R^{-1}(V)$, we see that $U \subsetneq V$, and

$$R : U \to V$$

is an unbranched analytic covering. By the Schwarz-Pick Theorem, it is an isometry of the hyperbolic metrics of U and V. On the other hand, by the same theorem, the proper inclusion $\iota : U \hookrightarrow V$ is a contraction of the hyperbolic metric. By the Chain Rule, for $z \in U$, we have

$$||DR(z)||_{V,V} = ||D\iota^{-1}(z)||_{V,U} ||DR(z)||_{U,V} = ||D\iota^{-1}(z)||_{V,U} > 1.$$

Note that the Julia set $J(R) \Subset U$ and, selecting a neighborhood $W \subset U$ of $J(R)$ which is compactly contained in V, we have

$$||DR(z)||_\mu > \lambda > 1 \text{ for all } z \in W,$$

where μ denotes the hyperbolic metric ρ_V. By the Chain Rule, the derivative of the n-th iterate

$$||DR^n(z)||_\mu > \lambda^n \text{ for } z \in J(R),$$

which concludes our proof. □

Let us make a useful note:

Proposition 3.6 *The constants λ and C of (3.1.6) can be estimated constructively.*

Proof. The algorithm for estimating C is easily derived from Proposition 2.4.

To estimate λ, note that the contraction coefficient of the inclusion $||\iota'(z)||_{U,V}$ can be bounded by a constant depending only on the value of

$$d = \mathrm{dist}_V(z, V \setminus U)$$

(the distance measured in the hyperbolic metric of V). Indeed, let us lift the inclusion $z \in U \hookrightarrow V$ to $z' \in U' \hookrightarrow \mathbb{U}$. Denote by v any of the points of $\mathbb{U} \setminus U'$ for which

$$\mathrm{dist}_{\mathbb{U}}(z', v) = d.$$

By applying a suitable fractional-linear transformation, send v to 0, and z' to $x \in (0,1)$. An explicit computation gives

$$d = \log \frac{1+x}{1-x}, \text{ so that } x = \frac{e^d - 1}{e^d + 1}.$$

By the Schwarz-Pick Theorem, the hyperbolic derivative of the inclusion $U' \hookrightarrow \mathbb{U}$ will become larger, if we make $\mathbb{U} \setminus U'$ smaller. More specifically, let us consider the domain $W = \mathbb{U} \setminus \{0\}$. Then, comparing the inclusions

$$\iota_1 : U' \hookrightarrow \mathbb{U},\ \iota_2 : W \hookrightarrow \mathbb{U},$$

we have

$$||D\iota(z)||_{U,V} = ||D\iota_1(x)||_{U',\mathbb{U}} \leq ||D\iota_2(x)||_{W,\mathbb{U}}.$$

The expression on the right can be estimated explicitly. It is equal to

$$a(x) = \frac{2|x \log x|}{1 - x^2} < 1.$$

We obtain a lower bound on the expansion factor λ_z at the point z as

$$\lambda(d) = 1/a(x) \text{ for } x = \frac{e^d - 1}{e^d + 1}.$$

Note that this bound decreases with d.

From Proposition 2.4, we can constructively obtain a uniform lower bound

$$d_l \leq \sup_{z \in U} \mathrm{dist}_V(z, V \setminus U).$$

The value of $\lambda(d_l) > 1$ is thus a constructive estimate for the expanding factor λ. $\square$

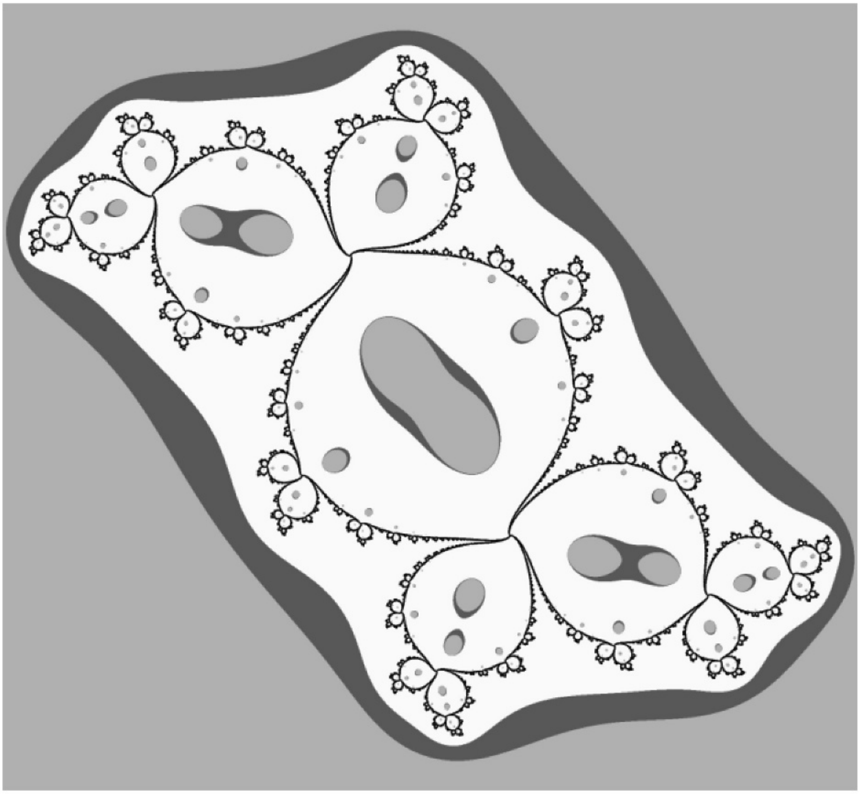

Fig. 3.2 Domains U (the complement of darker gray) and V (the complement of lighter gray) for a rabbit from the previous figure. The rabbit has two attracting orbits in $\hat{\mathbb{C}}$, the fixed point at ∞, and the period-3 cycle.

Proposition 3.7 (Preparatory step in the construction of M^ϕ) *There exists an algorithm which, given the coefficients of a hyperbolic rational map R of degree $d \geq 2$, outputs a planar domain $U \in \mathscr{C}$ such that:*

(I) $U \subsetneq R(U)$,
(II) $R(U) \cap \mathrm{Postcrit}(R) = \emptyset$,
(III) $J(R) \Subset U$.

Proof. We use the balls B_i around the attracting periodic orbits we found in Proposition 3.3. Let $\Omega_0 := \hat{\mathbb{C}} \setminus \cup B_i$. Define $\Omega_{i+1} := R^{-1}(\Omega_i)$ for all $i \geq 0$. By the properties of B_i, we have $\Omega_1 \Subset \Omega_0$. If we let $U_0 \in \mathscr{C}$ be any set such that $\Omega_1 \subset U_0 \subset \Omega_0$, U_0 will satisfy properties (I) and (III) above. To see that (I) holds observe that

$$U_0 \subset \Omega_0 = R(\Omega_1) \subset R(U_0).$$

(III) holds because $J_R \subset \Omega_1 \subset U_0$. Similarly, for any k we can compute $U_k \in \mathscr{C}$ such that $\Omega_{k+1} \subset U_k \subset \Omega_k$. For each such U_k conditions (I) and (III) hold just as they do for U_0.

Note that, if for some k

$$U_{k-2} \cap \mathrm{Postcrit}(R) = \emptyset, \tag{3.1.7}$$

we will be able to verify this. In this case $\Omega_{k-1} \cap \mathrm{Postcrit}(R) = \emptyset$, and thus $R(U_k) \cap \mathrm{Postcrit}(R) = \emptyset$, and $U = U_k$ satisfies condition (II). It remains to see that there is a k such that (3.1.7) holds. To this end, we use Fatou's result 2.13, which guarantees that all postcritical orbits leave Ω_0 in finitely many steps. Hence there is an ℓ such that $\Omega_\ell \cap \mathrm{Postcrit}(R) = \emptyset$, and $k := \ell + 2$ satisfies (3.1.7). □

We are now ready to compute $J(R)$ in polynomial time.

Proof (Theorem 3.4). At the preparatory stage of the computation, we obtain the domain U as in Proposition 3.7. Since the closure of the domain U does not intersect the postcritical set of R, we can compute a lower bound $s > 0$ on the distance from U to $\mathrm{Postcrit}(R)$.

Let us now run the algorithm of Theorem 3.1 to obtain a set $L \in \mathscr{C}$ with

$$\mathrm{dist}_H(L, J(R)) < s/8.$$

Computing several further preimages of U with sufficient precision, we can obtain a smaller domain $\widetilde{W} \Supset J(R)$ and such that:

- $R^2(\widetilde{W}) \Subset U$, and
- $\mathrm{dist}(z, L) < s/4$ for each $z \in \widetilde{W} \cup R(\widetilde{W})$;

and compute a dyadic number $\ell > 0$ such that

$$\mathrm{dist}(z, J(R)) > \ell \text{ for all } z \notin \widetilde{W}.$$

Set $V = R(U)$. Compute a dyadic $\varepsilon > 0$ such that $B(\widetilde{W}, \varepsilon) \Subset U$.

From Proposition 3.6 we find a lower bound $\lambda > 1$ on the expansion $||R'(z)||$ in the hyperbolic metric in V for $z \in R(\widetilde{W})$. We also construct a constant C from (3.1.6) as per Proposition 2.4. Thus, we have

$$||DR^n(z)|| > C\lambda^n,$$

for as long as the orbit of z stays in $R(\widetilde{W})$.

Suppose now that we are given a dyadic point $x \in \hat{\mathbb{C}}$, and a parameter m. Our goal is to output 1 if $d(x,J(R)) < 2^{-m}$, and 0 if $d(x,J(R)) > 2 \cdot 2^{-m}$. All the preliminary steps take time that depends on the hyperbolic function R but not on the precision parameter m. Consider the following subprogram; the logs are all base-2:

$i := 1$

while $i \le m/\log\lambda - \log C/\log\lambda + 1$ **do**

(1) Compute the values of

$$p_i \approx R^i(x) = R(R^{i-1}(x)) \text{ and } d_i \approx DR^i(x) = DR^{i-1}(x) \cdot DR(R^{i-1}(x))$$

with precision $\min(2^{-(m+3)}, \varepsilon/4)$.

(2) Check the inclusions $p_i \in \widetilde{W}$ and $p_i \in R(\widetilde{W})$:

- if $p_i \in \widetilde{W}$, go to step (5),
- if $p_i \notin R(\widetilde{W})$, proceed to step (3),
- if neither holds either option is fine.

(3) Check the inequality

$$\frac{\ell}{d_i} > 2^{-m}$$

with precision $2^{-(m+1)}$. If true, output 0 and exit the subprogram, else

(4) output 1 and exit the subprogram.

(5) $i \leftarrow i+1$

end while

(6) Output 1 and exit.

end

The program runs for at most $m/\log\lambda - \log C/\log\lambda + 1 = O(m)$ iterations each of which consists of a constant number of arithmetic operations with $O(m)$ bits of precision. Hence the running time of the program can be bounded by $O(m^2 \log m \log\log m)$ using efficient multiplication (even slightly faster, see [Fur07]).

Suppose the subprogram outputs 0 and exits on line (3). This case is illustrated in Figure 3.3(A). The fact that the subprogram has reached line (3) means that the ball $B(p_i, l)$ is disjoint from $J(R)$. Also by the construction of $\widetilde{W}$ this ball contains no postcritical points, and hence there is a neighborhood N_0 of x that maps conformally to $B(p_i, l)$ under R^i. By the invariance of $J(R)$, N_0 is disjoint from $J(R)$. By the Koebe One-Quarter Theorem, the distance from x to $J(R)$ is at least

$$\operatorname{dist}(x, J(R)) \ge \ell' = \frac{1}{4} \cdot \frac{\ell}{DR^i(x)} \ge 2^{-(m+3)}.$$

On the other hand, suppose the subprogram exits on line (4), a case illustrated in Figure 3.3(B). If this is true, surround the point $R^i(x)$ with the disks $B = B(R^i(x), s/2)$, and $\hat{B} = B(R^i(x), 3s/4) \Supset B$. By construction, $B \cap J(R) \neq \emptyset$.

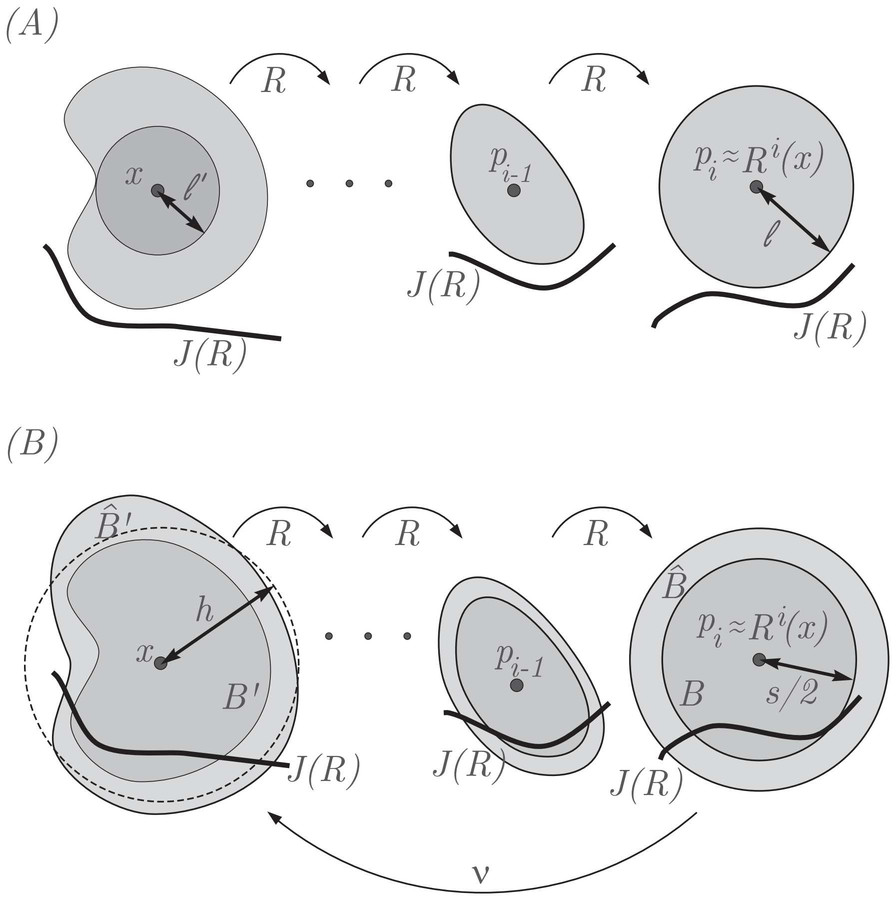

Fig. 3.3 A schematic figure illustrating the proof of correctness of the algorithm. Figure (A) illustrates exit on line (3) of the algorithm. Figure (B) illustrates exit on line (4).

On the other hand, as $R^2(\widetilde{W}) \subset U$, the disk $\hat{B}$ does not intersect with Postcrit(R). Hence there exists a well-defined branch of the inverse $\nu = R^{-i} : \hat{B} \mapsto \hat{B}' \ni x$. Denote by $B' \Subset \hat{B}'$ the image of B by this branch. Note that $B' \cap J(R) \neq \emptyset$.

We will now apply the Koebe Distortion Theorem to the restriction of ν from the larger disk $\hat{B}$ to the smaller one B. Namely, set $M(r) = (1+r)/(1-r)^3$. Note that the ratio of the radii of B and $\hat{B}$ is $r = 2/3$. By the Koebe Distortion Theorem

$$B' \subset B(x,h), \text{ where } h = \frac{s}{2DR^i(x)} \cdot M(2/3).$$

Putting this together with the negation of the inequality from line (3), we have

$$\mathrm{dist}(x, J(R)) < K \cdot 2^{-(m+3)}, \text{ where } K = \frac{8sM(2/3)}{\ell}. \tag{3.1.8}$$

Finally, suppose the sub-program exits on the last instruction. In this case, $x \in R^{-(i-1)}(\widetilde{W})$. On the other hand,

$$\mathrm{dist}(R^{-(i-1)}(\widetilde{W}), J(R)) < C^{-1}\lambda^{-m/\log\lambda + \log C/\log\lambda} = 2^{-m}.$$

In summary, converting all exponential estimates to base 2, there exists $M \in \mathbb{N}$ such that, for every $j \in \mathbb{N}$ sufficiently large, the subprogram can be used to distinguish between the cases:

- $\mathrm{dist}(x, J(R)) > K \cdot 2^{-j}$ (outputs 0), and
- $\mathrm{dist}(x, J(R)) < 2^{-j}$ (outputs 1).

This is not quite what we need, as we would like to distinguish the cases when this distance is $> 2^{-(m-1)}$ from when it is $< 2^{-m}$. To this end, we simply need to partition each pixel with side 2^{-n} into sub-pixels of size $2^{-n}/K$ and run the subprogram in the center of each of the sub-pixels. This only introduces a constant multiplicative overhead into the algorithm. □

The algorithms which use the estimate on the derivative of an iterate $R^i(z)$ to get an upper and lower bounds on the distance to $J(R)$ through the considerations of the Koebe One-Quarter Theorem and the Koebe Distortion Theorem, are known as *Distance Estimators*. They were first proposed by Milnor [Mil89] and Fisher [Fis89]. This approach can be very useful but, however, it has several obvious limitations. Firstly, a domain U whose preimages shrink to $J(R)$ cannot always be constructed (and indeed, does not always exist). But even when this obstacle can be overcome, the time bound on the rate of convergence of $R^{-m}(U)$ to $J(R)$ may be impractical. In the next section we will discuss a simple family of examples for which this bound becomes exponential.

3.2 Maps with parabolic orbits

Local dynamics of a parabolic orbit

We will describe here briefly the local dynamics of a rational mapping R with a parabolic periodic point p. By replacing R with its iterate, if needed, we may assume that $R(p) = p$, and $R'(p) = 1$. The map R then can be written as

$$R(z) = z + a(z-p)^{n+1} + O((z-p)^{n+2}), \tag{3.2.1}$$

for some $n \in \mathbb{N}$ and $a \neq 0$. Note that the integer $n+1$ is the local *multiplicity* of p as the solution of $R(z) = z$.

A complex number $\nu \in \mathbb{T}$ is called an *attracting direction* for p if the product $a\nu^n < 0$, and a *repelling direction* if the same product is positive. For each infinite orbit $\{R^k(z)\}$ which converges to the parabolic point, there is one of the n attracting directions ν for which the unit vectors

$$(R^k(z) - p)/|R^k(z) - p| \underset{k\to\infty}{\longrightarrow} \nu.$$

We say in this case that the orbit converges to p in the direction of ν. For each attracting direction ν, we say that a topological disk U is an *attracting petal* of R at p if the following properties hold:

- $\overline{U} \ni \{p\}$;
- $R^n(\overline{U}) \subset U \cup \{p\}$;
- an infinite orbit $\{R^k(z)\}$ is eventually contained in U if and only if it converges to p in the direction of ν.

Similarly, U is a *repelling petal* for R if it is an attracting petal for the local branch of R^{-1} which fixes p.

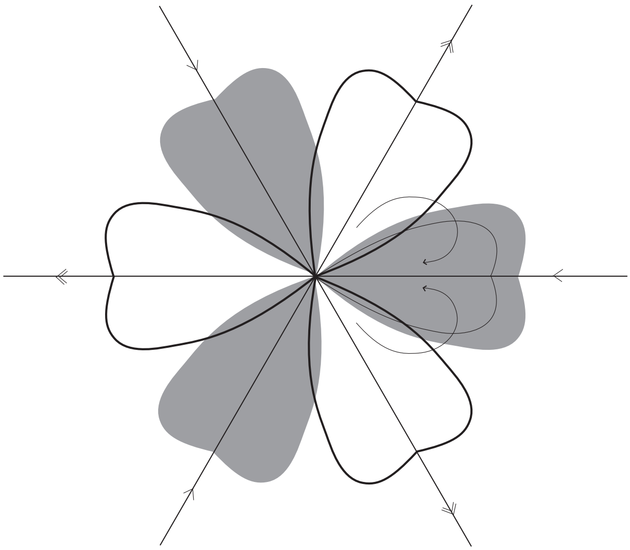

Fig. 3.4 A Leau-Fatou flower with three attracting petals (shaded) and three repelling petals (emphasized). The attracting and repelling directions are also indicated. The arrows show the direction of the orbits in one of the petals; the image of this petal is also indicated.

The petals form a *Leau-Fatou Flower* at p:

Theorem 3.8 *There exists a collection of n attracting petals P_i^a, and n repelling petals P_j^r such that the following holds. Any two repelling petals do not intersect,* *and every repelling petal intersects exactly two attracting petals. Similar properties hold for attracting petals. The union*

$$(\cup P_i^a) \cup (\cup P_j^r) \cup \{p\}$$

forms an open simply-connected neighborhood of p.

The proof of this statement is based on a multivalued change of coordinates

$$w = \kappa(z) = \frac{c}{(z-p)^n}, \text{ where } c = -\frac{1}{na}.$$

The map κ conformally transforms the infinite sector between two repelling directions into the plane with the negative real axis removed. In this sector, it changes the map R into

$$F(w) = w + 1 + O(1/\sqrt[n]{|w|}), \text{ as } w \to \infty.$$

Selecting a right half-plane $H_r = \{\operatorname{Re} z > r\}$ for a sufficiently large $r > 0$, we have

$$\operatorname{Re} F(w) > \operatorname{Re} w + 1/2, \text{ and hence } F(H) \subset H.$$

The corresponding attracting petal can then be chosen as the domain $\kappa^{-1}(H)$, using the appropriate branch of the inverse. Note that, given the coefficients of the rational mapping R, the description of the petal is constructive. Let us formulate this last statement in a language suitable for later references:

Lemma 3.9 *For each degree $d \geq 2$ there exists an oracle Turing Machine M^ϕ such that the following holds. Let R be a rational mapping of degree d with a parabolic periodic point p, with period m and multiplier* 1. *Let n be as in (3.2.1). The machine M^ϕ takes as input the values of m, n and a natural number k; it is given oracle access to the coefficients of R and the value of p. It outputs a set $L_k \in \mathscr{C}$ such that the following is true:*

- *$L_{k+1} \supset L_k$ and $\cup L_k = P$ is the union of attracting petals of R at p, covering all the attracting directions;*
- $\operatorname{dist}_H(L_k, P) < 2^{-k}$.

The dynamics inside a petal is described by the following:

Proposition 3.10 *Let P be an attracting or repelling petal of R. Then there exists a conformal change of coordinates Φ inside P, transforming $R(z)$ into the unit translation $z \mapsto z+1$. The image $\Phi(P)$ covers a right half-plane.*

The function Φ is called the *Fatou coordinate* , with the prefix *attracting* or *repelling* depending on the type of the petal P.

As an implication of Proposition 3.10, consider the quotient manifold $C_A \equiv P/_{z\sim R(z)}$, which parametrizes the *orbits* converging to the parabolic point through P. Then C_A is conformally isomorphic to the quotient of a right half-plane by the unit translation, which is the cylinder $\mathbb{C}/\mathbb{Z}$.

Suppose now that the multiplier of the fixed point p is a q-th root of unity, $R'(p) = e^{2\pi ip/q}$, where $(p,q) = 1$. A fixed petal for the iterate R^q corresponds to a cycle of q petals for R. It thus follows that q divides the number n of attracting/repelling directions of p as a fixed point of R^q. We make note of the following proposition, due to Fatou:

Proposition 3.11 *Each cycle of attracting petals of a rational mapping R captures an orbit of a critical point of R.*

This implies, in particular, that a quadratic polynomial f_c with a parabolic periodic point ζ with multiplier $e^{2\pi ip/q}$ has a Leau-Fatou flower at ζ with a single cycle of q attracting petals.

Computability of Julia sets in the presence of a parabolic orbit

A hyperbolic Julia set is computable (cf. Theorem 3.1) because it is easy to verify that an orbit belongs to the Fatou set of a hyperbolic rational mapping. A trapping neighborhood around every attracting orbit of such a mapping can be constructed algorithmically, and only those orbits which enter one of these neighborhoods do not lie in the Julia set.

An analogous approach in the presence of a parabolic cycle would require us to construct attracting petals, to detect the orbits which converge to the cycle. This construction cannot be made fully automated. Some non-uniform information will be required by the algorithm. For simplicity, let us formulate the computability statement only for Julia sets of parabolic quadratics. A more general theorem on the Julia set of a rational map whose Fatou set consists only of parabolic and attracting basins is easily obtained along the same lines.

Theorem 3.12 *There exists a Turing Machine M^ϕ with an oracle for a complex parameter c which computes the Julia set J_c of every parabolic quadratic polynomial f_c, given the following non-uniform information:*

- *the period m of the unique parabolic orbit of f_c;*
- *positive integers p and q with $(p,q) = 1$ such that the multiplier of the parabolic orbit of f_c is equal $e^{2\pi ip/q}$.*

Proof. Denote the parabolic orbit of f_c by

$$p_1 \mapsto p_2 \mapsto \cdots \mapsto p_m.$$

Note that the Taylor's expansion of $f_c^{q\cdot m}$ near each of the points p_i has the form

$$f_c^{q\cdot m}(z) = p_i + (z - p_i) + \alpha_{n+1}(z - p_i)^{n+1} + \ldots \alpha_{n+2}(z - p_i)^{n+2} + \ldots$$

Here n is the number of attracting (or repelling) directions. As we are in the quadratic case, there are exactly q attracting petals in the Leau-Fatou flower, so that $n = q$.

By Proposition 3.2, the roots of

$$f_c^m(z) = z$$

can be determined with an arbitrary accuracy. Among these solutions, repelling periodic points can be identified and excluded algorithmically. Thus the points $p_1, \dots, p_m$ can be identified with any desired precision. Hence, we can construct a sequence of domains L_k for the iterate f_c^{qm} provided by Lemma 3.9.

Now the proof of the theorem proceeds similarly to that of Theorem 3.1. By Proposition 2.16, $|c| \le 2$. Hence the ball $D = B(0, 4.1)$ has the property $f_c^{-1}(D) \Subset D$, and all orbits which originate outside of D converge to ∞. Fixing k, we obtain the picture of J_c with precision 2^{-k} as follows:

1. Set $t = 1$.
2. Compute $U_t \in \mathscr{C}$ which approximates $f_c^{-t}(D) \setminus f_c^{-t}(L_t)$ up to an error of $2^{-(k+3)}$ in Hausdorff metric.
3. Compute $V_t \in \mathscr{C}$ which approximates $\cup_{s=1}^{t} f_c^{-s}(p_1)$ up to an error of $2^{-(k+3)}$ in Hausdorff metric.
4. Check the inclusion $U_t \subset B(V_t, 2^{-(k+1)})$. If true, output V_t and halt. If false, increment $t \mapsto t+1$ and go to step 2.

The verification of the algorithm is straightforward, and is left to the reader. Note that by Proposition 2.9 the set $\cup_{s=1}^{\infty} f_c^{-s}(p_1)$ is dense in J_c, and thus the sequence $\{V_t\}$ approximates J_c well from below. □

3.3 Computing Julia sets with parabolic orbits efficiently

3.3.1 *The Distance Estimator in the parabolic case*

Julia sets with parabolic orbits are well-known examples for which the Distance Estimator algorithm of §3.1 becomes impractical (cf. the discussion in [Mil06]). As we have already noted, a key to the successful application of the algorithm in the case of a hyperbolic rational map R is that, for a point z which lies at a distance 2^{-n} of the Julia set $J(R)$, it would only take $O(n)$ iterates to magnify this distance to the order of 1. The situation becomes very different if there is a parabolic orbit in $J(R)$.

To fix the ideas, let us consider a very simple example – a quadratic polynomial $f(z) = z + z^2$ with a parabolic fixed point at the origin. Take a point $z_0 = 2^{-n}$ for some large value of n. On the one hand, $z_0 \notin J(f)$. Indeed, if we denote $z_k = f^k(z_0)$, then an easy induction shows that

$$z_k \ge 2^{-n} + k \cdot 2^{-2n} \longrightarrow \infty.$$

In fact, z_0 lies at a distance of approximately 2^{-2n} from $J(f)$.

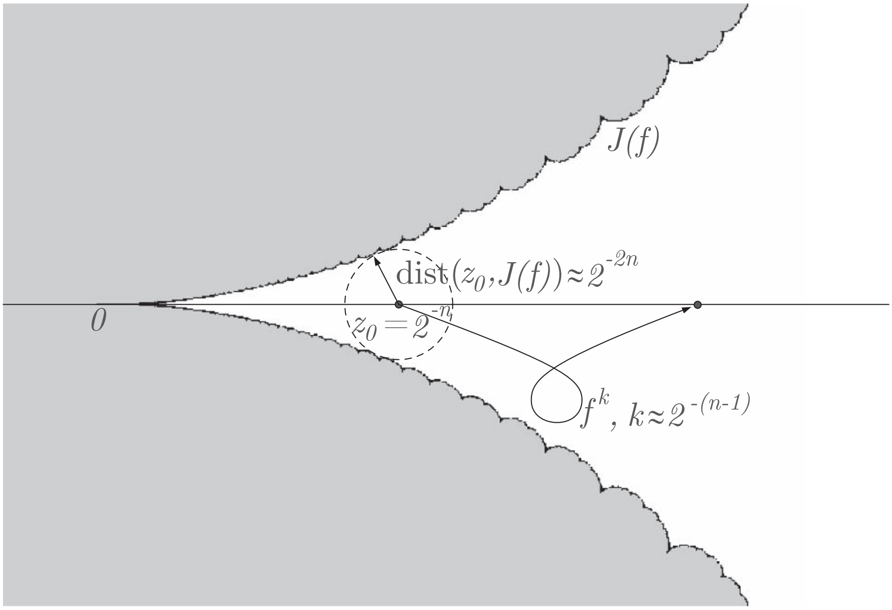

Fig. 3.5 Slow orbits in the neighborhood of the parabolic point.

On the other hand, if

$$z_k < 2^{-(n-1)}, \text{ then } z_{k+1} < z_k + 2^{-(2n-2)}.$$

Hence it will take the orbit of z_0 at least 2^{n-1} steps to reach $2 \cdot 2^{-n}$ (cf. Figure 3.5). We see that the orbit escapes to ∞, but it will take approximately 2^n steps to reach distance of order 1 from the origin. So if we apply the Distance Estimator algorithm to $f(z)$, it will become *exponential*- rather than polynomial-time.

Thus, a naïve approach to drawing a Julia set with parabolics leads to an impractical algorithm. To accelerate it, we will have to look at the dynamics of a rational map near a parabolic point more carefully. To avoid messy technicalities, let us again concentrate on the example of $f(z)$.

3.3.2 *Accelerating the map* $z \mapsto z + z^2$

Instead of iterating $f(z)$ starting at z_0 which is very near zero, let us write down the iterates of f on an arbitrary z symbolically. We only write the first 5 coefficients of each iteration.

$$\begin{cases} f^1(z) = z + 1\cdot z^2 + \ \ 0\cdot z^3 + \ \ 0\cdot z^4 + \ \ \ 0\cdot z^5 + \dots \\ f^2(z) = z + 2\cdot z^2 + \ \ 2\cdot z^3 + \ \ 1\cdot z^4 + \ \ \ 0\cdot z^5 + \dots \\ f^3(z) = z + 3\cdot z^2 + \ \ 6\cdot z^3 + \ \ 9\cdot z^4 + \ \ 10\cdot z^5 + \dots \\ f^4(z) = z + 4\cdot z^2 + 12\cdot z^3 + 30\cdot z^4 + \ \ 64\cdot z^5 + \dots \\ f^5(z) = z + 5\cdot z^2 + 20\cdot z^3 + 70\cdot z^4 + 220\cdot z^5 + \dots \\ \ \vdots \end{cases}$$

We can see some patterns in the coefficients of $f^k(z)$. For example, the coefficient of z is always 1 and the coefficient of z^2 is k. Higher coefficients are given by

$$f^k(z) = z + k\cdot z^2 + k(k-1)\cdot z^3 + \frac{(2k-3)k(k-1)}{2}\cdot z^4 + \frac{(3k-4)k(k-1)(k-2)}{3}\cdot z^5 + \dots \quad (3.3.1)$$

A few observations can be made about the formula:

- The coefficient of z^r is a polynomial in k of degree $r-1$ with leading coefficient 1;
- the coefficient of z^r is always between 0 and k^{r-1}.

Denote the coefficient of z^r by $c_r(k)$. First, let us show how to compute $c_r(k)$ explicitly in general. We know that $c_0(k) = 0$ and $c_1(k) = 1$. For $r \geq 2$, we use the recurrence $f^{k+1}(z) = f(f^k(z))$ to obtain

$$c_r(k+1) = c_r(k) + \sum_{j=1}^{r-1} c_j(k)c_{r-j}(k). \quad (3.3.2)$$

Thus we obtain an explicit recurrence

$$c_r(k) = \sum_{t=1}^{k-1}\sum_{j=1}^{r-1} c_j(t)c_{r-j}(t). \quad (3.3.3)$$

By solving the recurrence we obtain the formulas for the coefficients in (3.3.1). Formulas for the first n coefficients can be obtained with sufficiently high precision in time polynomial in n.

We would like to use the newly obtained symbolic coefficients of the k-th iteration of f to make big "leaps" in the iterations of f for values of z that are very close to 0 (where the iteration takes a long time to converge). We have the formula

$$f^k(z) = z + c_2(k)z^2 + c_3(k)z^3 + \dots \quad (3.3.4)$$

For a small z with $|z| \approx 2^{-n}$, we would like to make $2^{n-1} = \Omega(1/|z|)$ iterations in one step. We do this by plugging in $k = 2^{n-1}$. We can afford to take $\mathrm{poly}(n)$ terms of the sum (3.3.4), and thus we need all the subsequent terms to be insignificant.

Proposition 3.13 *For each k and for each r, $c_r(k) \leq k^{r-1}$.*

Proposition 3.13 implies that as long as $k < \frac{1}{2|z|}$, we will have $c_r(k)z^r < 2^{-r}/k$, and thus n terms would suffice in order to maintain precision of 2^{-n}. The proposition is proved by induction.

Proof (Proposition 3.13). $c_1(k) = 1$ and $c_2(k) = k$, hence the statement is true for $r = 1,2$. For higher r's we prove it by induction. Assume it is true up to $r-1$ for some $r \geq 3$. By the induction hypothesis,

$$c_r(k) = \sum_{t=1}^{k-1}\sum_{j=1}^{r-1} c_j(t)c_{r-j}(t) \leq \sum_{t=1}^{k-1}\sum_{j=1}^{r-1} t^{j-1}t^{r-j-1} = \sum_{t=1}^{k-1}\sum_{j=1}^{r-1} t^{r-2} = \sum_{t=1}^{k-1}(r-1)t^{r-2} \leq \sum_{t=1}^{k-1}((t+1)^{r-1} - t^{r-1}) = k^{r-1} - 1 < k^{r-1}.$$

□

Proposition 3.13 can now be used to compute a "long" iteration of a point z such that $|z|$ is small.

Proposition 3.14 *Suppose $|z| < 1/m$ for a sufficiently large m. Then we can compute the $\ell = \lfloor m/2 \rfloor$-th iterate of z and its derivative $df^\ell/dz(z)$ with a given precision 2^{-s} in time polynomial in s and* $\log\, m$.

Proof. We compute the first s coefficients $c_2(m), c_3(m), \ldots, c_{s+1}(m)$ with precision 2^{-s-1}. This can be done in time polynomial in s and $\log\, m$. Denote the approximate coefficients by $c'_2, c'_3, \ldots, c'_{s+1}$. We then approximate $f^m(z)$ by

$$f^m(z) \approx z + c'_2 z^2 + c'_3 z^3 + \ldots + c'_{s+1} z^{s+1}.$$

The error is bounded by

$$|c_2(m) - c'_2|z^2 + |c_3(m) - c'_3|z^3 + \ldots + |c_{s+1}(m) - c'_{s+1}|z^{s+1} + |c_{s+2}(m)z^{s+2}| + |c_{s+3}(m)z^{s+3}| + \ldots \leq 2^{-s-1}(z^2 + z^3 + \ldots) + m^{s+2}z^{s+2} + m^{s+3}z^{s+3} + \ldots \leq 2^{-s-1} + 2^{-s-1} = 2^{-s}.$$

We use Proposition 3.13 here to bound the tail terms $c_r(m)z^r$. □

The algorithm now works similarly to the hyperbolic case (Theorem 3.4), occasionally using Proposition 3.14 to perform a long iteration when the orbit is close to 0. We first construct a domain U similar to the initial domain used in Proposition 3.7:

Proposition 3.15 *We can compute a planar domain $U \in \mathscr{C}$ such that:*

(I) $\overline{U} \subsetneq \overline{f(U)}$, with finitely many intersection points at preimages of the parabolic point 0*,*

(II) $f(U) \cap \text{Postcrit}(f) = \emptyset$,
(III) $J(f) \subset \overline{U}$.

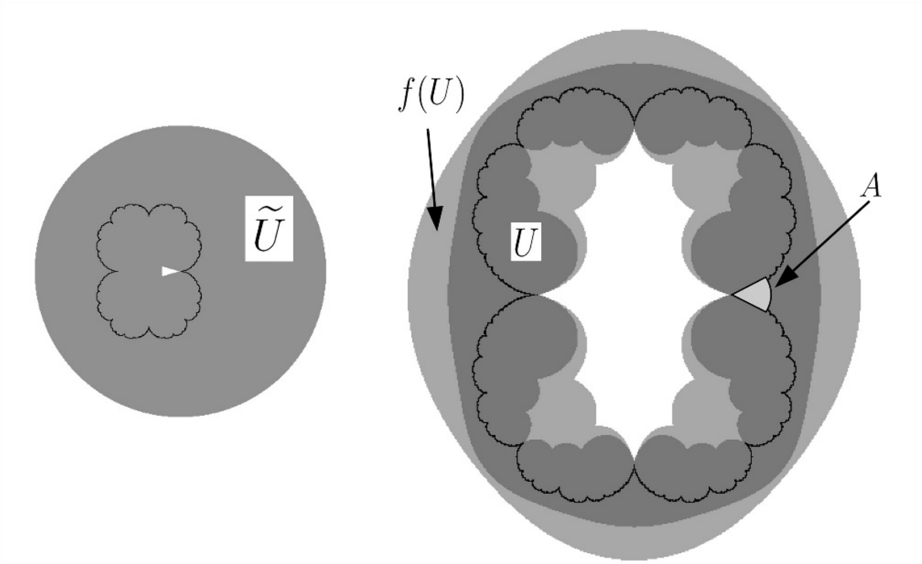

Fig. 3.6 The sets $\widetilde{U}$, $U \subset f(U)$ and A.

The set U is obtained by taking a domain $\widetilde{U}$ that is a sufficiently large disc with a wedge removed around the attracting direction of the parabolic point 0 (see Figure 3.6). All orbits originating in the wedge stay there and converge to the parabolic point 0. The orbit of the critical point $-1/2$ converges to 0 and eventually ends up in the wedge. Hence the inverse images of $\widetilde{U}$ will eventually consume the critical point. In our illustration, we take $U = f^{-3}(\widetilde{U})$. It then satisfies the requirements of Proposition 3.15. Furthermore, as in the hyperbolic case, by taking a few more inverse images under f, we can assure that every point in U is at least 32 times closer to $J(f)$ than to the postcritical set of f.

We can now apply a combination of the Distance Estimator algorithm with the "giant steps" from Proposition 3.14. For the algorithm, we will need to define a region A which is a wedge around the repelling direction of the map of some constant radius ε that contains $J(f) \cap B(0,\varepsilon)$ (see Figure 3.6 again). If a point z is ε-close to 0 ($|z| < \varepsilon$) but it is not in A, then we can estimate the distance $d(z,J(f))$ within a constant multiplicative error.

Suppose now that we are given a dyadic point $x \in \hat{\mathbb{C}}$, and a parameter m, and our goal is to output 1 if $d(x,J(f)) < 2^{-m}$, and 0 if $d(x,J(f)) > M \cdot 2^{-m}$ for some constant M. We use the following algorithm, where C is an appropriately chosen large constant (full details may be found in [Bra06]).

begin
$i = 1$, $x_0 = x$, $d_0 = 1$.
if $x_0 \notin U$, estimate $d(x_0, J(f))$ directly;
while $i \leq Cm$ **do**

(1) $x_i \leftarrow x_{i-1} + x_{i-1}^2$;
(2) $d_i \leftarrow 2x_{i-1} \cdot d_{i-1}$;
(3) Check the inequality $|x_i| < 2^{-Cm}$ with precision 2^{-Cm}; if the inequality holds halt and return 1, otherwise continue to (4);
(4) Check whether $x_i \in U$ with precision 2^{-2m}; if not, estimate $d(x, J(f))$ by $d(x_i, J(f))/d_i$, return the appropriate answer and halt;
(5) Check whether $|x_i| < \varepsilon$ with precision 2^{-2m}; if it is the case

(a) check whether $x_i \in A$ with precision 2^{-2m};
(b) if x_i is in A, make a "giant leap" of $\lfloor 1/(2|x_i|) \rfloor$ steps from x_i to obtain x_i' and d_i';
- if x_i' escapes U use binary search to find the smallest iterate $f^l(x_i)$ that escapes U; set $d_i \leftarrow Df^l(x_i)d_i$, $x_i \leftarrow f^l(x_i)$;
- otherwise, set $x_i \leftarrow x_i'$ and $d_i \leftarrow d_i'$;
- loop back to step (5);

(c) if x_i is not in A, estimate $d(x_i, J(f))$, estimate $d(x, J(f))$ by $d(x_i, J(f))/d_i$, return the appropriate answer and halt;

(6) $i \leftarrow i + 1$

end while

Output 1 and exit.

end

The proof of the fact that $d(x_i, J(f))/d_i$ estimates $d(x_i, J(f))$ within a constant multiplicative error whenever x_i is in U is similar to the hyperbolic case. Also, outside of A and finitely many preimages of A, ∂U and $\partial f(U)$ are bounded away from each other, thus giving an expansion by some $c > 1$ in the hyperbolic metric of $f(U)$. This means that if $d(x_0, J(f)) > 2^{-m}$, then the main loop may be executed at most Cm times for some constant C. Note that step (5)(c) does not decrease the hyperbolic distance from x_i to $J(f)$ by the same argument, although we have no estimate on the factor by which it increases this distance (only that it is ≥ 1).

Evidently, if the algorithm exists on line (4), i.e. when x_{i+1} is *very* close to $J(f)$, then x_0 must have been closer than 2^{-m} to $J(f)$.

Finally, assuming that $|x_i| > 2^{-Cm}$, it will take $O(m)$ iterations of step (5)(b) to escape the set A, which means that the total number of jumps in the algorithm is bounded by $O(m^2)$, which is *polynomial* in m.

3.3.3 Computing parabolic Julia sets in polynomial time: the general case

The construction above generalizes to any rational map $R(z)$ that only has parabolic and attracting orbits. Full details may be found in [Bra06]. We state:

Theorem 3.16 *Given*

- *a rational function $R(z)$ such that every critical orbit of R converges to an attracting or a parabolic orbit; and*
- *some finite combinatorial information about the parabolic orbit of R;*

there is an algorithm M that produces an image of the Julia set $J(R)$. It takes M at most time $C_R \cdot n^c$ to decide one pixel in $J(R)$ with precision 2^{-n}. Here c is some (small) constant and C_R depends on R but not on n.

The combinatorial information is required to identify the parabolic orbits, for instance, by specifying their periods, and approximate locations. It should also allow us to present an iterate R^q of R near each parabolic point p_i in a canonical form

$$R^q:\ p_i + z \mapsto p_i + z + z^{u_i+1} + a_{u_i+2} z^{u_i+2} + \ldots;$$

to do this we need to know q and u_i.

The algorithm works exactly as in the example above. It starts by creating a domain U such that $U \subset R(U)$ with only finitely many intersection points between ∂U and $\partial R(U)$ (at some preimages of the parabolic points). The set U is selected so that $U \cap \mathrm{Postcrit}(R) = \emptyset$.

To find out whether a point x is 2^{-n}-close to $J(R)$, the algorithm iterates it until the orbit escapes U while keeping track of the derivative. As in the special case, if the orbit reaches a set A which is a collection of wedges around the repelling directions of the parabolic orbits, it applies one long iteration to accelerate the computation. If the orbit lands extremely close to one of the parabolic points, then x must have been 2^{-n}-close by a derivative argument. Otherwise, it will take $O(n)$ long steps to escape A. Using a hyperbolic metric argument as above, one shows that at most $O(n)$ steps may be made outside A, bringing the total number of iterations before the algorithm terminates to $O(n^2)$.

The only possible complication is in computing the long iteration. Note that if our map was $g(z) = z + z^3$ instead of $z + z^2$, it would take $\approx 2^{2n}$ iterations to escape from $x_0 = 2^{-n}$, rather than $\approx 2^n$ iterations. Thus we would need a more powerful acceleration (that jumps $(1/z)^2$ steps rather than $1/z$ steps) in this case. To justify plugging in $k = \lfloor 1/(2z^2) \rfloor$ into the formula for $g^k(z)$ we need a generalization of Proposition 3.13.

Proposition 3.17 (cf. Lemma 5 in [Bra06]) *Let $u \geq 1$ be an integer. Set $g(z) = z + z^{u+1}$. Let $\alpha = 2u^3$. Write the k-th iterate of g:*

$$g^k(z) = z + c_{u+1}(k) z^{u+1} + c_{u+2}(k) z^{u+2} + \ldots$$

Then $c_r(k) \le (\alpha k)^{r/u}$.

In particular, for $g(z) = z + z^3$, $u = 2$ and $c_r(k) \le (16k)^{r/2}$. Thus we can take $k = \lfloor 1/(32z^2) \rfloor$ and the series will still converge. This allows for a jump of $\Omega(1/z^2)$ in one step, as required. Proposition 3.17 allows us to take even bigger jumps for higher values of u.

3.4 Lack of uniform computability of Julia sets

Our first interesting result in the negative direction answers the following natural question:

Is it possible to compute all Julia sets, or in particular all quadratic Julia sets, with a single oracle Turing Machine $M^\phi(n)$*?*

This is ruled out by Theorem 1.12, as the dependence $c \mapsto J(f_c)$ is discontinuous in the Hausdorff distance. For an excellent survey of the continuity problem see the paper of Douady [Dou94].

Theorem 3.18 ([Dou94]) *Denote by* $\mathbb{J}(c)$ *and* $\mathbb{K}(c)$ *the functions* $c \mapsto J_c$ *and* $c \mapsto K_c$ *respectively viewed as functions from* $\mathbb{C}$ *to* K_2^* *with the latter space equipped with Hausdorff distance. Then the following is true:*

(a) if c *is Siegel then* $\mathbb{J}(c)$ *is discontinuous at* c*, but* $\mathbb{K}(c)$ *is continuous at* c*;*
(b) if c *is parabolic then both* $\mathbb{J}(c)$ *and* $\mathbb{K}(c)$ *are discontinuous at* c*;*
(c) if c *is neither Siegel nor parabolic, then both* $\mathbb{J}(c)$ *and* $\mathbb{K}(c)$ *are continuous at* c*.*

The discontinuity of $\mathbb{J}$ at Siegel parameters is not difficult to prove:

Proposition 3.19 *Let* $c_* \in \mathscr{M}$ *be a parameter value for which* f_{c_*} *has a Siegel disk. Then the map* $\mathbb{J}(c)$ *is discontinuous at* c_**. More specifically, let* z_0 *be the center of the Siegel disk. For each* $s > 0$ *there exists* $\tilde{c} \in B(c_*, s)$ *such that* $f_{\tilde{c}}$ *has a parabolic periodic point in* $B(z_0, s)$*.*

Proof. Denote by Δ the Siegel disk around z_0, p its period, and θ the rotation angle. By the Implicit Function Theorem, for some $\varepsilon > 0$ there exists a holomorphic mapping $\zeta : B(c_*, \varepsilon) \to \mathbb{C}$ such that $\zeta(c_*) = z_0$ and $\zeta(c)$ is fixed under $(f_c)^p$. The mapping

$$\nu : c \mapsto D(f_c)^p(\zeta(c))$$

is holomorphic, and hence it is either constant or open.

If $\nu(c) \equiv d$ is constant, then there exists a maximal non-empty open set of parameters $A \ni c_*$ with a Siegel periodic point with the same period and multiplier. Since A is obviously closed in $\mathbb{C}$, it follows that every quadratic has a Siegel disk. This is not possible: for instance, $f_{1/4}$ has a parabolic fixed point, and thus no other non-repelling cycles, by the Fatou-Shishikura Bound. Therefore ν is open, and in particular there is a sequence of parameters $c_n \to c_*$ such that $\zeta(c_n)$ has multiplier $e^{2\pi i p_n/q_n}$. Since $\zeta(c_n)$ is parabolic, it lies in the Julia set of f_{c_n}. $\zeta(c_n) \to z_0$. Hence

$$\mathrm{dist}_H(J(f_{c_n}), J(f_{c_*})) \geq \mathrm{dist}(\zeta(c_n), \partial\Delta) > \mathrm{dist}(z_0, \partial\Delta)/2$$

for n large enough. □

Thus an arbitrarily small change of the multiplier of the Siegel point may lead to an implosion of the Siegel disk – its inner radius collapses to zero.

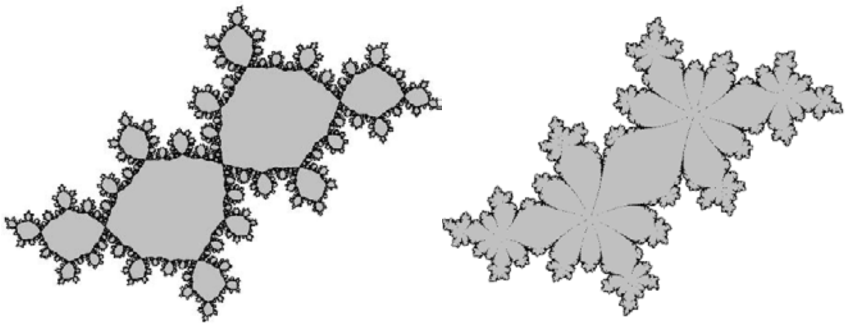

Fig. 3.7 An illustration of a Siegel implosion. On the left is the filled Julia set K_{c_*} (gray) and the Julia set J_{c_*} (black) of a quadratic polynomial with a Siegel fixed point ζ_0. The multiplier $f_{c_*}(\zeta_0) = e^{2\pi i\theta}$, where the rotation angle θ is the inverse golden mean, given by the infinite continued fraction $[1,1,1,1,1,\ldots]$. On the right is the filled Julia set of a nearby quadratic polynomial, whose fixed point is parabolic, with multiplier $[1,1,1,1,1] = 5/8$.

As an immediate consequence of Proposition 3.19 and Theorem 1.12 we have:

Proposition 3.20 *For any TM $M^\phi(n)$ with an oracle for $c \in \mathbb{C}$, denote by S_M the set of all values of c for which M^ϕ computes J_c. Then $S_M \neq \mathbb{C}$.*

In other words, a single algorithm for computing all quadratic Julia sets does not exist.

3.4.1 Discontinuity at a parabolic parameter

The discontinuity in $\mathbb{J}(c)$ which occurs at parabolic parameter values has found many interesting dynamical implications. The proof is very involved, and its outline may be found in [Dou94]. It is based on the Douady-Lavaurs theory of *parabolic implosion.* Let us briefly describe its mechanism for the case of a quadratic polynomial f_c.

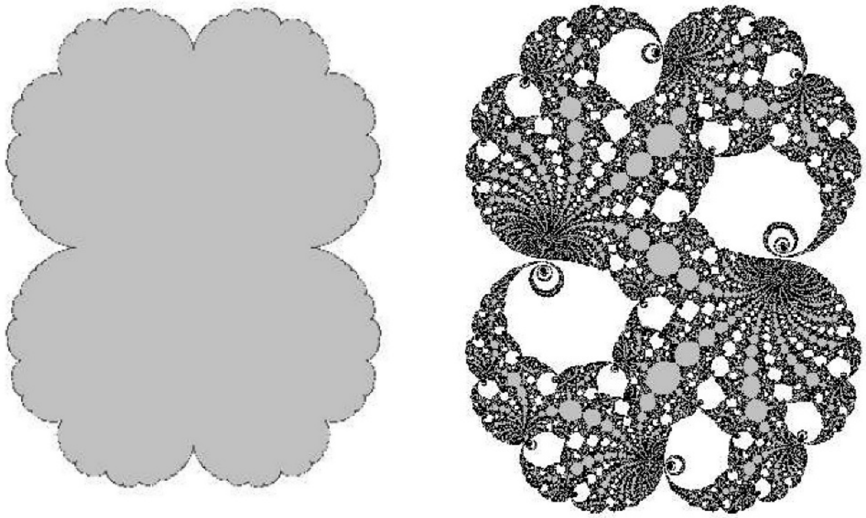

Fig. 3.8 Before and after a parabolic implosion. The Julia sets (black) and filled Julia sets (light gray) of a parabolic quadratic $f_{1/4}$ (left), and of $f_{1/4+\varepsilon}$ for a small complex ε.

Denote by ζ a parabolic periodic point of f_c with multiplier $e^{2\pi ip/q}$, and let $m \in \mathbb{N}$ be its period. Let P_A and P_R be attracting and repelling petals of f_c. Recall that, by Proposition 3.11, the cycle of images $f_c^{jm}(P_A \cup P_R)$, $j = 0, \ldots, q-1$ forms a full Leau-Fatou flower at ζ.

By Proposition 3.10, the quotient

$$C_A = P_A / f_c^{mq} \simeq \mathbb{C}/\mathbb{Z}.$$

The quotient C_A, is sometimes called the *attracting Fatou cylinder*. It parametrizes the orbits converging under the dynamics of the iterate f_c^m to the point ζ. A repelling Fatou cylinder $C_R \simeq \mathbb{C}/\mathbb{Z}$ is defined similarly as the quotient of a repelling petal.

Let τ be any conformal isomorphism $C_A \to C_R$. After uniformization,

$$C_A \underset{\approx}{\mapsto} \mathbb{C}/\mathbb{Z},\, C_R \underset{\approx}{\mapsto} \mathbb{C}/\mathbb{Z}$$

$\tau(z) \equiv z + q \bmod \mathbb{Z}$ for some $q \in \mathbb{C}$. Let $g_\tau : P_A \to P_R$ be any lift of τ; it necessarily commutes with f_c^{mq}. Consider the semigroup G generated by the dynamics of the pair (f_c, g_τ). The orbit Gz of a point $z \in \mathbb{C}$ is independent of the choice of the lift g_τ and only depends on τ.

Set

$$J_{(c,\tau)} = \{z \in \mathbb{C} \text{ such that } Gz \cap J_c \neq \emptyset\}.$$

It can be shown that this set is the boundary of

$$K_{(c,\tau)} = \{z \in \mathbb{C} \text{ such that } Gz \text{ is bounded}\}.$$

Notice that $K_{(c,\tau)} \subsetneq K_c$: some of the orbits which converge to ζ under f_c are thrown into the complement $(\mathbb{C}\setminus K_c) \cap P_R$ by g_τ. Holes which thus open in the set K_c motivate the use of the term "implosion".

The Douady-Lavaurs theory postulates:

Theorem 3.21 *For every τ as above and every $s > 0$ there exists $\tilde{c} \in B(c,s)$ such that $B(J_{\tilde{c}}, s) \supset J_{(c,\tau)}$.*

Thus the Julia set of f_c "explodes" under the perturbation from c to $\tilde{c}$.

Chapter 4
Positive Results

4.1 Computability of filled Julia sets

Computability of Julia sets of rational functions can be rather subtle, and will lead us to some surprising findings in the next chapter. The situation is much simpler, however, with *filled* Julia sets of polynomial mappings. In this section we show:

Theorem 4.1 *For any polynomial $p(z)$ there is an oracle Turing Machine $M^{\phi}(n)$ that, given an oracle access to the coefficients of $p(z)$, outputs a 2^{-n}-approximation of the filled Julia set $K_p \equiv K(p(z))$*[1].

Moreover:

Theorem 4.2 *In the case when $p(z) = z^2 + c$ is quadratic, only two oracle machines suffice to compute all non-parabolic filled Julia sets: one for $c \in \mathscr{M}$, and one for $c \notin \mathscr{M}$.*

For a given polynomial $p(z)$ we construct a machine computing the corresponding filled Julia set K_p. We will use some combinatorial information about p in the construction, and so the algorithm will, in general, vary with the polynomial. Note that all the information we will need can be encoded using a finite number of bits.

- Information that would allow us to compute the non-repelling orbits of the polynomial with an arbitrary precision, as well as their type: attracting, parabolic, Siegel, or Cremer. By the Fatou-Shishikura bound, there are at most $\deg p - 1$ of them.
  By Proposition 3.2, such information could, for example, consist of the list of periods k_i of such orbits; and for each i a finite collection of dyadic balls $\{D_i^j\}_{j=1}^{k_i}$ separating the points of the corresponding orbit from the other solutions of the equation $p^{k_i}(z) = z$.

[1] Theorem 4.1 answers in the affirmative the question posed to us by J. Milnor, after we first demonstrated the existence of non-computable quadratic Julia sets in [BY06].

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- For each parabolic periodic point with period m and multiplier p/q, the values of m, p, q.
- In the case of a Siegel disc D, information that would allow us to identify a repelling periodic point ζ_D in the same connected component of K_p as D. Again, by Proposition 3.2, it is sufficient to know both its period and a small enough dyadic ball around it, which separates it from all other periodic points with the same period.

4.1.1 Computing K_p

We are given a dyadic point $d \in \mathbb{D}$ and an $n \in \mathbb{N}$. Our goal is to output 1 if $B(d, 2^{-n}) \cap K_p \neq \emptyset$ and to output 0 if $B(d, 2 \cdot 2^{-n}) \cap K_p = \emptyset$. We do it by constructing five machines. They are guaranteed to terminate each on a different condition, always with a valid answer. Together they cover all possible cases.

Lemma 4.3 *There are five oracle machines M_{ext}, M_{jul}, M_{attr}, M_{par}, M_{sieg} such that*

1. *if d is at distance $\geq \frac{4}{3} \cdot 2^{-n}$ from K_p, $M_{ext}(d,n)$ will halt and output* 0. *If d is at distance $\leq 2^{-n}$ from K_p, $M_{ext}(d,n)$ will never halt;*
2. *if d is at distance $\leq \frac{5}{3} \cdot 2^{-n}$ from $J_p = \partial K_p$, $M_{jul}(d,n)$ will halt and output* 1. *If d is at distance $\geq 2 \cdot 2^{-n}$ from J_p, $M_{jul}(d,n)$ will never halt;*
3. *$M_{attr}(d,n)$ halts and outputs* 1 *if and only if d is inside the basin of an attracting orbit of p;*
4. *$M_{par}(d,n)$ halts and outputs* 1 *if and only if d is inside the basin of a parabolic orbit of p;*
5. *$M_{sieg}(d,n)$ halts and outputs* 1 *if the orbit of d reaches a Siegel disc, and d is at distance $\geq \frac{4}{3} \cdot 2^{-n}$ from J_p. It never halts if d is at distance $\geq 2 \cdot 2^{-n}$ from K_p.*

Proof (Theorem 4.1,given Lemma 4.3). By the Fatou-Sullivan classification it is clear that for each (d,n) at least one of the machines halts. Moreover, by the definition of the machines, they always output a valid answer whenever they halt. Hence running the machines in parallel and returning the output of the first machine to halt gives the algorithm for computing K_p. □

We now prove Lemma 4.3.

Proof (Lemma 4.3). We give a simple construction for each of the five machines.

1. M_{ext}: Take a large ball B such that $p^{-1}(B) \Subset B$. Outside of B all orbits converge to infinity. Intuitively, we pull the ball back under p to get a good approximation of K_p. Let B_k be a $2^{-(n+3)}$-approximation of the set $p^{-k}(B)$. Output 0 iff $B_k \cap B(d, \frac{7}{6} \cdot 2^{-n}) = \emptyset$ for some k. It is not hard to see that this algorithm satisfies the conditions on M_{ext}.

2. M_{jul}: By Proposition 3.2 for each k we can compute all periodic orbits of $p(z)$ in B with periods $j \le k$, as roots of the equation
$$p^j(z) - z = 0$$
with an arbitrarily high precision. Moreover, by our assumptions, we have the means to distinguish the non-repelling orbits from the repelling ones.
Let C_k be a finite collection of complex numbers with dyadic rational real and imaginary parts which approximate the repelling periodic orbits with periods up to k with precision $2^{-(n+3)}$. Output 1 iff $\mathrm{dist}(d, C_k) < \frac{11}{6} \cdot 2^{-n}$. The repelling periodic orbits are all in J_p and are dense in this set. Hence the algorithm satisfies the conditions on M_{jul}.
3. M_{attr}: For each k, and for each $0 \le i < j \le k$, compute a $1/k^2$-approximation $C_k^{ij} \in \mathscr{C}$ of
$$p^{j-i}(B(p^i(d), 1/k)),$$
and check whether
$$C_k^{ij} \subset B(p^i(d), 1/(2k)).$$
If this is the case we terminate and output 1. In other words, we look for a small ball $B(p^i(d), 1/k)$ around an image of d such that it is mapped strictly into itself under an iteration p^{j-i} of p. By the Schwarz Lemma, this implies that p has an attracting orbit that passes through $B(p^i(d), 1/k)$, to which the orbit of d converges.
Conversely, if the orbit of d converges to an (super-)attracting periodic orbit ζ of period m, then the convergence happens at a (super-)geometric rate. Thus there is a $z_0 \in \zeta$, $n_0 \in \mathbb{N}$, and $\tau < 1$ such that
$$|p^{mt}(d) - z_0| < \tau^{mt}$$
for every $t > n_0$. Moreover, there is an $\varepsilon > 0$ such that if
$$|x - z_0| < \varepsilon$$
then
$$|p^{mt}(x) - z_0| < \tau^{mt}|x - z_0|$$
for all t. Choose a $k > 3n_0$ such that
$$4/k < \varepsilon \text{ and } \tau^{k/3} < 1/(4k).$$
Then there is a t such that
$$k/3 < mt < k/2.$$
Let $i = mt$ and $j = 2mt$. We have
$$|p^i(d) - z_0| < 1/(4k), \text{ thus } B(p^i(d), 1/k) \subset B(z_0, 2/k) \subset B(z_0, \varepsilon).$$

Hence

$$p^{j-i}(B(p^i(d),1/k)) \subset p^{j-i}(B(z_0,2/k)) \subset B(z_0,2\tau^{j-i}/k)$$
$$\subset B(z_0,1/k^2) \subset B(p^i(d),1/k^2+1/(4k)) \subset B(p^i(d),1/(2k)-1/k^2).$$

The machine will therefore terminate and output 1.

4. M_{par}: We make use of Lemma 3.9. Since we can produce arbitrarily good approximations of every parabolic periodic point ζ of $p(z)$, we do not need an oracle for the value of this point. Let L_k^ζ be the sets from Lemma 3.9 corresponding to the point ζ. Let $z_k = p^k(d)$ computed with precision $2^{-(k+2)}$. We output 1 if z_k is inside L_k^ζ for some ζ and at least 2^{-k}-away from its boundary.
5. M_{sieg}: This is the most interesting case. It is not hard to see (cf. Figure 2.5) that, for each k, we can compute a union $E_k \in \mathscr{C}$ of dyadic balls such that

$$\bigcup_{i=0}^{k} p^i\left(B(d,\frac{4}{3}\cdot 2^{-n})\right) \subset E_k \subset \bigcup_{i=0}^{k} p^i\left(B(d,\frac{5}{3}\cdot 2^{-n})\right).$$

Let ζ_* be the center of the Siegel disc (one of the centers, in case of an orbit), and let y be the given periodic point in the connected component of ζ_*. We terminate and output 1 if E_k separates ζ_* from y in $\mathbb{C}$ (or covers either one of them) for some k.
Clearly, if d is inside the Siegel disc, then the forward images of $B(d,\frac{4}{3}\cdot 2^{-n})$ will cover an annulus in the disc that will separate ζ_* from the boundary of the disc, and in particular from y. Hence M_{sieg} will terminate and output 1.
On the other hand, if the distance from d to K_p is $\geq 2\cdot 2^{-n}$, then $E_k \cap K_p = \emptyset$ for all k. In particular, E_k cannot separate ζ_* from y, since they belong to the same connected component of K_p.

□

The proof is simplified in the case of a quadratic polynomial.

Proof (Theorem 4.2). If we assume that $p(z) = f_c(z)$ then by the Fatou-Shishikura bound, there is at most one non-repelling orbit. By our assumption, it is not parabolic. Moreover, if it is a Siegel orbit, then the Julia set is connected. Therefore, *any* repelling periodic orbit will be in the same connected component of K_p as the Siegel disk.

If $c \notin \mathscr{M}$, we run M_{ext} and M_{jul}. One and only one of them is guaranteed to halt and output a correct answer.

For $c \in \mathscr{M}$ we will use a modified Turing Machine $\widehat{M}_{sieg}$. It will compute the set E_k as before. If E_k separates the plane into two or more components, it will use the algorithm from Proposition 3.2 to search for two periodic points of period at most k in different components separated by E_k. If a k is found for which such two orbits are located, or if E_k covers a periodic orbit, $\widehat{M}_{sieg}$ will terminate and output 1. Since all periodic points are in K_c, and K_c is connected, this would clearly be a valid answer.

If $c \in \mathscr{M}$, then we run M_{ext}, M_{jul}, M_{hyp}, and $\widehat{M}_{sieg}$. As before, it is easy to see that one of them will terminate, and its output will be a correct one. □

Using Theorem 1.12 we obtain an interesting corollary:

Corollary 4.4 (Continuity of K_c) *Denote by $\mathscr{P}$ the set of c's for which J_c is parabolic. The function*

$$\mathbb{K} : c \mapsto K_c = K(z^2 + c)$$

is continuous in the Hausdorff metric on the set $\mathscr{M} \setminus \mathscr{P}$.

This result is well-known (see [Dou94]). However, it is quite remarkable that we have arrived at it using considerations of computability, rather than an analytic construction.

4.2 Computability of Julia sets in the absence of rotation domains

Similar ideas were used in [BBY07a] to prove the following theorem:

Theorem 4.5 *Let f be a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ without rotation domains. Then its Julia set is computable in the spherical metric by an oracle Turing machine M^ϕ with the oracle representing the coefficients of f. The algorithm uses the following non-uniform information about each parabolic periodic point ζ of f with period m and multiplier $e^{2\pi i p/q}$:*

- *a dyadic ball $B(w,r) \ni p$ such that $B(w,2r)$ does not contain any other points periodic with period m;*
- *the values of m, p, and q.*

Proof. We will show how to approximate $J(f)$ with any desired precision in the spherical metric on $\hat{\mathbb{C}}$. Note, that this metric is equivalent to the metric induced on $\hat{\mathbb{C}}$ by the Euclidean distance in $\mathbb{R}^3$ via the stereographic projection (see Definition 2.1.1). The spherical metric leads to a natural definition of Hausdorff distance between compacta in $\hat{\mathbb{C}}$. As before, we will denote this distance d_H.

For every natural n we can compute a sequence of rationals $\{q_i\}$ such that

$$B(J(f), 2^{-(n+2)}) \Subset \bigcup_{i=1}^{\infty} B(q_i, 2^{-(n+1)}) \Subset B(J(f), 2^{-n}). \tag{4.2.1}$$

To do this, for each $k > n+2$, we compute 2^{-k}-approximations of the periodic points of f in $\hat{\mathbb{C}}$ with periods at most k using Proposition 3.2. Let $M > 0$ be some bound on $|D^2 f^m(z)|$ in the area of an approximate periodic orbit r_i with period m. Then $|Df^m(r_i)| > 1 + 2^{-k}M$ implies that $|Df^m(w)| > 1$ for the periodic point w which r_i approximates. In this case we add the point r_i to our sequence of rationals. Clearly, for each repelling periodic point of f we will eventually obtain in this way a rational point which approximates it with precision at least $2^{-(n+3)}$. Since such points are contained in $J(f)$, and are dense there, our sequence has the desired property.

Of course, we can similarly *eventually* find every attracting orbit $\bar{\zeta}$ of f with an arbitrary precision. In this case, we will compute a set $D_{\bar{\zeta}}$ – a union of k dyadic balls – for this orbit such that $f^k(D_{\bar{\zeta}}) \Subset D_{\bar{\zeta}}$. Set $D = \cup_{\bar{\zeta}} D_{\bar{\zeta}}$.

Finally, for each parabolic periodic point ζ of f let L_k^{ζ} be the sets from Lemma 3.9. Set $L_k = \cup_{\zeta} L_k^{\zeta}$.

We are now ready to present an algorithm to find a set $C_m \in \mathscr{C}$ with $\mathrm{dist}_H(C_m, J(f)) < 2^{-m}$. Fix $m \in \mathbb{N}$. Our algorithm to find $C_m \in \mathscr{C}$ works as follows. At the k-th step:

- compute the finite union $B_k = \cup_{i=1}^{k} B(q_i, 2^{-(m+1)}) \in \mathscr{C}$;
- compute with precision $2^{-(m+3)}$ the complement of the preimage

$$f^{-k}(D \cup L_k),$$

  that is, find $W_k \in \mathscr{C}$ such that

$$d_H(\overline{W_k}, \overline{\hat{\mathbb{C}} \setminus (f^{-k}(D \cup L_k))}) < 2^{-(m+3)},$$

  where D is the union of $D_{\bar{\zeta}}$'s discovered so far by the algorithm.
- if $W_k \subset B_k$ output $C_m = B_k$ and terminate. Otherwise, go to step $k+1$.

By the Fatou-Sullivan classification, the algorithm will eventually terminate. Now suppose that the algorithm terminates at step k. Since $W_k \subset B_k$ and $J(f) \subset B(W_k, 2^{-(m+3)})$ we have $J(f) \subset B(C_m, 2^{-(m+3)})$. On the other hand, $\cup\{q_i\} \subset J(f)$, and thus $B_k = C_m \subset B(J(f), 2^{-(m+1)})$.

□

Using the result we have just obtained together with Theorem 1.12, we see that the examples of discontinuity in the dependence $c \mapsto J_c$ which we discussed in §3.4 are the only possible ones:

Corollary 4.6 (Continuity of J_c) *The dependence $c \mapsto J_c$ is continuous with respect to the Hausdorff distance at all values of c which are neither parabolic nor Siegel.*

4.3 Computable Julia sets of Siegel quadratics

What can we say about computability of Julia sets with Siegel disks or rotation domains? For simplicity, let us concentrate on one particular example: the family of quadratic polynomials given by the formula

$$P_\theta(z) = z^2 + e^{2\pi i\theta} z, \text{ where the parameter } \theta \in \mathbb{R}/\mathbb{Z}.$$

Each P_θ has a neutral fixed point at the origin, and so, by the Fatou-Shishikura bound, every other periodic cycle of P_θ is repelling. In the next chapter we will describe a precise characterization of those values of $\theta \in \mathbb{R}\setminus\mathbb{Q}$ for which 0 is a Siegel fixed point of P_θ. For the moment let us simply assume that this is the case, and denote by $\Delta_\theta \ni 0$ the Siegel disk.

Let us give a simple necessary and sufficient condition for computability of J_θ.

Definition 4.3.1 *The inner radius of the Siegel disk Δ_θ is the distance*

$$\rho_\theta = \mathrm{dist}(0, \Delta_\theta).$$

That is, ρ_θ is the radius of the largest Euclidean circle about the origin which can be inscribed into Δ_θ.

Denote by J_θ the Julia set $J(P_\theta)$. As $\rho_\theta = \mathrm{dist}(0, J_\theta)$, the following is evident:

Proposition 4.7 *Suppose that J_θ is computable by a Turing Machine M^ϕ with an oracle for θ. Then the same is true for ρ_θ.*

The converse also holds:

Theorem 4.8 *Suppose that ρ_θ is computable by a Turing Machine M^ϕ with an oracle for θ. Then so is J_θ.*

Proof. The algorithm for producing the 2^{-n} approximation of the Julia set is as follows. First, compute a large disk D around 0 with $P_\theta^{-1}(D) \Subset D$. Then

(I) compute a set $D_k \in \mathscr{C}$ which is a $2^{-(n+3)}$-approximation of the preimage $P_\theta^{-k}(D)$;
(II) set W_k to be the round disk with radius $\rho_\theta - 2^{-k}$ about the origin. Compute a set $B_k \in \mathscr{C}$ which is a $2^{-(n+3)}$-approximation of $P_\theta^{-k}(W_k)$;
(III) if D_k is contained in a $2^{-(n+1)}$-neighborhood of B_k, then output a $2^{-(n+1)}$-neighborhood of $D_k \setminus B_k$, and stop. If not, go to step (I).

A proof of the validity of the algorithm is completely straightforward. The idea is that D_k is an "upper bound" on the filled Julia set K_θ, and W_k is a "lower bound". The increasing sequence $\{W_k\}$ fills in the interior of K_θ, approximating it from the inside, and $\{D_k\}$ approximates it from the outside. When the two approximations meet, we know that set $D_k \setminus B_k$ must be approximating $J_\theta = \partial K_\theta$. We let the reader fill in the details of the argument. A step in the approximation process is illustrated in Figure 4.3. □

Here is a particular class of examples of Siegel Julia sets for which ρ_θ is computable. Recall that an irrational θ whose continued fraction expansion is given by $\theta = [a_1, a_2, \ldots]$ is said to be of bounded type if $\sup\{a_n\} < \infty$.

Proposition 4.9 *Let θ be of bounded type. Then J_θ is computable by a TM with an oracle for θ.*

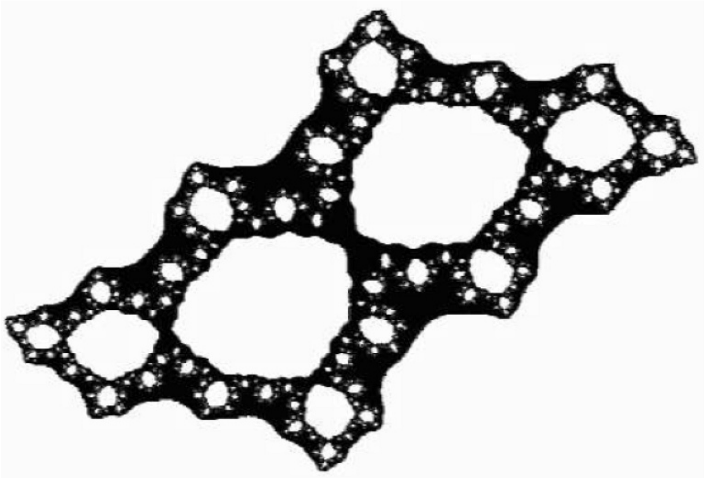


Fig. 4.1 A figure produced by the algorithm of Theorem 4.8 for $\theta = (\sqrt{5}-1)/2$.

For θ of bounded type, the boundary $\partial\Delta_\theta$ is a quasi-fractal Jordan curve passing through the critical point p_θ of P_θ. Although it cannot be smooth because of this, its shape is well-controlled. Recall that a simple closed curve $\gamma : \mathbb{T} \to \mathbb{C}$ in the plane is called M-quasisymmetric, if the following holds.

Let a, b be a pair of points in $\mathbb{T}$. Then

$$\operatorname{diam}(\gamma([a,b])) \le M|a-b|.$$

The image of an M-quasisymmetric curve is called a *quasicircle*.

Proposition 4.10 *For each $B \in \mathbb{N}$ there exists $M(B) > 1$ such that the following holds. Assume that all terms in the continued fraction of $\theta \in (0,1) \setminus \mathbb{Q}$ are bounded by B. Then $\partial\Delta_\theta$ is a quasicircle with $M \le M(B)$.*

This statement is due to Douady, Ghys, Herman, and Shishikura, and the proof can be found in [Dou88].

4.3.1 Approximating the boundary of Δ_θ by the critical orbit.

The same work of Douady, Ghys, Herman, and Shishikura yields the following statement:

Proposition 4.11 *Let θ be of bounded type, and denote by p_n/q_n its continued fraction convergents. Let $B > 0$ be an upper bound on $\sup q_{n+1}/q_n$. There exist constants $K > 0$, $\tau < 1$ which depend only on B, such that*

$$\operatorname{dist}_H(\Omega_n, \partial\Delta_\theta) < K\tau^n, \text{ where } \Omega_n = \{P_\theta^i(p_\theta),\ i = 0, \ldots, q_{n+2}\}.$$

Here $p_\theta = -e^{2\pi i\theta}/2$ *is the critical point of* P_θ.

Proposition 4.11 gives a recipe for computing the inner radius, approximated by

$$r_n = \min_{i=0,\ldots,q_{n+2}} |P_\theta^i(p_\theta)|,$$

and thus implies Proposition 4.9.

Proof of Proposition 4.11

Siegel quadratic Julia sets of bounded type may be constructed by means of quasiconformal surgery due to Douady, Ghys, Herman, and Shishikura (cf. [Dou88]) on a Blaschke product

$$B_\theta(z) = e^{2\pi i\tau(\theta)} z^2 \frac{z-3}{1-3z}.$$

This map homeomorphically maps the unit circle $\mathbb{T}$ onto itself with a single (cubic) critical point at 1. The angle $\tau(\theta)$ can be selected uniquely and constructively in such a way that the rotation number of the restriction $\rho(B_\theta|_{\mathbb{T}}) = \theta$.
For each n, the points

$$\{1, B_\theta(1), B_\theta^2(1), \ldots, B_\theta^{q_{n+1}-1}(1)\}$$

form the *n-th dynamical partition* of the unit circle. The following result is due to Swiatek and Herman (for the proof see e.g. Theorem 3.1 of [dFdM99]):
Theorem 4.12 (Universal real *a priori* **bound)** *There exists an explicit constant* $C > 1$ *independent of* θ *and* n *such that the following holds. Let* $\theta \in \mathbb{R}\setminus\mathbb{Q}$ *and* $n \in \mathbb{N}$*. Then any two adjacent intervals* I *and* J *of the n-th dynamical partition of* B_θ *are C-commensurable:*

$$C^{-1}|I| \leq |J| \leq C|I|.$$

Proposition 4.13 ([Her86]) *For each bounded type* $\theta = [a_0, \ldots, a_k, \ldots]$ *the Blaschke product* B_θ *is M-quasisymmetrically conjugate to the rotation* $R_\theta : x \mapsto x + \theta \bmod \mathbb{Z}$*. The quasisymmetric constant may be taken as* $M = (2\max a_i)^{10C^2}$.

Let us now consider the mapping Ψ which identifies the critical orbits of B_θ and P_θ by

$$\Psi : B_\theta^i(1) \mapsto P_\theta^i(p_\theta).$$

We have the following (for details see, for example, Theorem 3.10 of [YZ01]):

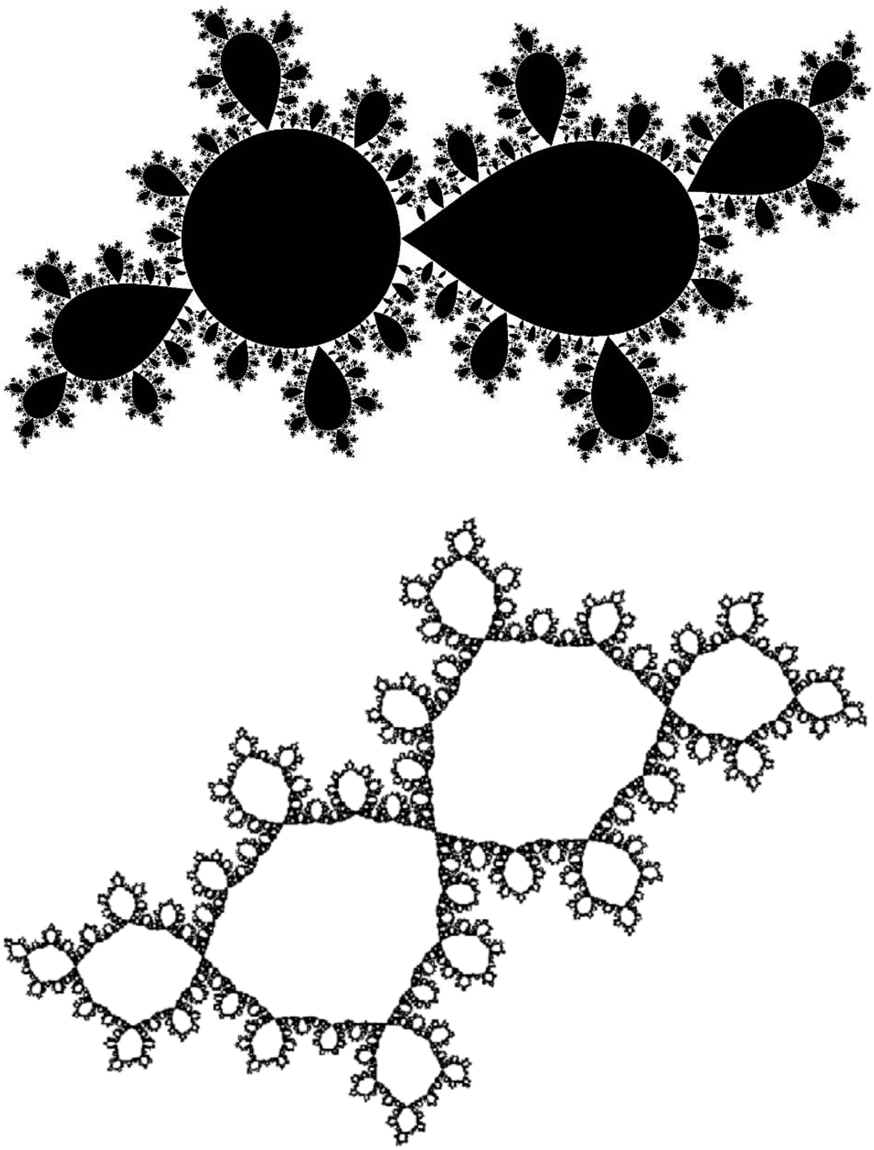


Fig. 4.2 For the Blaschke product B_θ denote by U the component of the first preimage of the unit disk $\mathbb{U}$ which is contained in $\mathbb{C}\backslash\mathbb{U}$. The top figure consists of the closed unit disk and all the points whose orbits under B_θ will eventually land in $\bar{U}$. Quasiconformal surgery transforms it into the filled Julia set K_θ, and its boundary into J_θ (the bottom figure). The image of $\mathbb{U}$ is the Siegel disk Δ_θ, the image of U is $-\Delta_\theta$.

Theorem 4.14 (Douady, Ghys, Herman, Shishikura) *The mapping Ψ extends to a K-quasiconformal homeomorphism of the plane $\mathbb{C}$ which maps the unit disk $\mathbb{D}$ onto the Siegel disk Δ_θ. The constant K may be taken as the quasiconformal dilatation of any global quasiconformal extension of the M-quasisymmetric conjugacy of Proposition 4.13. In particular, we can ensure that $K \leq 2M$.*

Elementary combinatorics implies that each interval of the n-th dynamical partition contains at least two intervals of the $(n+2)$-nd dynamical partition. This in conjunction with Theorem 4.12 implies that the size of an interval of the $(n+2)$-nd dynamical partition of B_θ is at most τ^n where

$$\tau = \sqrt{\frac{C}{C+1}}.$$

Hence, setting

$$\Omega_n = \{P_\theta^i(p_\theta),\ i = 0, \ldots, q_{n+2}\},$$

by Theorem 4.14,

$$\mathrm{dist}_H(\Omega_n, \partial\Delta_\theta) < K\tau^n. \qquad \square$$

4.4 Robust computability

In practice, a Julia set $J(R)$ is often computed without regard for a round-off error. This can be thought of as iterating a point $z \in \hat{\mathbb{C}}$ by a sequence of rational mappings R_i whose coefficients are close (but not necessarily equal) to those of R.

Note here the strong similarity to *random iteration* of rational maps, as described, for example in [FS91].

In our context, the appropriate question to ask here is the following:

Given a point z and a rational map R, can we determine whether z is close to the Julia set of some nearby rational map R'?

More specifically, restricting ourselves to quadratic polynomials, let $\mathfrak{J}$ be the subset of $\mathbb{C} \times \mathbb{C}$ given by

$$\mathfrak{J} = \overline{\{(z, c)\ :\ z \in J_c\}}.$$

We formulate the question:

Is the set $\mathfrak{J}$ computable?

We thank M. Shub for suggesting this question to us. The answer is “yes”:

Theorem 4.15 *Let $d > 0$ be any computable real. Then the compact set*

$$\mathfrak{J} \cap \mathbb{C} \times \overline{B(0,d)}$$

is a computable subset of $\mathbb{C} \times \mathbb{C}$.

Informally, we may think of the projection of $\mathfrak{J} \cap \mathbb{C} \times (c-\varepsilon, c+\varepsilon)$ to the first coordinate as the picture that a computer could produce when ignoring the round-off error.

We prove Theorem 4.15 by showing that $\mathfrak{J}$ is *weakly* computable (Definition 1.4.1).

We will need the following lemma.

Lemma 4.16 *For any point (z,c) in the complement of the closure $\overline{\mathfrak{J}}$, z converges to an attracting periodic orbit of $f_c : z \mapsto z^2 + c$.*

The proof of the lemma will require us to recall the nature of discontinuities in the function $\mathbb{J}(c)$, particularly the theory of parabolic implosion, as was described in §3.4.1. We postpone it until the end of the section.

The following lemma allows us to “cover” all points that belong to $\mathfrak{J}$.

Lemma 4.17 *There is an algorithm $A_1(n)$ that on input n outputs a sequence of dyadic points $p_1, p_2, \ldots \in \mathbb{C} \times \mathbb{C}$ such that*

$$B(\mathfrak{J}, 2^{-(n+3)}) \subset \bigcup_{j=1}^{\infty} B(p_j, 2^{-(n+2)}) \subset B(\mathfrak{J}, 2^{-(n+1)}).$$

Proof. The repelling periodic orbits of f_c are dense in J_c. Hence the set

$$S_{rep} = \{(z,c)\ :\ z \text{ is in a repelling periodic orbit of } f_c\}$$

is dense in $\mathfrak{J}$. S_{rep} is a union of a countable number of algebraic curves S^m_{rep} given by the constraints

$$\begin{cases} f_c^m(z) = z \\ |(f_c^m)'(z)| > 1. \end{cases}$$

For each m we can compute a finite number of points $p_1^m, \ldots, p_{r_m}^m$ approximating S^m_{rep} such that

$$B(S^m_{rep}, 2^{-(n+3)}) \subset \bigcup_{j=1}^{r_m} B(p_j^m, 2^{-(n+2)}) \subset B(S^m_{rep}, 2^{-(n+1)}).$$

We have

$$\overline{\mathfrak{J}} = \overline{S_{rep}} = \overline{\bigcup_{m=1}^{\infty} S^m_{rep}}.$$

Hence the computable sequence $p_1^1, \ldots, p_{r_1}^1, p_1^2, \ldots, p_{r_2}^2, \ldots, p_1^m, \ldots, p_{r_m}^m, \ldots$ satisfies the conditions of the lemma. □

Corollary 4.18 *There is an oracle machine $M_1^{\phi_1,\phi_2}(n)$, where ϕ_1 is an oracle for $z \in \mathbb{C}$ and ϕ_2 is an oracle for $c \in \mathbb{C}$, such that $M_1^{\phi_1,\phi_2}$ always halts whenever $d((z,c),\mathfrak{J}) < 2^{-(n+4)}$ and never halts if $d((z,c),\mathfrak{J}) \geq 2^{-n}$.*

Proof. Query the oracles for a point $p \in \mathbb{C} \times \mathbb{C}$ such that $d(p,(z,c)) < 2^{-(n+4)}$. Then run the following loop:

$i \leftarrow 0$
do
  $i \leftarrow i+1$
  generate p_i using $A_1(n)$ from Lemma 4.17
while $d(p,p_i) > 2^{-(n+2)}$

If $d((z,c),\mathfrak{J}) < 2^{-(n+4)}$, then $d(p,\mathfrak{J}) < 2^{-(n+3)}$, and hence by Lemma 4.17 there is an i such that $d(p,p_j) \leq 2^{-(n+2)}$, and the loop terminates. If $d((z,c),\mathfrak{J}) > 2^{-n}$, then $d(p,\mathfrak{J}) > 2^{-n} - 2^{-(n-4)} > 1.5 \cdot 2^{-(n+1)}$. Hence, by Lemma 4.17, $p \notin B(p_i, 2^{-(n+1)})$ for all i, and the loop will never terminate. □

The following lemma allows us to exclude points outside $\overline{\mathfrak{J}}$ from $\mathfrak{J}$.

Lemma 4.19 *There is an oracle machine $M_2^{\phi_1,\phi_2}$, where ϕ_1 is an oracle for $z \in \mathbb{C}$ and ϕ_2 is an oracle for $c \in \mathbb{C}$, such that $M_2^{\phi_1,\phi_2}$ halts if and only if z converges to an attracting periodic orbit (or to ∞) under $f_c : z \mapsto z^2 + c$.*

Proof. M_2 is systematically looking for an attracting cycle of f_c. It also iterates f_c on z with increasing precision and for increasingly many steps until we are sure that either one of the following two things holds:

1. the orbit of z converges to ∞; or
2. we find an attracting orbit of f_c and the orbit of z converges to it.

If the search is done systematically, the machine will eventually halt if one of the possibilities above holds. It obviously won't halt if neither holds. □

Proof (Theorem 4.15). **The algorithm is:** Run the machines $M_1^{\phi_1,\phi_2}(n)$ from Corollary 4.18 and $M_2^{\phi_1,\phi_2}$ from Lemma 4.19 in parallel. Output 1 if M_1 terminates first and 0 if M_2 terminates first.

First we observe that $M_1(n)$ only halts on points that are 2^{-n}-close to $\mathfrak{J}$, in which case 1 is a valid answer according to Definition 1.4.1. Similarly, M_2 only halts on points that are outside $\mathfrak{J}$, in which case 0 is a valid answer. Hence if the algorithm terminates, it outputs a valid answer. It remains to see that it does always terminate. Consider two cases.

Case 1: $(z,c) \in \overline{\mathfrak{J}}$. In this case $d((z,c),\mathfrak{J}) = 0 < 2^{-(n+4)}$, and the first machine is guaranteed to halt.

Case 2: $(z,c) \notin \overline{\mathfrak{J}}$. By Lemma 4.16, z converges to an attracting periodic orbit of f_c in this case, and hence the second machine is guaranteed to halt. □

Proof of Lemma 4.16

Suppose $z \notin J_c$ and the orbit of z does not belong to an attracting basin. By the Fatou-Sullivan classification (see e.g. [Mil06]), there exists $k \in \mathbb{N}$ such that $w \equiv f_c^k(z)$ belongs to a Siegel disk or to the immediate basin of a parabolic orbit. Our aim is to show that for an arbitrary small $\delta > 0$, there exists a pair $(\tilde{z}, \tilde{c}) \in \mathbb{C} \times \mathbb{C}$ with

$$|z - \tilde{z}| < \delta, \ |c - \tilde{c}| < \delta, \text{ and for which } \tilde{z} \in J_{\tilde{c}}.$$

We will treat the Siegel case first.

The case when w lies in a Siegel disk. Let us denote by Δ the Siegel disk containing w, and let $m \in \mathbb{N}$ be its period, that is, the mapping

$$f_c^m : \Delta \to \Delta$$

is conjugated by a conformal change of coordinates $\phi : \Delta \to \mathbb{D}$ to an irrational rotation of $\mathbb{D}$.

By Proposition 3.19, we have the following. Denote $\zeta = \phi^{-1}(0) \in \Delta$ the center of the Siegel disk. For each $s > 0$ there exists $\tilde{c} \in B(c,s)$ such that $f_{\tilde{c}}$ has a parabolic periodic point $\tilde{\zeta}$ of period m in $B(\zeta,s)$. In particular, $J_{\tilde{c}}$ is connected, and $B(\zeta,s) \cap J_{\tilde{c}} \neq \emptyset$.

Consider now the f_c^m-invariant analytic circle

$$S_r = \phi^{-1}(\{z = re^{2\pi i\theta}, \ \theta \in [0, 2\pi)\})$$

which contains w. Let $\varepsilon > 0$ be such that

$$B(w, \varepsilon) \subset f_c^k(B(z, \delta)) \cap \Delta.$$

Set $B \equiv B(w, \varepsilon/2)$ and let $n \in \mathbb{N}$ be such that the union

$$\bigcup_{0 \le i \le n} f_c^{mi}(B) \supset S_r.$$

By Proposition 3.19 for all $\delta > 0$ small enough, there exist $\tilde{c} \in B(c, \delta)$ for which $J_{\tilde{c}}$ is connected and there is a point of $J_{\tilde{c}}$ inside the domain bounded by S_r. Since repelling periodic orbits of f_c are dense in $\partial\Delta$, again for δ small enough, there are points of $J_{\tilde{c}}$ on the outside of S_r as well, and so there exists a point $\xi \in J_{\tilde{c}} \cap S_r$. By construction, there exists $j \in \mathbb{N}$ such that $f_c^j(B(z, \delta)) \ni \xi$. By the invariance of Julia sets, if $\tilde{c}$ is close enough to c we have $B(z, \delta) \cap J_{\tilde{c}} \neq \emptyset$, and the proof is complete.

The case when w lies in a parabolic basin. Denote by ζ the parabolic periodic point of f_c whose immediate basin contains w, and let $m \in \mathbb{N}$ be its period. We employ the notations of §3.4.1.

Recall that, by Theorem 3.21, for every $s > 0$ there exists $\tilde{c} \in B(c,s)$ such that $B(J_{\tilde{c}}, s) \supset J_{(c,\tau)}$.

Since $\zeta \in J_c$, and J_c is connected, there exists a point $u \in J_c \cap P_R$, where P_R is a repelling petal. Let $\hat{w} \in C_A$ be the orbit of w, and let $\hat{u} \in C_R$ be the orbit of u. Choose the translation $\tau : C_A \to C_R$ so that $\tau(\hat{w}) = \hat{u}$. Then $J_{(c,\tau)} \ni w$, which implies $J_{(c,\tau)} \ni z$. The claim follows by Theorem 3.21. $\square$

Chapter 5
Negative Results

5.1 Occurrence of Siegel disks and Cremer points in the quadratic family

Let us discuss in more detail the occurrence of Siegel disks in the quadratic family. First we formulate the following strengthening of Siegel's Theorem 2.12, proved by Brjuno in the early 1970's . Recall that, for an irrational number $\theta \in (0,1)$, represented by an infinite continued fraction

$$\theta = [r_1, r_2, \ldots, r_n, \ldots]$$

with positive terms, we denote its rational convergents by $p_n/q_n = [r_1, \ldots, r_n]$.

Theorem 5.1 ([Brj71]) *Let R be an analytic map with a periodic point $z_0 \in \hat{\mathbb{C}}$ of period p. Suppose that the multiplier of the cycle is $\lambda = e^{2\pi i\theta}$, with the rational convergents of θ satisfying*

$$B(\theta) = \sum_n \frac{\log(q_{n+1})}{q_n} < \infty. \tag{5.1.1}$$

Then the local linearization equation (2.2.1) holds, and hence z_0 is a Siegel point.

As we have noted before, a quadratic polynomial with a fixed Sigel disk with rotation angle θ can after an affine change of coordinates be written as

$$P_\theta(z) = z^2 + e^{2\pi i\theta} z. \tag{5.1.2}$$

In 1987 Yoccoz [Yoc95] proved the following converse to Brjuno's Theorem:

Theorem 5.2 ([Yoc95]) *Suppose that for $\theta \in [0,1)$ the polynomial P_θ has a Siegel point at the origin. Then $B(\theta) < \infty$.*

The numbers satisfying (5.1.1) are called Brjuno numbers; the set of all Brjuno numbers will be denoted $\mathscr{B}$. It is evident that $\cup\mathscr{D}(k) \subset \mathscr{B}$ and thus the set $\mathscr{B}$ has full

M. Braverman and M. Yampolsky. *Computability of Julia Sets*.
Algorithms and Computation in Mathematics,

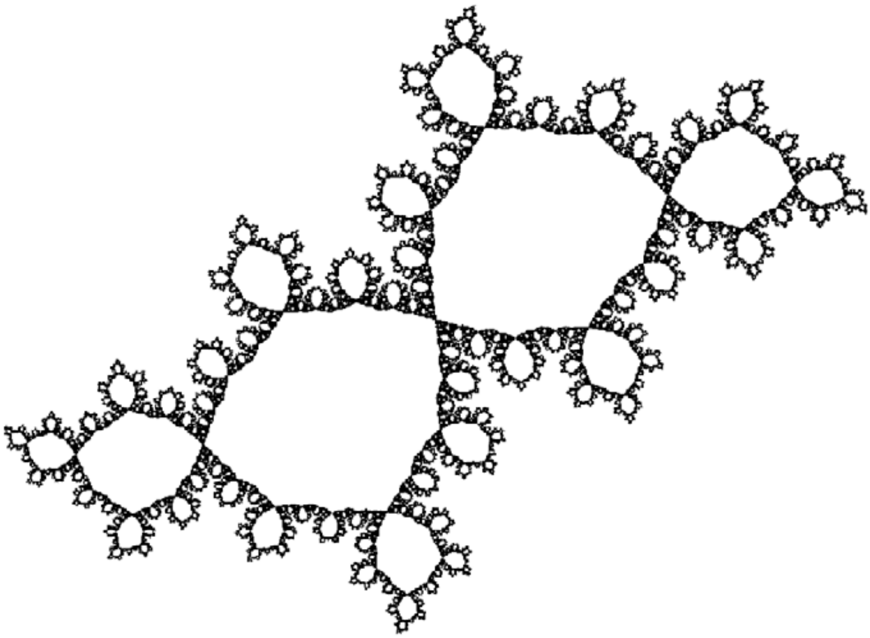


Fig. 5.1 The Julia set of P_θ for $\theta = [1,1,1,1,\ldots]$ (the inverse golden mean).

measure in the unit circle. On the other hand, it can be shown that its complement is dense-G_δ.

The sum of the series (5.1.1) is called the Brjuno function. For us a different characterization of $\mathscr{B}$ will be more useful. Inductively define $\theta_1 = \theta$ and $\theta_{n+1} = \{1/\theta_n\}$. In this way, if $\theta = [r_1, r_2, \ldots]$, then

$$\theta_n = [r_n, r_{n+1}, r_{n+2}, \ldots].$$

We define the *Yoccoz's Brjuno function* as

$$\Phi(\theta) = \sum_{n=1}^{\infty} \theta_1 \theta_2 \cdots \theta_{n-1} \log \frac{1}{\theta_n} = \log \frac{1}{\theta_1} + \theta_1 \log \frac{1}{\theta_2} + \theta_1 \theta_2 \log \frac{1}{\theta_3} + \ldots$$

One can verify that

$$B(\theta) < \infty \Leftrightarrow \Phi(\theta) < \infty.$$

The value of the function Φ is related to the size of the Siegel disk in the following way.

Definition 5.1.1 *Let $W \subset \mathbb{C}$ be a simply-connected domain in the complex plane, and let $w \in W$ be an arbitrary choice of a marked point. Consider the unique conformal isomorphism*

$$\phi_{(W,w)} : \mathbb{U} \to W \tag{5.1.3}$$

which maps 0 *to* w *and has a positive derivative at* 0*. Then the conformal radius of the marked domain* (W,w) *is*

$$r(W,w) = \phi'_{(W,w)}(0).$$

For example, if the domain V is a disk of radius r,

$$(V,v) = (B(0,r),0),$$

then $\phi_{(V,v)}(z) = r \cdot z$, and $r(V,v) = r$. Note that by the Koebe One-quarter Theorem, for any domain (W,w) we have

$$W \supset B(w, r(W,w)/4).$$

Definition 5.1.2 *Let* $P(\theta)$ *be a quadratic polynomial with a Siegel disk* $\Delta_\theta \ni 0$*. The conformal radius of the Siegel disk* Δ_θ *is*

$$r(\theta) = r(\Delta_\theta, 0).$$

For all other $\theta \in [0,\infty)$ *we set* $r(\theta) = 0$*.*

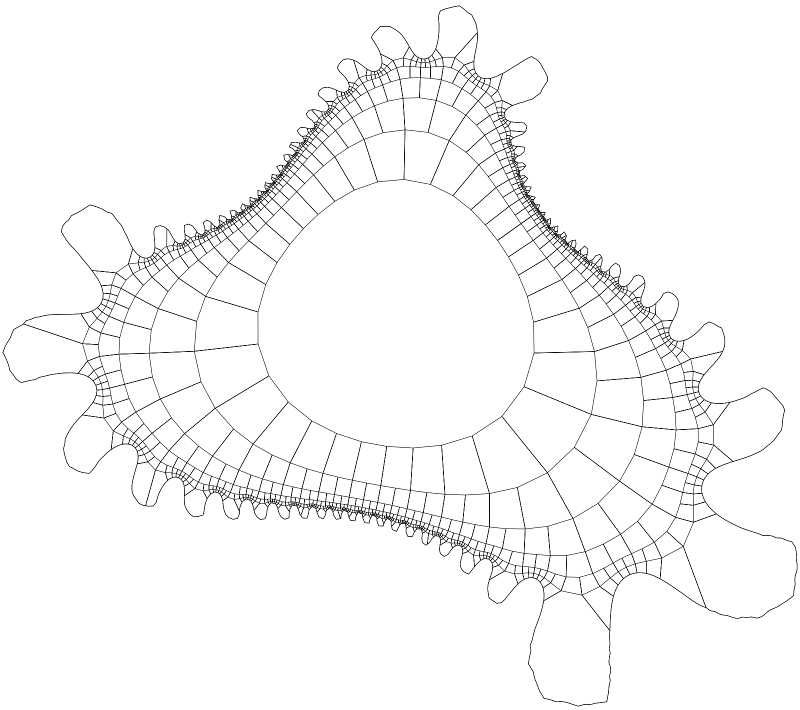


Fig. 5.2 The image of the polar grid by the Riemann Mapping of Δ_θ with $\theta = [3,20,200,1,1,1,\ldots]$.

Yoccoz [Yoc95] has shown that the sum

$$\upsilon(\theta) \equiv \Phi(\theta) + \log r(\theta) \tag{5.1.4}$$

is bounded from below independently of $\theta \in \mathscr{B}$. Recently, Buff and Chéritat have greatly improved this result by showing that:

Theorem 5.3 ([BC06b]) *The function* $\upsilon : \theta \mapsto \Phi(\theta) + \log r(\theta)$ *extends to* $\mathbb{R}$ *as a 1-periodic continuous function.*

We remark that the following stronger conjecture exists (see [MMY97]):

Marmi-Moussa-Yoccoz Conjecture. [MMY97] *The function*

$$\upsilon : \theta \mapsto \Phi(\theta) + \log r(\theta)$$

is Hölder with exponent $1/2$*: that is, there is a constant* $C > 0$ *such that for any* $x, y \in [0,1]$,

$$|\upsilon(x) - \upsilon(y)| < C \cdot |x - y|^{1/2}.$$

We have actually demonstrated the following in [BY06]:

Theorem 5.4 *There exists* $\theta_0 \in \mathscr{B}$ *such that the function* $\theta \mapsto \Phi(\theta)$ *is non-computable on the domain consisting of a single point* $\{\theta_0\}$ *by a Turing Machine with an oracle access to* θ.

Assuming that the Marmi-Moussa-Yoccoz Conjecture holds, Theorem 5.4 would be sufficient to demonstrate that $r(\theta)$ is not computable for some values of $\theta \in \mathbb{T}$; which in turn, by Proposition 5.13 below, would imply non-computability of $J(P_\theta)$:

Conditional Implication 1. *If the function*

$$\upsilon : \theta \mapsto \Phi(\theta) + \log r(\theta)$$

has a computable modulus of continuity, then it is uniformly computable on the entire interval $[0,1]$.

The proof of the above implication uses the following result of Buff and Chéritat ([BC06b]).

Lemma 5.5 ([BC06b]) *For any rational point* $\theta = \frac{p}{q} \in [0,1]$ *denote, as before,*

$$P_\theta(z) = e^{2\pi i\theta} z + z^2,$$

and let the Taylor expansion of $P_\theta^{\circ q}(z)$ *at* 0 *start with*

$$P_\theta^{\circ q}(z) = z + Az^{q+1} + \dots, \textit{ for } q \in \mathbb{N}$$

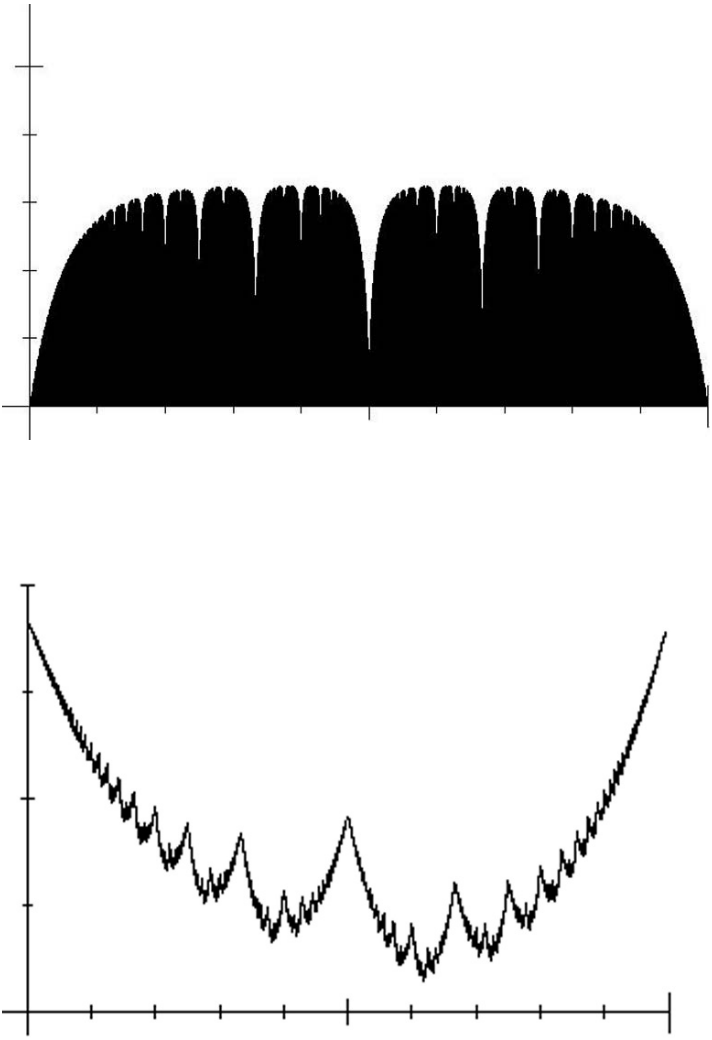


Fig. 5.3 The top figure is an attempt to visualize the (non-computable!) function Φ, by plotting the heights of $\exp(-\Phi(\theta))$ over a grid of Brjuno irrationals. The lower figure is the graph of the (conjecturally computable) function $\upsilon(x)$.
Both figures courtesy of Arnaud Chéritat

Let $L(\theta) = \left(\frac{1}{qA}\right)^{1/q}$. *Denote by* Φ_{trunc} *the modification of* Φ *applied to rational numbers where the sum is truncated before the infinite* *term. Then we have the following explicit formula for computing* $\upsilon(\theta)$:

$$\upsilon(\theta) = \Phi_{trunc}(\theta) + \log L(\theta) + \frac{\log 2\pi}{q}. \tag{5.1.5}$$

Equation (5.1.5) allows us to compute the value of υ easily at every rational $\theta \in \mathbb{Q} \cap [0,1]$ with an arbitrarily good precision. Assuming that υ has a computable modulus of continuity, and putting together Lemma 5.5 and Theorem 1.7, we have the Conditional Implication 1.
The following conditional result follows:

Conditional Implication 2. *Suppose that Conditional Implication 1 holds. Let $\theta \in [0,1]$ be such that $\Phi(\theta)$ is finite. Then there is an oracle Turing Machine M_1^{ϕ} computing $\Phi(\theta)$ with an oracle access to θ if and only if there is an oracle Turing Machine M_2^{ϕ} computing $r(\theta)$ with an oracle access to θ.*

Proof. Suppose that M_1^{ϕ} computes $\Phi(\theta)$ for some θ. Let M^{ϕ} be the machine uniformly computing the function υ. Then we can use M_1^{ϕ} and M^{ϕ} to compute $\log r(\theta) = \upsilon(\theta) - \Phi(\theta)$ with an arbitrarily good precision. We can then use this construction to give a machine M_2^{ϕ} which computes $r(\theta)$.
The opposite direction is proved analogously. □

Dependence of the conformal radius of a Siegel disk on the parameter

In this section we will show that the conformal radius of a Siegel disk varies continuously with the Julia set. To that end we will need a preliminary definition:

Definition 5.1.3 *Let (U_n, u_n) be a sequence of topological disks $U_n \subset \mathbb{C}$ with marked points $u_n \in U_n$. The kernel or Carathéodory convergence $(U_n, u_n) \to (U, u)$ means the following:*

- $u_n \to u$;
- *for any compact $K \subset U$ and for all n sufficiently large, $K \subset U_n$;*
- *for any open connected set $W \ni u$, if $W \subset U_n$ for infinitely many n, then $W \subset U$.*

The geometric meaning of this convergence is as follows. For a pointed domain (U,u) denote by

$$\phi_{(U,u)} : \mathbb{U} \to U$$

the unique conformal isomorphism with $\phi_{(U,u)}(0) = u$, and $(\phi_{(U,u)})'(0) > 0$. We again denote $r(U,u) = |(\phi_{(U,u)})'(0)|$ the conformal radius of U with respect to u.

By the Riemann Mapping Theorem, the correspondence

$$(U,u) \mapsto \phi_{(U,u)}$$

establishes a bijection between marked topological disks properly contained in $\mathbb{C}$ and univalent maps $\phi : \mathbb{U} \to \mathbb{C}$ with $\phi'(0) > 0$. The following theorem is due to Carathéodory; a proof may be found in [Pom92]:

Theorem 5.6 (Carathéodory Kernel Theorem) *Pointed simply-connected domains (U_n,u_n) converge to (U,u) in the sense of Carathéodory if and only if the Riemann mappings*

$$\phi_{(U_n,u_n)} \to \phi_{(U,u)} \text{ uniformly on compact subsets of } U.$$

We can now state and prove the following:

Proposition 5.7 *The conformal radius of a quadratic Siegel disk varies continuously with respect to the Hausdorff distance on Julia sets.*

Proof. To fix the ideas, consider the family P_θ with $\theta \in \mathscr{B}$ and denote by Δ_θ the Siegel disk of P_θ. It is evident that the Hausdorff convergence $J(P_{\theta_n}) \to J(P_\theta)$ implies the Carathéodory convergence of the pointed domains

$$(\Delta_{\theta_n}, 0) \to (\Delta, 0).$$

The proposition follows from this and the Carathéodory Kernel Theorem. □

Recall that for a pointed domain (U,u) the *inner radius* is given by

$$\rho(U,u) = \operatorname{dist}(u, \partial U).$$

To understand how the conformal radius of the domain is affected by the change in the inner radius, consider the example of the slit disk

$$\mathbb{U}_x \equiv \mathbb{U} \setminus (x,1) \text{ for } x \in (0,1)$$

(see Figure 2.1). A direct computation shows that

$$r(\mathbb{U}_x, 0) = \frac{4x}{(1+x)^2} < 1,$$

and so the conformal radius decreases linearly with the size of the slit. In fact, the domain $\mathbb{U}_x$ is extremal in the following sense:

Proposition 5.8 *Let $W \ni 0$ be a simply-connected open subset of $\mathbb{U}$. Assume that $\rho(W,0) \le x$. Then*

$$r(W,0) \le \frac{4x}{(1+x)^2}.$$

The proof uses a standard complex-analytic technique, which goes beyond the scope of this book (see e.g. [Ahl06]).

Let us now state a quantitative version of Proposition 5.7:

Lemma 5.9 *Let U be a simply-connected bounded subdomain of $\mathbb{C}$ containing the point 0 in the interior. Suppose $V \subset U$ is a simply-connected subdomain of U, and $\partial V \subset B(\partial U, \varepsilon)$. Then*

$$r(U,0) - r(V,0) \le 4\sqrt{r(U,0)}\sqrt{\varepsilon}.$$

Moreover, let $F(x) = 4x/(1+x)^2$. Then

$$r(V,0) \le r(U,0) F\left(\frac{\rho(V,0)}{\rho(U,0)}\right).$$

Proof. We begin with the proof of the first inequality (cf. Proposition 1 of [RZ04]). Let us set $\phi_U \equiv \phi_{(U,0)} : \mathbb{U} \to U$ and $\phi_V \equiv \phi_{(V,0)} : \mathbb{U} \to V$ as in (5.1.3). It is not difficult to see that the Koebe Distortion Theorem (§2.1) implies

$$\operatorname{dist}(\phi_U(z), \partial U) \ge \frac{1}{16}(1-|z|)^2 \phi_U'(0). \tag{5.1.6}$$

Set

$$D_V \equiv \phi_U^{-1}(V) \subset \mathbb{U}.$$

By (5.1.6) we have

$$D_V \supset B(0,r) \tag{5.1.7}$$

for any r satisfying

$$\frac{1}{16}(1-r)^2 r(U,0) \ge \varepsilon. \tag{5.1.8}$$

Fix any such r, and set

$$h_V \equiv \phi_{(D_V,0)}.$$

The inclusion (5.1.7) together with the Schwarz Lemma implies that

$$1 > h_V'(0) \ge r.$$

Since

$$r(U,0) h_V'(0) = r(V,0),$$

we have

$$r(U,0) - r(V,0) \le (1-r) r(U,0). \tag{5.1.9}$$

To complete the proof of the first inequality of the lemma, it remains to rewrite the inequality (5.1.8) as

$$(1-r) \geq 4\sqrt{\frac{\varepsilon}{r(U,0)}},$$

and substitute into (5.1.9).

Let us prove the second statement, The domain U contains a round disk $B = B(0,\rho(U,0))$. Applying the Schwarz Lemma to the restriction of ϕ_U^{-1} to the disk B, we have

$$|\phi_U^{-1}(z)| \leq \frac{|z|}{\rho(U,0)}.$$

Let $z \in \partial V \cap B$ be such that

$$|z| = \rho(V,0).$$

Then

$$x = |\phi_U^{-1}(z)| \leq \frac{\rho(V,0)}{\rho(U,0)}.$$

Let us apply Proposition 5.8 to estimate:

$$r(\phi_U^{-1}(V),0) = r(V,0)/r(U,0) \leq F(x),$$

and the claim immediately follows. $\square$

We also state for future reference the following proposition:

Proposition 5.10 *Let $\{\theta_i\}$ be a sequence of Brjuno numbers such that $\theta_i \to \theta$ and $\overline{\lim}\, r(\theta_i) = l > 0$. Then θ is also a Brjuno number and $r(\theta) \geq l$.*

Proof. Denote $\phi_i \equiv \phi_{(\Delta_{\theta_i},0)}$. By the Schwarz Lemma, the inverse $\psi_i \equiv (\phi_i)^{-1}$ linearizes P_{θ_i} on Δ_{θ_i}:

$$\psi_i \circ P_{\theta_i} \circ \psi_i^{-1}(z) = e^{2\pi i \theta_i} z.$$

Note first that, by the Koebe Distortion Theorem, the family of maps $\{\phi_i\}$ is equicontinuous on any proper subset of the unit disk. Further, $K_{\theta_i} \subset B(0,2)$, and hence there is a uniform upper bound on $\{\phi_i\}$. By the Arzelà-Ascoli Theorem, we can pass to a subsequence to assure that

$$\phi_i \to \phi \text{ locally uniformly, and } \phi'(0) = l.$$

By continuity, ϕ^{-1} is a linearizing coordinate for P_θ, so θ is a Brjuno number. Moreover, $\phi(\mathbb{U}) \subset \Delta_\theta$, and so by the Schwarz Lemma $r(\theta) \geq l$. $\square$

5.2 Computability of Julia sets of Siegel quadratics and negative results

5.2.1 Computability of $r(\theta)$ is equivalent to computability of J_θ

.

Of course, the change in parametrization from c to θ makes it natural to talk about computability of $J(P_\theta)$ by a TM with an oracle for θ, rather than for c. However, these notions are obviously equivalent, as $c = c(\theta)$ is found by the formula

$$c = c(\theta) = \lambda/2 - \lambda^2/4, \text{ where } \lambda = e^{2\pi i\theta}. \tag{5.2.1}$$

Let us abbreviate $J(P_\theta)$ as J_θ in what follows. To address the question of computability of J_θ for $\theta \in \mathscr{B}$ we first make note of the following result:

Proposition 5.11 *Suppose $r(\theta)$ is computable by a Turing Machine M^ϕ with an oracle to θ. Then so is J_θ.*

Proposition 5.11 follows from Theorem 4.8, and the following lemma:

Lemma 5.12 *Suppose $r(\theta)$ is computable by a Turing Machine M^ϕ with an oracle for θ. Then so is the inner radius $\rho(\Delta_\theta, 0) \equiv \rho_\theta$.*

Proof. The algorithm works as follows:

(I) For $k \in \mathbb{N}$ compute a set $D_k \in \mathscr{C}$ which is a 2^{-m}-approximation of the preimage $P_\theta^{-k}(D)$, for some sufficiently large disk D;

(II) evaluate the conformal radius $r(D_k, 0)$ with precision $2^{-(m+1)}$ (this can be done, for example, by using one of the numerous existing methods for computing the Riemann Mapping of a computable domain, see [BB85]);

(III) as before, let

$$F(x) = 4x/(1+x)^2, \text{ for } x \in [0,1].$$

Note that this function is monotone, and let $\psi(w) = F^{-1}(w)$. This function is computable, and $\psi(1) = 1$.

Evaluate

$$p = \psi(r(\theta)/r(D_k, 0))$$

with precision $2^{-(m+5)}/\rho(D,0)$. If

$$|1-p| < 2^{-(m+3)}/\rho(D,0),$$

then compute the inner radius $\rho(D_k, 0) \equiv r_k$ around 0 with precision $2^{-(m+1)}$ and output this number. Else, increment k and return to step (I).

Termination. Let $K = K(P_\theta)$ be the filled Julia set of P_θ. Then

$$\cap_{k=0}^{\infty} D_k = K \supset \Delta_\theta$$

and

$$D_0 \supset D_1 \supset D_2 \supset \dots .$$

Hence for every $\delta > 0$ there will be a step $k = k(\varepsilon)$ after which

$$\mathrm{dist}(\partial D_k, J_\theta) < \delta.$$

Since $\partial \Delta_\theta \subset J_\theta$, by Lemma 5.9 this implies that

$$|r(D_k,0) - r(\Delta_\theta,0)| = |r(D_k,0) - r(\theta)| < 4\sqrt{r(D,0)}\sqrt{\delta} \underset{\delta\to 0}{\longrightarrow} 0.$$

Since for every large enough k, the value of

$$\psi(r(\theta)/r(D_k,0)) > 1 - 2^{-(m+4)}/\rho(D,0),$$

the algorithm will eventually terminate on step (III).

Correctness. Now suppose the algorithm has terminated on step (III). As $\Delta_\theta \subset D_k$, Lemma 5.9 implies that

$$1 \geq \frac{\rho_\theta}{\rho(D_k,0)} \geq 1 - \frac{2^{-(m+1)}}{\rho(D,0)},$$

and so

$$|\rho(D_k,0) - \rho_\theta| \leq 2^{-(m+1)}.$$

□

As a converse to Proposition 5.11, we have the following:

Proposition 5.13 *Suppose that the function $\theta \mapsto J(P_\theta)$ is computable at a point θ_0 by a TM M^ϕ with an oracle access to θ_0, then the same is true for $r(\theta_0)$.*

Proof. Using the output of the TM computing $J(P_{\theta_0})$ in an obvious way, for each $\varepsilon > 0$ we can obtain a simply-connected domain $V \in \mathscr{C}$ such that

$$V \subset \Delta_{\theta_0} \text{ and } \partial V \subset B(\partial \Delta_{\theta_0}, \varepsilon).$$

It is elementary to verify that for every $\theta \in \mathbb{T}$, the set $J(P_\theta) \subset B(0,2)$. This implies, by the Schwarz Lemma, that the conformal radius $r(\theta_0) < 2$. Hence, by Lemma 5.9,

$$|r(V,0) - r(\theta_0)| < \delta = 8\sqrt{\varepsilon}.$$

Using any constructive version of the Riemann Mapping Theorem (see e.g. [BB85]), we can compute $r(V,0)$ to precision δ, and hence know $r(\theta_0)$ up to an error of 2δ. Given that δ can be made arbitrarily small, we have shown that $r(\theta_0)$ is computable. □

By Propositions 5.11 and 5.13 we have:

Theorem 5.14 *The conformal radius $r(\theta)$ is computable by a Turing Machine with an oracle for θ if and only the same is true for the Julia set J_θ.*

Note that both directions of Theorem 5.14 are explicitly constructive. That is, the Julia set J_θ can be computed for all θ if provided with an oracle for both θ and $r(\theta)$. To formalize the converse statement we need the following natural definition of an oracle representing a subset of $\mathbb{R}^k$.

Definition 5.2.1 *A function $\psi : \mathbb{D}^k \times \mathbb{D} \to \{0,1\}$ is said to be an oracle for a compact set $K \subset \mathbb{R}^k$ if ψ is from the family* (1.3.1) *for the set K.*

The proofs of Propositions 5.11 and 5.13 are constructive, yielding the following constructive version of Theorem 5.14.

Theorem 5.15 *There is an oracle TM M^{ϕ_1,ϕ_2} that, given an oracle ϕ_1 for $\theta \in (0,1)\backslash\mathbb{Q}$ and an oracle ϕ_2 for $r(\theta)$, computes the Julia set $J_\theta = J(z^2 + e^{2\pi i\theta}z)$.*
Conversely, there is an oracle TM $M^{\phi,\psi}$ that, given an oracle ϕ for $\theta \in (0,1)\backslash\mathbb{Q}$ and an oracle ψ for J_θ, computes the conformal radius $r(\theta)$.

Note that, for example, when θ corresponds to a Cremer Julia set, the oracle for $r(\theta) = 0$ becomes redundant, and J_θ is computable by a single machine M^{ϕ_1} with an oracle for θ. We already know this fact from Theorem 4.2.

5.2.2 *Conformal radius of a Siegel quadratic with a computable θ*

The theorem we formulate below characterizes the values of $r(\theta)$ which correspond to *computable* parameters θ:

Theorem 5.16 *Let $r \in (0, r_{sup})$ be a real number. Then $r = r(\theta)$ is the conformal radius of a Siegel disk of the Julia set J_θ for some computable number θ if and only if r is right-computable.*

Before proving this theorem, let us formulate a corollary:

Corollary 5.17 *There exist computable values of the parameter c, such that the Julia set J_c is not computable by a TM M^ϕ with oracle access to c.*

Proof. By Proposition 1.3 there exist right computable numbers $r_* \in [0, r_{\mathrm{sup}}]$ which are not in $\mathbb{R}_{\mathscr{C}}$. By Theorem 5.16, $r_* = r(\theta_*)$ for $\theta_* \in \mathbb{R}_{\mathscr{C}}$. Since θ_* itself is computable, r_* is non-computable by a TM with an oracle access to θ_*. By Theorem 5.14, the Julia set J_{θ_*} is non-computable by a TM with an oracle access to θ_*. The claim follows by (5.2.1). □

Proof (The "only if" direction of Theorem 5.16). We assume that θ is computable, and show that $r(\theta)$ is right-computable. Recall that repelling periodic orbits are dense in the Julia set J_θ.

By Proposition 3.2 we can algorithmically find an arbitrarily good approximation of the set of all periodic points with period not greater than n by dyadic rationals. Moreover, for every repelling point p with period $k \leq n$ we can *eventually* verify that it is of a repelling type, by checking the inequality

$$|DP_\theta^k(p)| > 1 + 2^{-l},$$

for $l \in \mathbb{N}$.

Hence there exists a growing sequence of sets $H_1 \subset \cdots \subset H_n \subset H_{n+1} \subset \cdots$ with the properties:

- each H_n consists of repelling periodic orbits of P_θ with period at most n;
- the union $\cup H_n$ is dense in J_θ;
- we can algorithmically find an arbitrarily good approximation of H_n by dyadic rationals.

A set H_n gives a lower bound on the set J_θ. By "connecting the dots" in H_n, we can get an upper bound on Δ_θ, and use that to estimate $r(\theta)$ from above. The estimates improve as n grows, since periodic orbits in H_n fill out more details in $\partial\Delta_\theta$.

To put this plan into action, first note that

$$\overline{\cup H_n} = J_\theta \text{ is connected.}$$

Hence, for every l, there exists n_l such that the set $B(H_{n_l}, 2^{-(l+1)})$ is connected. Of course, such a number n_l can be found algorithmically.

Since J_θ separates 0 from ∞, the same is true for $B(H_{n_l}, 2^{-(l+1)})$ provided l is sufficiently large. Hence we can compute a strictly increasing sequence $\{n_l\}_{l=l_0}^{\infty} \subset \mathbb{N}$, and a set $U_l \subset \mathscr{C}$ with the property

$$B(H_{n_l}, 2^{-(l+1)}) \subset U_l \subset B(H_{n_l}, 2^{-l})$$

such that $\mathbb{C} \setminus \overline{U_l}$ has a simply-connected component W_l containing 0.

Using any constructive algorithm for computing the conformal radius (such as, for instance, the one in [BB85]) we can approximate the k-th term of the sequence

$$t_k = r(W_k, 0).$$

By Lemma 5.9,

$$t_k \to r(\theta).$$

However, this sequence need not to decrease in a monotone way. Set

$$G_k = \{z \,:\, B(z, 2^{-k+1}) \subset W_k\} \subset W_k.$$

Clearly, G_k is either empty or simply-connected. The Schwarz Lemma implies that $r(G_k,0) < 4$. On the other hand, by construction we have

$$G_{k+1} \cap B(J_\theta, 2^{-k}) = \emptyset,$$

and thus, by definition,

$$G_{k+1} \subset \{\text{the connected component of } \mathbb{C}\backslash B(J_\theta, 2^{-k}) \text{ around } 0\} \subset W_k.$$

By Lemma 5.9, this implies

$$t_k = r(W_k,0) \geq r(G_{k+1},0) \geq r(W_{k+1},0) - 8 \cdot 2^{-(k-1)/2} = t_{k+1} - 2^{3-(k-1)/2}.$$

To turn t_k into a decreasing subsequence, we add a correction term:

$$R_k = r(W_k,0) + 2^{5-(k-1)/2} \searrow r(\theta).$$

We can algorithmically compute a sequence ℓ_k of dyadic approximation of R_k such that $|\ell_k - R_k| < 2^{-k}$. Set

$$r_k = \ell_k + 3 \cdot 2^{-k}.$$

Then $\{r_k\}$ is a computable sequence of dyadic numbers. We have

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} \ell_k = \lim_{k\to\infty} R_k = \lim_{k\to\infty} t_k = r(\theta)$$

and, for each k,

$$r_k = \ell_k + 3 \cdot 2^{-k} \geq R_k + 2 \cdot 2^{-k} \geq R_{k+1} + 4 \cdot 2^{-(k+1)} \geq \ell_{k+1} + 3 \cdot 2^{-(k+1)} = r_{k+1}.$$

This shows that $r(\theta)$ is right-computable.

□

Remark 5.2.1 *Note that we know that $H_n \to J_\theta$ is Hausdorff metric, which allows us to conclude that $r_k \to r(\theta)$. However, we do not have (and* cannot *have) an estimate on the rate of convergence of H_n to J_θ, and thus cannot obtain an estimate on the rate of convergence of $r_k \to r(\theta)$ and compute $r(\theta)$.*

PROOF OF THE "IF" DIRECTION OF THEOREM 5.16. Now comes the hard part of the proof of Theorem 5.16. Given a computable sequence $\{r_n\}$ such that $r_n \searrow r$ we claim that we can construct a θ such that $r = r(\theta)$. Before we proceed with the argument, let us describe the plan of attack.

An outline of the proof. The strategy of the argument lies in computing a sequence θ_n of parameters such that

1. $\{\theta_n\}$ converges to θ effectively: $|\theta_n - \theta| < 2^{-n}$. Thus to compute θ with precision 2^{-n} it suffices to compute θ_{n+1} with precision $2^{-(n+1)}$;
2. the sequence $\{r(\theta_n)\}$ behaves similarly to the sequence $\{r_n\}$:

$$r(\theta_n) \approx r_n.$$

In particular,

$$\lim r(\theta_n) = \lim r_n = r;$$

3. Finally, the sequence $\{\theta_n\}$ should be constructed in a fashion that would make passing to the limit possible: we want

$$r(\theta) = r(\lim \theta_n) = \lim r(\theta_n) = r,$$

which would then complete the proof.

In our construction, all θ_n's will correspond to *golden* Siegel parameters, that is, their continued fraction expansions will have the form

$$\theta_n = [I_n, 1, 1, \ldots].$$

How will we carry out one step of the construction? For this, given a golden rotation angle

$$\theta_{n-1} = [I_{n-1}, 1, 1, \ldots] \text{ such that } r(\theta_{n-1}) \approx r_{n-1}$$

and, given $r_n < r_{n-1}$ we will need to construct $\theta_n = [I_n, 1, 1, \ldots]$ such that

$$|\theta_n - \theta_{n-1}| < 2^{-n} \text{ and } r(\theta_n) \approx r_n.$$

Moreover, to facilitate an inductive argument, we will ask that the initial segment of the continued fraction I_n be an extension of I_{n-1}.

Note that we have no control over the way the sequence $\{r_n\}$ descends to r. In particular, the drop $r_{n-1} - r_n$ may be arbitrarily large compared to 2^{-n}, which is the amount by which we are allowed to change θ_{n-1} to obtain θ_n. Thus our main task may be summarized as follows:

make an arbitrarily large or small drop in the value of $r(\theta_{n-1})$ while only changing θ_{n-1} by a controlled amount ($\leq 2^{-n}$).

This goal will be accomplished as follows. Suppose that I_{n-1} has k elements in it. We choose a position $m > k$ in the continued fraction expansion of θ_{n-1}, and denote by

$$\theta^N = [I_{n-1}, 1, 1, \ldots, 1, N, 1, \ldots],$$

where N is located in the m-th position. The number m can be chosen so large that any change beyond position m in the continued fraction expansion does not change θ_{n-1} by more than 2^{-n}. In particular, for any N,

$$|\theta^N - \theta_{n-1}| < 2^{-n}.$$

When $N = 1$, we have

$$\theta^N = \theta_{n-1} \text{ and } r(\theta^N) = r(\theta_{n-1}) \approx r_{n-1}.$$

On the other hand, when $N \to \infty$, the parameters θ^N approach a rational number, and the Siegel disk implodes. Hence

$$\lim_{N\to\infty} r(\theta^N) = 0.$$

A precise control of the behavior of $r(\theta^N)$ can be obtained using the representation of the function $\Phi(\theta)$ and continuity of the function $\upsilon(\theta)$. Thus, we will see that when we increase N gradually from $N = 1$ to $N = \infty$, the value $r(\theta^N)$ decreases also gradually, in small steps, from $r(\theta_{n-1}) \approx r_{n-1}$ to 0. Thus it will reach a point where $r(\theta^N) \approx r_n$. We then take

$$\theta_n = \theta^N.$$

The above strategy has its obvious gaps. We do not know exactly at what rate $r(\theta^N)$ descends to 0 as $N \to \infty$. Moreover, since the set of parameters N is discrete, we cannot just use an "intermediate value theorem" to claim that the value r_n will be attained at some point. Nonetheless, as we will see in this section, the process described here can be made to work.

In Fig. 5.4 we see an illustration of this process for

$$\theta_{n-1} = [1,1,20,1,1,\dots] \approx 0.511838.$$

In this case $I_{n-1} = [1,1,20]$. The corresponding Siegel disk is illustrated in Fig. 5.4(a). We choose $m = 6$, so that θ^N would have the form $[1,1,20,1,1,N,1,\dots]$. It is not hard to see that, for any such N,

$$|\theta_{n-1} - \theta^N| < 2^{-13}.$$

In Fig. 5.4(b)–(f), the Siegel disks corresponding to θ^N are shown for $N = 10$, 100, 1000 and 10000. One sees that the Siegel disk gradually shrinks, thus allowing us to attain any conformal radius between $r(\theta_{n-1})$ and 0 within a certain degree of accuracy.

The proof

The technical side of the argument will rely on the following three lemmas. The first one is Lemma 3.1 of [BY06], and the second one is Lemma 4.2 of [BBY06]. We will postpone their proofs until later in the chapter (§5.4).

Lemma 5.18 *For any initial segment $I = [a_0, a_1, \dots, a_n]$, write $\omega = [a_0, a_1, \dots, a_n, 1, 1, 1, \dots]$. Then for any $\varepsilon > 0$, there is an $m > 0$ and an integer N such that, if we write $\beta = [a_0, a_1, \dots, a_n, 1, 1, \dots, 1, N, 1, 1, \dots]$, where the N is located in the $n+m$-th position, then*

$$\Phi(\omega) + \varepsilon < \Phi(\beta) < \Phi(\omega) + 2\varepsilon.$$

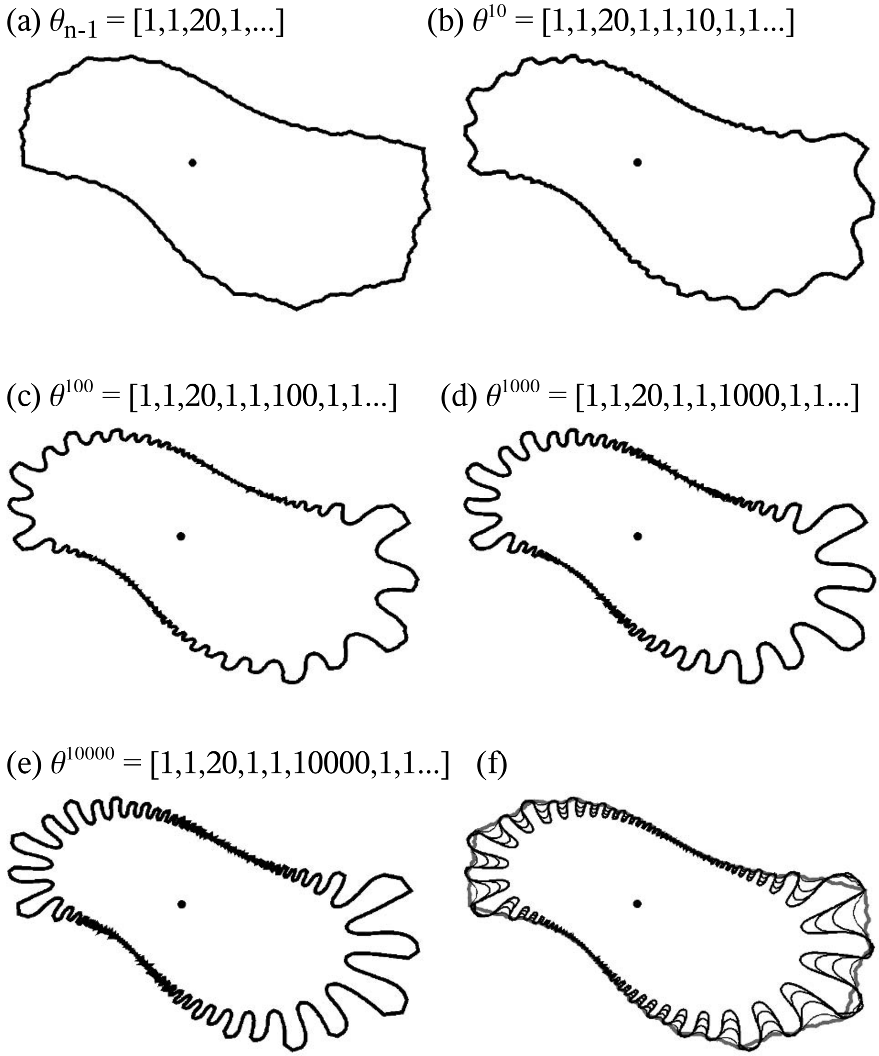


Fig. 5.4 To illustrate the idea of the argument, consider the Siegel disks $\Delta_{\theta_{n-1}}$ for θ_{n-1} given by the continued fractions $[1,1,20,1,\ldots]$ (a), and Δ_{θ^N} for $N = 10, 100, 1000, 10000$ (b)–(e). The conformal radius $r(\theta^N)$ slowly tends to 0 as $N \to \infty$, as illustrated by combining the images (f).

Lemma 5.19 *With ω as above, for any $\varepsilon > 0$ there is an $m_0 > 0$, which can be computed from $(a_0, a_1, \dots, a_n)$ and ε, such that for any $m \geq m_0$ and any tail $I = [a_{n+m}, a_{n+m+1}, \dots]$*

$$\Phi(\beta^I) > \Phi(\omega) - \varepsilon,$$

where

$$\beta^I = [a_1, a_2, \dots, a_n, 1, 1, \dots, 1, a_{n+m}, a_{n+m+1}, \dots].$$

Lemma 5.20 *Let $\omega = [a_1, a_2, \dots]$ be a Brjuno number, that is, $\Phi(\omega) < \infty$. Write $\omega_k = [a_1, a_2, \dots, a_k, 1, 1, \dots]$. Then for every $\varepsilon > 0$ there is an m such that, for all $k \geq m$,*

$$\Phi(\omega_k) < \Phi(\omega) + \varepsilon.$$

Using Proposition 4.9, this implies computability of $r(\theta)$ for θ of bounded type, and we can get a computable version of Lemmas 5.18 and 5.19.

Lemma 5.21 *For any given initial segment $I = [a_0, a_1, \dots, a_n]$ and $m_0 > 0$, write $\omega = [a_0, a_1, \dots, a_n, 1, 1, 1, \dots]$. Then for any $\varepsilon > 0$, we can uniformly compute $m > m_0$, an integer t and an integer N such that, if we write $\beta = [a_0, a_1, \dots, a_n, 1, 1, \dots, 1, N, 1, 1, \dots]$, where the N is located in the $n+m$-th position, we have*

$$r(\omega) - 2\varepsilon < r(\beta) < r(\omega) - \varepsilon, \tag{5.2.2}$$

$$\Phi(\beta) > \Phi(\omega) \tag{5.2.3}$$

and, for any

$$\gamma = [a_0, a_1, \dots, a_n, 1, 1, \dots, 1, N, 1, \dots, 1, c_{n+m+t+1}, c_{n+m+t+2}, \dots],$$

$$\Phi(\gamma) > \Phi(\omega) - 2^{-n}. \tag{5.2.4}$$

Proof. We first show that such m and N *exist*, and then give an algorithm to compute them. By Lemma 5.18 we can increase $\Phi(\omega)$ by any controlled amount by modifying one term arbitrarily far in the expansion.

By Theorem 5.3,

$$\upsilon : \theta \mapsto \Phi(\theta) + \log r(\theta)$$

extends to a continuous function over the reals. Hence for any ε_0 there is a δ such that

$$|\upsilon(x) - \upsilon(y)| < \varepsilon_0 \text{ whenever } |x - y| < \delta.$$

In particular, there is an m_1 such that

$$|\upsilon(\beta) - \upsilon(\omega)| < \varepsilon_0 \text{ whenever } m \geq m_1.$$

This means that, if we choose m large enough, a controlled increase of Φ closely corresponds to a controlled drop of r by a corresponding amount, and hence there are $m > m_0$ and N such that (5.2.2) holds. (5.2.3) is satisfied almost automatically. The only problem is to *computably* find such m and N.

To this end, we apply Proposition 4.9. Together with Theorem 5.14, it implies that for any specific m and N we can compute $r(\beta)$. This means that we can find the suitable m and N by enumerating all the pairs (m,N) and exhaustively checking (5.2.2) and (5.2.3) for all of them. We know that eventually we will find a pair for which (5.2.2) and (5.2.3) hold.

Finally, t exists and can be computed by Lemma 5.19. □

Lemma 5.20 yields the following lemma.

Lemma 5.22 *The supremum of $r(\theta)$ over all angles is equal to the supremum over the angles whose continued fraction expansion has only finitely many terms that are not* 1:

$$r_{sup} = \sup_{\theta=[a_1,a_2,\dots,a_k,1,1,\dots]} r(\theta).$$

Proof. Let $\varepsilon > 0$ be an arbitrarily small positive number. By the definition of r_{sup} there is a $\theta = [a_1, a_2, \dots]$ such that $\log r(\theta) > \log r_{sup} - \varepsilon$. Write

$$\theta_k = [a_1, a_2, \dots, a_k, 1, 1, \dots].$$

Lemma 5.20 states that there is an m such that $\Phi(\theta_k) < \Phi(\theta) + \varepsilon$ for $k \geq m$. Moreover, there is a δ such that whenever $|\phi - \theta| < \delta$ we have $|\upsilon(\phi) - \upsilon(\theta)| < \varepsilon$.

Now $\theta_k \to \theta$ and hence there is an $n \geq m$ such that $|\theta_n - \theta| < \delta$. θ_n has the required form, and we have

$$\log r(\theta_n) = \upsilon(\theta_n) - \Phi(\theta_n) > \upsilon(\theta) - \Phi(\theta) - 2\varepsilon = \log r(\theta) - 2\varepsilon > \log r_{sup} - 3\varepsilon.$$

This shows that we can make $r(\theta_n)$ as close to r_{sup} as we like. □

We are given

$$r = \lim \searrow r_n < r_{sup},$$

and hence there is an s and an $\varepsilon > 0$ such that

$$r_s < r_{sup} - 2\varepsilon.$$

By Lemma 5.22, there exists

$$\gamma_0 = [a_1, a_2, \dots, a_n, 1, 1, \dots]$$

such that

$$r_s + \varepsilon/2 < r(\gamma_0) < r_s + \varepsilon.$$

We are now ready to give an algorithm for computing a rotation number θ for which

$$r(\theta) = \lim \searrow r_n.$$

The algorithm works as follows. At stage k it produces a finite initial segment $I_k = [a_0, \dots, a_{m_k}]$ such that the following properties hold:

(1) $I_0 = [a_1, a_2, \ldots, a_n]$;
(2) I_k has at least k terms, i.e. $m_k \geq k$;
(3) for each k, I_{k+1} is an extension of I_k;
(4) for each k, define $\gamma_k = [I_k, 1, 1, \ldots]$. Then

$$r_{s+k} + 2^{-(k+1)}\varepsilon < r(\gamma_k) < r_{s+k} + 2^{-k}\varepsilon;$$

(5) for each k, $\Phi(\gamma_k) > \Phi(\gamma_{k-1})$;
(6) for each k and for any extension

$$\beta = [I_k, b_{m_k+1}, b_{m_k+2}, \ldots],$$

$\Phi(\beta) > \Phi(\gamma_k) - 2^{-k}$.

The first three properties are very easy to verify. The last three are checked using Lemma 5.21. By this Lemma we can decrease $r(\gamma_{k-1})$ by any given amount (possibly in more than one step) by extending I_{k-1} to I_k. Here we use the facts that the r_k's are computable and non-increasing.

Denote

$$\theta = \lim_{k\to\infty} \gamma_k.$$

The continued fraction expansion of θ is the limit of the initial segments I_k. This algorithm gives us at least one term of the continued fraction expansion of θ per iteration, and hence we would need at most $O(n)$ iterations to compute θ with precision 2^{-n} (in fact, much fewer iterations would suffice). The initial segment of γ_0 can also be computed as in the proof of Lemma 5.21. It remains to prove that, in fact, θ is the rotation number we are looking for.

Lemma 5.23 *The following equalities hold:*

$$\Phi(\theta) = \lim_{k\to\infty} \Phi(\gamma_k) \quad \textit{and} \quad r(\theta) = \lim_{k\to\infty} r(\gamma_k) = r.$$

Proof. By the construction, the limit $\theta = \lim \gamma_k$ exists. We also know that the sequence $r(\gamma_k)$ converges to the number $r = \lim \searrow r_k$, and that the sequence $\Phi(\gamma_k)$ is monotone non-decreasing, and hence converges to a value ψ (*a priori* we could have $\psi = \infty$). By Proposition 5.10, we have $r(\theta) \geq r > 0$, and so $\Phi(\theta) < \infty$. On the other hand, by the property we have established through the construction, we know that

$$\Phi(\theta) > \Phi(\gamma_k) - 2^{-k} \text{ for all } k.$$

Hence

$$\Phi(\theta) \geq \psi.$$

In particular, $\psi < \infty$.

From [BC06b] we know that $\upsilon : \theta \mapsto \Phi(\theta) + \log r(\theta)$ is continuous, and thus

$$\psi + \log r = \lim(\Phi(\gamma_k) + \log r(\gamma_k)) = \lim \upsilon(\gamma_k) = \upsilon(\gamma) = \Phi(\theta) + \log r(\theta). \quad (5.2.5)$$

Hence we must have $\psi = \Phi(\theta)$ and $r = r(\theta)$, which completes the proof. $\square$

5.2.3 *How difficult is it to produce a θ for which J_θ is non-computable?*

As we have seen, a value of θ for which J_θ is non-computable can be produced constructively. As we will see below, under a reasonable assumption, it is not even hard to do so:

Conditional Implication 3. *Assume that the 1-periodic continuous function $\upsilon : \theta \mapsto \Phi(\theta) + \log r(\theta)$ has a computable modulus of continuity (1.2.1); this follows, for instance, from the Marmi-Moussa-Yoccoz Conjecture. Suppose there is a computable sequence $r_1, r_2, \ldots$ of dyadic numbers such that*

- *$\{r_i\}$ is non-increasing, $r_1 \geq r_2 \geq \ldots$, and*
- *$\lim_{i\to\infty} r_i = r$.*

Then there is a poly-time *computable θ (and hence a poly-time computable $c = c(\theta)$) such that $r(\theta) = r$.*

Proof. By the assumption, there is a computable function $\mu : \mathbb{N} \to \mathbb{N}$ such that

$$|\upsilon(\theta_1) - \upsilon(\theta_2)| < 2^{-n} \text{ whenever } |\theta_1 - \theta_2| < 2^{-\mu(n)}.$$

The proof goes along the lines of the proof of the "if" direction of Theorem 5.16. We outline the modifications made to the proof here and leave the details to the reader. The key difference is that in the proof of Theorem 5.16 we used Lemma 5.21 to perform a step in decreasing the conformal radius from $r(\gamma_{k-1})$ to $r(\gamma_k)$. The algorithm there is basically an exhaustive search, which, of course, could take much longer than polynomial time in the precision of γ_k to compute. By assuming that υ has a computable modulus of continuity, we can deal with $\Phi(\gamma_{k-1})$ and $\Phi(\gamma_k)$ instead of the $r(\bullet)$'s. We have an explicit formula for Φ that converges well, and we can compute the continued fractions coefficients to make $\Phi(\gamma_k)$ close to whatever we want relatively fast.

The step of going from γ_{k-1} to γ_k is as follows. First, we do the following computations:

- compute d_k which is the "drop" in r we are trying to achieve; we want
$$d_k/2 < \log(r(\gamma_{k-1})) - \log(r(\gamma_k)) < d_k;$$
- compute using the function μ a value δ_k such that $|\upsilon(x) - \upsilon(y)| < d_k/8$ whenever $|x - y| < \delta_k$.

We have no *a priori* bound on how long these computations would take, but we would still like to be computing θ in polynomial time. To achieve this, we use 1's in the continued fraction expansion of θ to "pad" the computation.

When asked about the value of θ with precision 2^{-n} which is higher than what the known terms of the expansion $[I_{k-1}]$ can provide, we do the following:

- try to compute d_k and δ_k as above, but run the computation for at most n steps;
- if the computation does not terminate, output an answer consistent with the initial segment $[I_{k-1}, \underbrace{1, 1, \dots, 1}_{2n}]$;
- if the computation terminates in less than n steps proceed as described below.

Note that so far the computation is polynomial in n. For some sufficiently large n the computation will terminate in n steps, at which point we will have computed d_k and δ_k. If necessary, we then add more 1's to the initial segment to assure that

$$|\gamma_{k-1} - \gamma_k| < \delta_k.$$

Recall that our goal is to assure that

$$d_k/2 < \log(r(\gamma_{k-1})) - \log(r(\gamma_k)) < d_k.$$

With the current initial segment for γ_k we have $|\gamma_{k-1} - \gamma_k| < \delta_k$, and hence in the difference

$$\log(r(\gamma_{k-1})) - \log(r(\gamma_k)) = \Phi(\gamma_k) - \Phi(\gamma_{k-1}) + (\upsilon(\gamma_{k-1}) - \upsilon(\gamma_k)),$$

the last term is bounded by $d_k/8$. This means that for the current step it suffices to increase $\Phi(\gamma_k)$ relative to $\Phi(\gamma_{k-1})$ by between $\frac{5}{8}d_k$ and $\frac{7}{8}d_k$.

Let M be the total length of I_{k-1} and the 1's we have added, and let us extend the continued fraction by putting $N \in \mathbb{N}$ in the $M+1$-st term, and all 1's further. Thus, I_k has the form $[I_{k-1}, 1, \dots, 1, N, 1, \dots, 1]$. Increasing M if necessary, we can ensure an approximate equality

$$\Phi(\gamma_k) \approx \Phi(\gamma_{k-1}) + \alpha(N) \log N$$

up to an error of $\frac{1}{32}d_k$. Let p_M/q_M be the M-th convergent of the resulting continued fraction. Recall that on an input n we need to compute θ with precision 2^{-n} in time polynomial in n. If

$$2^{-n} > 1/\sqrt{q_M},$$

then we do not need to know anything about N to compute the required approximation.

Suppose that

$$2^{-n} < 1/\sqrt{q_M}, \text{ which means that } n > \log q_M/2.$$

We have time polynomial in $\log q_M$ to perform the computation.

Note that $M = O(\log q_M)$. It is also not hard to see that

$$\alpha(N) < 2^{-M/2},$$

and so in order to have a change by $\approx 3d_k/4$ we must have

$$N > e^{\Omega(2^{M/2})},$$

hence by making M sufficiently large (depending on the value of d_k), we can guarantee that

$$N > e^{2^{M/3}}.$$

This means that we can approximate $\alpha(N)$ with the truncated function Φ at the M-th convergent of the continued fraction. Write

$$p_M/q_M = [a_1, a_2, \dots, a_M],$$

and denote

$$\beta = [a_1, a_2, \dots, a_M] \cdot [a_2, a_3, \dots, a_M] \cdot \dots \cdot [a_{M-1}, a_M] \cdot [a_M].$$

Then β approximates $\alpha(N)$ within a very small relative error. In particular, we can assure that

$$\beta \cdot \left(1 - \frac{1}{32}\right) < \alpha(N) < \beta \cdot \left(1 + \frac{1}{32}\right).$$

In time polynomial in $\log q_M$ we can compute the exact expression for β using rational arithmetic: $\beta = p/q$. Now we can estimate N and write it as $e^{6d_k/8\beta}$ in time polynomial in $\log(q_M)$. From there we can continue by adding enough 1's to get I_k and $\gamma_k = [I_k, 1, 1, \dots]$. By construction, it would give us the necessary decrease in the value of $r(\gamma_k)$. □

5.3 The complexity of Julia sets

5.3.1 Siegel Julia sets with an arbitrarily high complexity

We will show that in addition to non-computable Julia sets, it is possible to construct Julia sets J_θ of an arbitrarily high computational complexity.

Theorem 5.24 *Let $t : \mathbb{N} \to \mathbb{N}$ be a computable function. Then we can compute a parameter c, such that the Julia set J_c is computable, but is not computable by a TM M^ϕ with an oracle access to c in time bounded by $t(n)$.*

Moreover, assuming the function υ (5.1.4) is computable, the parameter c can be computed in polynomial time.

We will prove the conditional second statement of the theorem first.

For a direct proof of the first statement, using tools similar to the ones used in Theorem 5.16, see [BBY06]. Here, we will give a proof *using* a slight extension of Theorem 5.16.

We first note that Proposition 1.4 extends to a dense set of reals:

Proposition 5.25 *Let $t : \mathbb{N} \to \mathbb{N}$ be a computable function. Then there is a dense set $\mathscr{S}_t$ of computable numbers such that for any $x \in \mathscr{S}_t$ the complexity of x has a strict lower bound of $t(n)$.*

Proof. By Proposition 1.4 a single computable number α with this property exists. For any rational number q, the sum $\alpha + q$ is also a computable real. Moreover, every TM which computes $\alpha + q$ does so in time strictly greater than $t(n)$ for all n sufficiently large.

□

Since every computable real number is right-computable, we know that a number x as in Proposition 5.25 can be the conformal radius of a Siegel disk Δ_θ, with a computable parameter θ. Moreover, by Conditional Implication 3, we can choose such a parameter θ to be poly-time computable:

Proposition 5.26 *Let $t : \mathbb{N} \to \mathbb{N}$ be a computable function. Assuming υ is computable, there exists θ, computable in time $O(n^c)$ for some c, such that $r(\theta)$ is not computable in time $t(n)$.*

We will also need an explicit complexity bound on the time it takes to compute $r(\theta)$, assuming we have an algorithm that runs in time $T(n)$ for computing J_θ. Such a bound is given in [BBY07b]:

Lemma 5.27 (Theorem 1.10 in [BBY07b]) *Suppose there is an algorithm that computes J_θ in time $T(n)$. Then $r(\theta)$ can be computed in time*

$$T'(n) = 2^{O(n)} \cdot T(O(n)).$$

We can now prove the second statement of Theorem 5.24:

Proof (The second statement in Theorem 5.24.). Suppose the computable function $t(n)$ is given. For convenience we may assume that $t(n)$ is non-decreasing. Let

$$t_1(n) = (2^{n^2} t(n^2))^{c+1},$$

where c is the constant from Proposition 5.26. Let $r = r(\theta) \in \mathbb{R}_{\mathscr{C}}$ be a number that is not computable in time $t_1(n)$. According to Proposition 5.25 such a number exists, and according to Proposition 5.26, the parameter θ is computable in time $O(n^c)$.

Assume, for the sake of a contradiction, that J_θ is computable in time $t(n)$. Then by Lemma 5.27, $r(\theta)$ can be computed in time

$$t_2(n) = 2^{O(n)} \cdot t(O(n)) < 2^{n^2} t(n^2),$$

for sufficiently large n with oracle queries to the number θ. We know that θ is computable in time $O(n^c)$ and the queries are of precision bounded by $t_2(n)$ – it suffices to evaluate θ with precision $2^{-t_2(n)}$. Hence the running time of the algorithm together with the time it takes to respond to queries is bounded by

$$O(t_2(n)^c) < t_2(n)^{c+1} < t_1(n)$$

for sufficiently large n, and we have arrived at a contradiction. □

We will now prove the first statement in Theorem 5.24 using reasoning similar to the discussion above. We cannot just take an $r(\theta)$ that requires a long time to compute, and say that J_θ takes a long time to compute. This is because the computation is with an oracle for θ. Thus even if $r(\theta)$ is a very hard number, it may be the case that it becomes very easy with an oracle for θ. More work is needed to bypass this difficulty. First, we will need a slight extension of Theorem 5.16.

Lemma 5.28 *Suppose that for a computable number $r \in (0, r_{sup})$ there is a computable decreasing sequence $\{r_n\}$ such that*

- $r_n = \frac{a_n}{2^n}$ *for an integer a_n;*
- $r_n \geq r_{n+1}$*;*
- *for all n,* $|r - r_n| < 2^{-n+2}$.

Suppose further that we are given a computable function $T_1(n)$. Then there is a TM M that computes θ with $r = r(\theta)$. Moreover, M needs only the first n terms of the sequence $\{r_n\}$ to evaluate θ with precision $2^{-T_1(n)}$.

We will not present a proof of the lemma here. It is proved similarly to Theorem 5.16. The only difference now is that when computing the n-th iterate $\theta_n = [I_n, 1, \dots]$ we will insert $2\,T_1(n)$ 1's after I_n so that θ_n would be a $2^{-T_1(n)}$-approximation of θ.

Note that for each n there are finitely many possible *finite* sequences $r_1, \dots, r_n$ as in Lemma 5.28, and they can be easily enumerated. Hence:

Corollary 5.29 *Under the assumptions of Lemma 5.28, we can compute a function $T_2(n)$ such that each evaluation of θ_n takes time bounded by $T_2(n)$.*

Proof (The first statement in Theorem 5.24.). Suppose that the statement is not true. Then there is a computable function $T_3(n)$ such that for any c such that the the Julia set J_c has a Siegel disk and is computable in time $O(T_3(n))$ with oracle access to c. By Lemma 5.27, for any θ such that $J_{c(\theta)}$ has a Siegel disk, $r = r(\theta)$ is computable in time $O(T_4(n))$, where

$$T_4(n) = 2^{O(n)} \cdot T_3(O(n)),$$

with an oracle access to θ.

Let $r \in (0, r_{sup})$ be a computable number. We will show how to compute r efficiently. Suppose that we know r with precision 2^{-n-2}. Then we can compute an initial sequence of $r_1, r_2, \dots, r_n$ so that it can be extended to a sequence as in Lemma 5.28. Then by Corollary 5.29, there exists a computable function $T_2(n)$, such that, for

$$T_1(n) = T_4(2n+2)^2,$$

a $2^{-T_1(n)}$-approximation of θ with $r = r(\theta)$ can be computed in time $T_2(n)$ from the sequence $r_1, \dots, r_n$. By our assumption, a 2^{-2n-2}-approximation of r can be computed from the approximation of θ in time $T_4(O(n))$.

Hence a 2^{-2n-2}-approximation of r can be computed from a 2^{-n-2}-approximation of r in time

$$O(T_4(O(n))+T_2(n)).$$

Hence r is computable in time

$$O(n\cdot(T_4(O(n))+T_2(n))).$$

By Proposition 5.25 we can find a computable number $r\in(0,r_{sup})$ that is not computable in time

$$O(n\cdot(T_4(O(n))+T_2(n))),$$

thus arriving at a contradiction. □

5.3.2 The mystery of Cremer Julia sets.

By the Fatou-Shishikura bound, and the Fatou-Sullivan classification, the filled Julia set K_c of a Cremer quadratic polynomial has no interior. As we have seen in §4.1 this means that $J_c=K_c$ is always computable. It is easy, using the Brjuno condition (5.1.1), to construct a Cremer parameter value c. Yet no informative pictures of Cremer Julia sets have ever been produced. The topological structure of a Cremer Julia set is known to be quite pathological. It is not well understood in general, although a lot is already known (see e.g. [BBCO] and references therein for some state-of-the-art results).

Dynamically, a Cremer polynomial can be described by a sequence of very small perturbations of parabolic maps. This approach, pioneered by Douady, was originally developed by Sørensen [Sør98]. While throwing some light on the nature of the topological complexity of a Cremer Julia set, it also helps to understand the computational difficulty. Parabolic dynamics is very slow, and hence orbits outside J_c but near the Cremer point may take an enormous number of iterates to escape a small neighborhood of J_c. This renders the Distance Estimator algorithm practically useless in a Cremer case. Strangely, however, we do not know of *any* Cremer Julia sets with a complexity provably higher than polynomial. Two natural questions thus emerge:

Questions:

(I) Is there any Cremer quadratic Julia set with a low complexity bound? More specifically, is there a practical algorithm to draw pictures of at least one Cremer Julia set?

(II) Does there exist any Cremer quadratic Julia set with a complexity higher than polynomial? Moreover, are there Cremer Julia sets of an arbitrarily high complexity?

The existence of an efficient way to compute Julia sets of parabolics (as described in §3.3) suggests that the answer to the first question could be "Yes". It may also suggest a possible strategy for constructing such an algorithm. It is probable that the second question also has an affirmative answer. However, we cannot rule out the annoying possibility that there is an easy way to compute every Cremer Julia set, which has so far remained undetected by the dynamicists.

5.4 Proofs of the main technical lemmas

We present outlines of proofs for Lemmas 5.18, 5.19 and 5.20. The complete proofs of the intermediate lemmas can be found in [BBY06] and [BY06]. For convenience, we restate the lemmas here:

Lemma 5.18 For any initial segment $I = [a_1, a_2, \dots, a_n]$, write $\omega = [a_1, a_2, \dots, a_n, 1, 1, 1, \dots]$. Then for any $\varepsilon > 0$, there is an $m > 0$ and an integer N such that if we write $\beta = [a_1, a_2, \dots, a_n, 1, 1, \dots, 1, N, 1, 1, \dots]$, where the N is located in the $n+m$-th position, then

$$\Phi(\omega) + \varepsilon < \Phi(\beta) < \Phi(\omega) + 2\varepsilon.$$

Lemma 5.19 With ω as above, for any $\varepsilon > 0$ there is an $m_0 > 0$, which can be computed from $(a_1, a_2, \dots, a_n)$ and ε, such that for any $m \geq m_0$, and for any tail $I = [a_{n+m}, a_{n+m+1}, \dots]$

$$\Phi(\beta^I) > \Phi(\omega) - \varepsilon$$

where

$$\beta^I = [a_1, a_2, \dots, a_n, 1, 1, \dots, 1, a_{n+m}, a_{n+m+1}, \dots].$$

Lemma 5.20 Let $\omega = [a_1, a_2, \dots]$ be a Brjuno number, that is, $\Phi(\omega) < \infty$. Write $\omega_k = [a_1, a_2, \dots, a_k, 1, 1, \dots]$. Then for every $\varepsilon > 0$ there is an m such that, for all $k \geq m$,

$$\Phi(\omega_k) < \Phi(\omega) + \varepsilon.$$

To prove the lemmas we will need some preliminary bounds. Let $\eta_1(\omega) := \omega$, and

$$\eta_{i+1}(\omega) := \left\{ \frac{1}{\eta_i(\omega)} \right\}, \text{ so that } \eta_i(\omega) = [a_i, a_{i+1}, \dots].$$

Write

$$\Phi^-(\omega) = \Phi(\omega) - \eta_1(\omega)\eta_2(\omega)\dots\eta_{n+m-1}(\omega)\log\frac{1}{\eta_{m+n}(\omega)}.$$

The value of the integer $m > 0$ is yet to be determined. Write

$$\beta^N = [a_1, a_2, \dots, a_n, 1, 1, \dots, 1, N, 1, 1, \dots].$$

The following estimates are proved by induction.

Lemma 5.30 *For any N, the following holds:*

1. For $i \leq n+m$ we have

$$\left|\log \frac{\eta_i(\beta^N)}{\eta_i(\beta^{N+1})}\right| < 2^{i-(n+m)}/N;$$

2. for $i < n+m$,

$$\left|\log \frac{\eta_i(\beta^N)}{\eta_i(\beta^1)}\right| < 2^{i-(n+m)};$$

3. for $i < n+m$,

$$\left|\log \frac{\log \frac{1}{\eta_i(\beta^N)}}{\log \frac{1}{\eta_i(\beta^{N+1})}}\right| < 2^{i-(n+m)+1};$$

4. for $i < n+m-1$,

$$\left|\log \frac{\log \frac{1}{\eta_i(\beta^N)}}{\log \frac{1}{\eta_i(\beta^1)}}\right| < 2^{i-(n+m)+1}.$$

These estimates yield the following.

Lemma 5.31 *For any ω of the form as in lemma 5.18 and for any $\varepsilon > 0$, there is an $m_0 > 0$ such that, for any N and any $m \geq m_0$,*

$$|\Phi^-(\beta^N) - \Phi^-(\beta^1)| < \frac{\varepsilon}{4}.$$

Proof. (Sketch) The $\sum$ in the expression for $\Phi(\beta^1)$ converges, and hence there is an $m_1 > 1$ such that the tail of the sum

$$\sum_{i \geq n+m_1} \eta_1 \eta_2 \ldots \eta_{i-1} \log \frac{1}{\eta_i} < \frac{\varepsilon}{16}.$$

It can be shown that

- for a sufficiently large $m_0 > m_1$, if $m > m_0$, then for any N the influence on the sum of the "head" elements is very small:

$$\left| \sum_{i=1}^{n+m_1-1} \eta_1(\beta^N) \ldots \eta_{i-1}(\beta^N) \log \frac{1}{\eta_i(\beta^N)} - \right.$$

$$\left. - \sum_{i=1}^{n+m_1-1} \eta_1(\beta^1) \ldots \eta_{i-1}(\beta^1) \log \frac{1}{\eta_i(\beta^1)} \right| < \frac{\varepsilon}{16};$$

- for the "tail" terms, for $i \geq n+m_1$ such that $i \neq n+m$,

$$\frac{\eta_1(\beta^N) \ldots \eta_{i-1}(\beta^N) \log \frac{1}{\eta_i(\beta^N)}}{\eta_1(\beta^1) \ldots \eta_{i-1}(\beta^1) \log \frac{1}{\eta_i(\beta^1)}} \leq e.$$

After the change each term of the tail could increase by a factor of e at most. The value of the "tail" starts at the interval $(0, \frac{\varepsilon}{16}]$. Hence it remains in the interval $(0, \frac{e\varepsilon}{16}]$, and the change in the tail is bounded by

$$\frac{e\varepsilon}{16} < \frac{3\varepsilon}{16}.$$

So the total change in Φ^- is bounded by

$$\text{change in the "head"} \ + \ \text{change in the "tail"} \ < \frac{\varepsilon}{16} + \frac{3\varepsilon}{16} = \frac{\varepsilon}{4}.$$

□

Lemma 5.31 immediately yields:

Lemma 5.32 *For any ε and for the same $m_0(\varepsilon)$ as in Lemma 5.31, for any $m \geq m_0$ and N,*

$$|\Phi^-(\beta^N) - \Phi^-(\beta^{N+1})| < \frac{\varepsilon}{2}.$$

Write

$$\Phi^1(\omega) = \eta_1(\omega)\eta_2(\omega)\ldots\eta_{n+m-1}(\omega)\log\frac{1}{\eta_{m+n}(\omega)} = \Phi(\omega) - \Phi^-(\omega).$$

Using the estimates 5.30 one can prove the following:

Lemma 5.33 *For sufficiently large m, for any N,*

$$\Phi^1(\beta^{N+1}) - \Phi^1(\beta^N) < \frac{\varepsilon}{2}.$$

Since $\Phi = \Phi^- + \Phi^1$, summing the inequalities in Lemmas 5.32 and 5.33 yields:

Lemma 5.34 *For sufficiently large m, for any N,*

$$\Phi(\beta^{N+1}) - \Phi(\beta^N) < \varepsilon.$$

It is immediate from the formula of $\Phi(\beta^N)$ that:

Lemma 5.35

$$\lim_{N\to\infty} \Phi(\beta^N) = \infty.$$

We are now ready to prove Lemma 5.18.

Proof (Lemma 5.18.). Choose m large enough for Lemma 5.34 to hold. Increase N by one at a time starting with $N = 1$. We know that $\Phi(\beta^1) = \Phi(\omega) < \Phi(\omega) + \varepsilon$ and, by Lemma 5.35, there exists an M with $\Phi(\beta^M) > \Phi(\omega) + \varepsilon$. Let N be the smallest such M. Then $\Phi(\beta^{N-1}) \leq \Phi(\omega) + \varepsilon$, and by Lemma 5.34

$$\Phi(\beta^N) < \Phi(\beta^{N-1}) + \varepsilon \leq \Phi(\omega) + 2\varepsilon.$$

Hence

$$\Phi(\omega)+\varepsilon<\Phi(\beta^N)<\Phi(\omega)+2\varepsilon.$$

Choosing $\beta=\beta^N$ completes the proof. □

The second part of the following lemma follows by the same argument as Lemma 5.31 by taking $N\geq 1$ to be an arbitrary real number, not necessarily an integer. The first part is obvious, since the tail of ω has only 1's.

Lemma 5.36 *For an $\omega=\beta^1$ as above, for any $\varepsilon>0$ there is an $m_0>0$, such that for any $m\geq m_0$, and for any tail $I=[a_{n+m},a_{n+m+1},\ldots]$ if we write*

$$\beta^I=[a_1,a_2,\ldots,a_n,1,1,\ldots,1,a_{n+m},a_{n+m+1},\ldots],$$

then

$$\sum_{i\geq n+m}\eta_1(\beta^1)\eta_2(\beta^1)\ldots\eta_{i-1}(\beta^1)\log\frac{1}{\eta_i(\beta^1)}<\varepsilon,$$

and

$$\sum_{i=1}^{n+m-1}\left|\eta_1(\beta^I)\ldots\eta_{i-1}(\beta^I)\log\frac{1}{\eta_i(\beta^I)}-\eta_1(\beta^1)\ldots\eta_{i-1}(\beta^1)\log\frac{1}{\eta_i(\beta^1)}\right|<\varepsilon.$$

Moreover, such an m_0 can be computed from $(a_1,a_2,\ldots,a_n)$.

We can now prove Lemma 5.19.

Proof (Lemma 5.19). Applying Lemma 5.36 with $\frac{\varepsilon}{2}$ instead of ε, we get

$$\Phi(\beta^I)-\Phi(\omega)=\sum\{\text{“head”}(\beta^I)-\text{“head”}(\omega)\}+\sum\{\text{“tail”}(\beta^I)-\text{“tail”}(\omega)\}>$$
$$-\frac{\varepsilon}{2}-\sum\{\text{“tail”}(\omega)\}>-\frac{\varepsilon}{2}-\frac{\varepsilon}{2}=-\varepsilon.$$

□

Proof (Sketch of the proof of Lemma 5.20). We divide the sum for $\Phi(\omega)$,

$$\Phi(\omega)=\underbrace{\sum_{i=1}^{s}\eta_1(\omega)\ldots\eta_{i-1}(\omega)\log\frac{1}{\eta_i(\omega)}}_{\text{“head”}}+\underbrace{\sum_{i=s+1}^{\infty}\eta_1(\omega)\ldots\eta_{i-1}(\omega)\log\frac{1}{\eta_i(\omega)}}_{\text{“tail”}},$$

so that “tail” $<\varepsilon/16$. Using the estimates from Lemma 5.30 one can show that modifying ω to ω_k for some appropriately chosen $k\gg s$ will satisfy:

- $\displaystyle\sum_{i=1}^{s}\eta_1(\omega_k)\ldots\eta_{i-1}(\omega_k)\log\frac{1}{\eta_i(\omega_k)}<\sum_{i=1}^{s}\eta_1(\omega)\ldots\eta_{i-1}(\omega)\log\frac{1}{\eta_i(\omega)}+\varepsilon/16,$

  since the relative error in the “head” terms can be made arbitrarily small;

- for $i > s$,

$$\eta_1(\omega_k)\dots\eta_{i-1}(\omega_k)\log\frac{1}{\eta_i(\omega_k)} < 9\cdot\eta_1(\omega)\dots\eta_{i-1}(\omega)\log\frac{1}{\eta_i(\omega)} + 2^{-(i-1)/2}.$$

Note that for $i > k$ the $2^{-(i-1)/2}$ term alone dominates the expression on the left.

Finally for a k as above,

$$\Phi(\omega_k) < \text{“head”}(\omega_k) + \text{“tail”}(\omega_k) < \text{“head”}(\omega) + \varepsilon/16 + 9\cdot\text{“tail”}(\omega) + 2^{2-s/2} <$$

$$\text{“head”}(\omega) + \varepsilon/16 + 9\varepsilon/16 + 2^{2-s/2} < \Phi(\omega) + \varepsilon,$$

for a sufficiently large s.

□

5.5 Number-theoretic properties of c and computability of J_c

We have shown that there exist computable parameters c for which J_c is not computable. Assuming that υ has a computable modulus of continuity, some such c's can even be constructed in polynomial time – that is, they are computationally *easy*. It is reasonable to ask, whether there also exist parameters with this property, which are *easy to write down*. A classical interpretation of this question is

Do there exist quadratics with algebraic parameters and non-computable Julia sets?

We again concentrate our attention on the quadratic polynomials $f_c(z) = z^2 + c$ which have a Siegel disk at the origin, that is, maps which after a change of coordinates can be written as

$$P_\theta(z) = z^2 + e^{2\pi i\theta}z \text{ with } c = \lambda/2 - \lambda^2/4, \text{ where } \lambda = e^{2\pi i\theta}.$$

Let us disregard for the moment the unknown computable properties of the function υ, and discuss the computability of the value of the Yoccoz' Brjuno function Φ, rather than that of the Julia set.

We recall the formula

$$\Phi(\theta) = \sum_{n=1}^{\infty} \theta_1\theta_2\cdots\theta_{n-1}\log\frac{1}{\theta_n},$$

where $\theta_1 = \theta$, and $\theta_i = \{1/\theta_{i-1}\}$. In other words, if θ is expanded into an infinite continued fraction $[a_1, a_2, \dots]$ with $a_i \in \mathbb{N}$, then

$$\theta_i = [a_i, a_{i+1}, \dots].$$

The rotation angle θ can itself be considered as the parameter of our quadratic map, and we can ask what happens if it is an algebraic number. It is not difficult to see that in this case $\Phi(\theta)$ is computable, starting with the classical result of Liouville:

Proposition 5.37 *Suppose $\zeta \in \mathbb{R}\backslash\mathbb{Q}$ is a root of an algebraic equation of degree k with integer coefficients. Then ζ belongs to the Diophantine class $\mathscr{D}(k)$.*

Now let us prove:

Proposition 5.38 *If $\theta \in \mathscr{D}(k)$ then $\Phi(\theta)$ is a finite computable real.*

Proof. As usual, p_n/q_n denotes the n-th convergent of θ, and recall that

$$q_n = a_n q_{n-1} + q_{n-2}. \tag{5.5.1}$$

Standard considerations starting with (5.5.1) imply that

$$\theta_1\theta_2\cdots\theta_n \asymp \frac{1}{q_n}.$$

By (5.5.1),

$$\log a_n \lesssim \log q_n.$$

As

$$q_n \lesssim (q_{n-1})^k,$$

we have the following estimate for the n-th term in the series $\Phi(\theta)$:

$$v_n \equiv \theta_1\theta_2\cdots\theta_{n-1}\log\frac{1}{\theta_n} \asymp \frac{\log a_n}{q_{n-1}} \lesssim \frac{\log q_n}{q_{n-1}} \lesssim \frac{k\log q_{n-1}}{q_{n-1}}.$$

The denominators q_{n-1} grow at least exponentially,

$$q_{n-1} > w^n \text{ for some } w > 1.$$

Thus, we have

$$v_n \lesssim u^{-n}$$

for any $1 < u < w$ and the claim of the theorem follows. □

A much more interesting question is what happens when the parameter c, or equivalently, $\lambda = e^{2\pi i\theta}$ is algebraic. The answer is the same:

Theorem 5.39 *Suppose $\lambda = e^{2\pi i\theta}$ with $\theta \in \mathbb{R}\backslash\mathbb{Q}$ is an algebraic parameter on the unit circle. Then the value of the Yoccoz's Brjuno function $\Phi(\theta)$ is a finite computable real number.*

As the first step toward the proof, consider the difference:

$$z_n \equiv \left|\theta - \frac{p_n}{q_n}\right| = \left|\frac{1}{2\pi i}\log\lambda - \frac{p_n}{q_n}\right|. \tag{5.5.2}$$

On the one hand, by the standard facts about continued fractions, we have

$$z_n \asymp \frac{1}{a_{n+1}q_n^2} \asymp \frac{1}{q_n q_{n+1}}. \tag{5.5.3}$$

On the other hand,

$$z_n \asymp \left|\log\lambda - 4\frac{p_n}{q_n}\log A\right|, \tag{5.5.4}$$

where A is the algebraic number $i = \exp(\pi i/2)$.

We will make use of a version of Baker's lower bound on linear combinations of logs of algebraic numbers [Bak75]. Namely:

Theorem 5.40 *[Bak75] Consider a linear combination*

$$L = \left|\log\alpha_1 + \frac{p}{q}\log\alpha_2\right|,$$

where α_1, α_2 are algebraic numbers, and $p/q \in (0,1)$ is a rational number written in lowest terms. Then there exists a constant $C > 1$ which depends only on α_1, α_2 such that, assuming $L \neq 0$, L has a lower bound

$$L \geq q^{-C}.$$

Applying the bound of Theorem 5.40 to (5.5.4), we have

$$z_n \gtrsim q_n^{-C} \tag{5.5.5}$$

for $C > 1$.

Putting together equations (5.5.3) and (5.5.5), we have a Diophantine estimate

$$q_{n+1} \lesssim (q_n)^{C-1}. \tag{5.5.6}$$

Theorem 5.39 follows from this and Proposition 5.38.

Assuming that the function υ is in fact computable, we obtain the analogues of Proposition 5.38 and Theorem 5.39 that deal with the Julia sets rather than with the Yoccoz's Brjuno function $\Phi(\theta)$.

Conditional Implication 4. *Assume that υ is computable. Suppose that either the number $\theta \in \mathbb{R}\setminus\mathbb{Q}$ or the number $\lambda = e^{2\pi i\theta}$ is algebraic. Then the Julia set J_λ corresponding to the polynomial $f_\lambda(z) = z^2 + \lambda z$ is computable.*

5.6 Quadratics with non-computable Julia sets are rare

It is a standard fact that the complement of the set of all Diophantine numbers (the *Liouville* numbers) is very sparse:

Theorem 5.41 *The set of Liouville numbers in $\mathbb{T}$ has Hausdorff dimension zero (so, in particular, its Lebesgue measure is also null).*

Using this together with Proposition 5.38, we obtain:

Corollary 5.42 *The function $\Phi : \mathbb{T} \to \mathbb{R}$ is computable on a set whose complement has Hausdorff dimension zero.*

We thus have a conditional statement:

Conditional Implication 5. *Assuming that υ has a computable modulus of continuity, the set of all $\theta \in \mathbb{T}$ for which the Julia set J_θ is not computable by a TM with an oracle access to the value of θ has zero Hausdorff dimension.*

Using techniques of Complex Dynamics, we can take this one step further:

Conditional Implication 6. *Assuming that υ has a computable modulus of continuity, the set of all parameters c for which the Julia set J_c is not computable by a TM with an oracle for c has Hausdorff dimension zero.*

Proof of Conditional Implication 6.

The proof involves deep results of Douady-Hubbard renormalization theory. We will not attempt to review this theory here, and point the expert reader to the original article of Douady and Hubbard [DH85] and to the works of Lyubich [Lyu97] and [Lyu99].

Let H be a hyperbolic component of the Mandelbrot set $\mathscr{M}$, and denote by $\mathscr{M}_H$ the corresponding small copy of $\mathscr{M}$. For the main component H_0 of $\mathscr{M}$ we have $\mathscr{M}_{H_0} = \mathscr{M}$. Further, denote

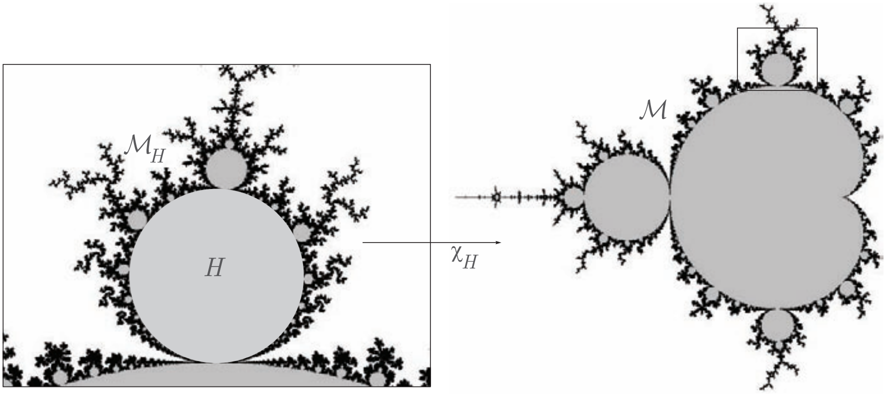
$$\chi_H : \mathscr{M}_H \to \mathscr{M}$$

the Douady-Hubbard embedding, induced by renormalization and straightening. Note that χ_H maps hyperbolic components of $\mathscr{M}_H$ conformally to those of $\mathscr{M}$, beginning with $H \mapsto H_0$. It further has a quasiconformal extension to the whole complex plane, with a possible exclusion of a small disk around the root of H. Its restriction to the boundary of each hyperbolic component of $\mathscr{M}_H$ is real-analytic (again, except possibly at the root of H).

Let us recall the construction of χ_H. Let $k = k(H)$ be the period of H. Then for every $c \in \mathscr{M}_H$, the iterate f_c^k has a quadratic-like restriction

$$F_c \equiv f_c^k|_{U_c}$$

to a topological disk around the origin. The straightening map ψ_c is a quasiconformal homeomorphism of the plane which conjugates F_c to $f_{\chi_H(c)}$. In particular, it maps the filled Julia set



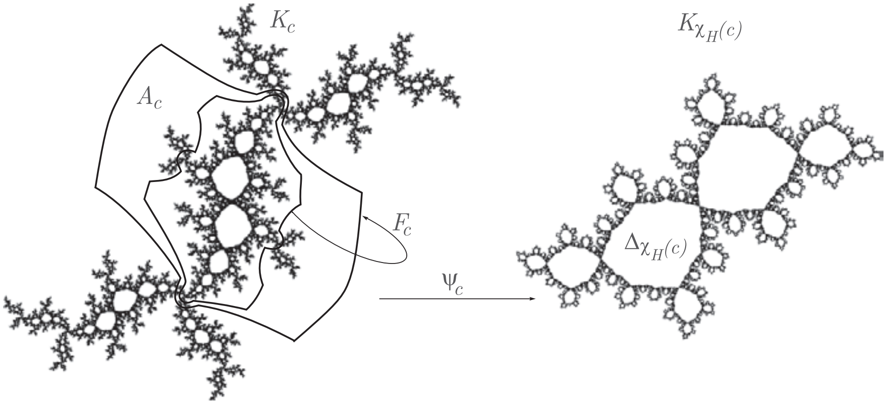


Fig. 5.5 Above on the right is the Mandelbrot set $\mathscr{M}$. On the left, a small copy of $\mathscr{M}$ is seen in an enlargement of a fragment of $\mathscr{M}$. The hyperbolic component H corresponds to Douady's rabbit. Below is an example of a map ψ_c for $c \approx -0.05624 - 0.80907i \in \mathscr{M}_H$. The parameter $c' = \chi_H(c)$ corresponds to the map with a Siegel fixed point whose rotation number is equal to the golden mean. On the left, the inner and outer boundaries of the fundamental annulus A_c of the quadratic-like mapping $F_c = f_c^3$ are indicated.

$$K(F_c) \mapsto K(f_{\chi_H(c)}),$$

and is analytic on the interior $\overset{\circ}{K}(F_c)$.

We are not able to argue that the map ψ_c can be obtained constructively in general – even when J_c is computable. However, the fundamental annulus

$$A_c = F_c(U_c) \setminus U_c$$

can be computed with an arbitrary precision, given an oracle for c. Therefore, we have:

Proposition 5.43 *There exists a TM with an oracle for c which computes a bound $\kappa = \kappa_c$ for the quasiconformal dilatation of the conjugacy ψ_c.*

As a corollary, using the Hölder bound on quasiconformal homeomorphisms of the sphere, we have:

Proposition 5.44 *With an oracle for c we can compute bounds for the moduli of continuity of ψ_c and ψ_c^{-1}.*

We now prove:

Proposition 5.45 *Let $c \in \partial H$ be such that J_c possess a Siegel disk Δ_c, and denote by ρ_c its inner radius. Assume that the inner radius $\rho' = \rho_{\chi_H(c)}$ is computable with an oracle for $\chi_H(c)$. Then ρ_c is computable with an oracle for c.*

Proof. First note, that knowing c, we can compute the rotation angle θ of the Siegel disk Δ_c. Indeed, we can use Proposition 3.2 to compute the center p_c of Δ_c with an arbitrary precision, and then estimate

$$\theta = \frac{1}{2\pi i} \log Df_c^k(p) \bmod \mathbb{Z}.$$

Of course, the rotation number of the Siegel disk $\Delta_{\chi_H(c)}$ is the same θ. Hence, an oracle for c allows us to compute ρ'.

Let P_n be the periodic orbits of F_c whose periods do not exceed n. Denote the corresponding set of periodic orbits of $f_{\chi_H(c)}$ by

$$P_n' = \psi_c(P_n).$$

As ρ' is computable, and by Proposition 3.2, we can determine n such that

$$\partial \Delta_{\chi_H(c)} \subset B(P_n', \varepsilon)$$

for every $\varepsilon > 0$. By Proposition 5.44 we can for every $l \in \mathbb{N}$ compute $n = n(l)$ such that

$$\partial \Delta_c \subset B(P_n, 2^{-(l+1)}).$$

To compute ρ_c with precision 2^{-l}, it remains to compute W_n with $\mathrm{dist}_H(W_n, P_n) < 2^{-(l+2)}$, and evaluate $\mathrm{dist}(p_c, W_n)$ with precision $2^{-(l+2)}$. □

Since $\chi_H^{-1}|_{\partial H}$ preserves the property of a set to have zero Hausdorff dimension, the Conditional Implication 6 readily follows from Conditional Implication 5.

It is likely true that υ has a computable modulus of continuity, and hence Conditional Implication 6 holds. Non-computable examples occur only for parameters c such that a Siegel disk is present in J_c. Such parameters are contained in the set of boundaries $\mathscr{B}$ of the hyperbolic components in $\mathscr{M}$. The set $\mathscr{B}$ is a countable union of algebraic curves, and hence has Hausdorff dimension 1. Thus we immediately obtain an unconditional statement that is weaker than Conditional Implication 6:

Theorem 5.46 *The set of parameters c for which J_c is not computable with an oracle for c has a Hausdorff dimension of at most* 1.

In fact, a stronger statement can be shown unconditionally:

Theorem 5.47 *The linear Lebesgue measure of the set of quadratic parameters c for which J_c is not computable with an oracle for c, is equal to zero.*

We sketch the proof below. Its foundation is a theorem of Petersen and Zakeri [PZ04]. Their results strengthen a theorem of Douady, Ghys, Herman, and Shishikura [Dou88], which we have employed in §4.3.1, as follows. Assume that θ is an irrational number in $(0,1)$ for which

$$\theta = [a_1, a_2, \dots] \text{ with the property } \log a_n = O(\sqrt{n}). \tag{5.6.1}$$

Then the Julia set of $P_\theta(z) = z^2 + e^{2\pi i\theta} z$ can be obtained by a *David surgery* [Dav88] on a cubic Blaschke product

$$B_\theta(z) e^{2\pi\tau(\theta)} z^2 \frac{z-3}{1-3z}.$$

In particular, there exists a David map ζ which maps the unit disk onto Δ_θ.

Notably, such a map has an explicit bound on its modulus of continuity, of the form

$$|\zeta(z) - \zeta(z_0)| \leq \mathrm{const}/\left(\log|z - z_0|\right)^\alpha.$$

The proof of this statement can be found in [GMSM04]. Peter Jones has pointed out to us that it is implicitly contained in the papers of Lehto [Leh70, Leh71].

This implies:

Proposition 5.48 *Assume that θ satisfies condition (5.6.1). Then the inner radius of the Siegel disk Δ_θ (and hence also J_θ) is computable with an oracle for θ.*

As shown in [PZ04], the numbers satisfying (5.6.1) form a set of full Lebesgue measure on the unit circle. The claim of Theorem 5.47 follows by Proposition 5.45.

As we have previously seen, a Julia set $J_\theta = J(z^2 + e^{2\pi i\theta} z)$ for $\theta \notin \mathbb{Q}$ is computable by a single Turing Machine, given an oracle access to θ and the conformal radius $r(\theta)$ of the Siegel disk ($r(\theta) = 0$ for a Cremer Julia set). The inner radius ρ_θ can obviously be computed from J_θ, and hence it can be computed with an oracle access to θ and $r(\theta)$. Thus, Proposition 5.45, together with the constructiveness of Propositions 5.43 and 5.44, implies:

Theorem 5.49 *There is an oracle TM $M^{\phi_1,\phi_2}(m)$ that gets an integer m as an input and the parameters $c \in \mathbb{C}$ and $r(\theta) \in \mathbb{R}$ from the oracles so that whenever the following conditions are satisfied:*

1. *the map $z \mapsto z^2 + c$ has a neutral periodic orbit of period m with multiplier $e^{2\pi i\theta}$;*
2. *$\theta \notin \mathbb{Q}$;*
3. *$r(\theta)$ is the conformal radius of the Siegel disk of J_θ;*

$M^{\phi_1,\phi_2}(m)$ computes the Julia set J_c.

The fact that $r(\theta)$ can be computed from an oracle for the set J_θ (see Definition 5.2.1) implies that the oracle for $r(\theta)$ in Theorem 5.49 can be replaced with an oracle for J_θ.

Chapter 6
Computability versus Topological Properties of Julia Sets

6.1 How can the boundary of a computable set be non-computable?

To provide some intuition why the filled Julia set is computable even when the Julia set is not, we propose a "toy" example. As a first step, let us denote by $W(\theta, w)$ the closed wedge in the unit disc $\mathbb{U}$ around direction θ with width w at the base, which penetrates the disc to depth $1/2$ (as shown in Figure 6.1(a)).

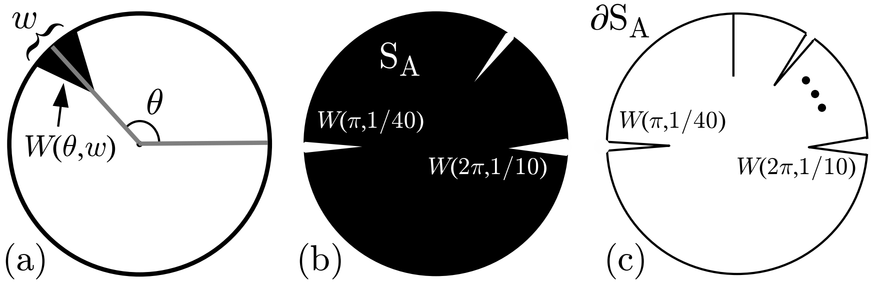


Fig. 6.1 Toy model of an non-computable Julia set. Here $A(1,1){=}1$, $A(2,1){=}1$, $B(3){=}0$, $A(4,100) = 1$ and $A(5,1) = 1$.

Let $A : \mathbb{N} \times \mathbb{N} \to \{0,1\}$ be a computable predicate such that the predicate $B(x) = \exists y\, A(x,y)$ is not computable. We can choose A such that, for each x, $A(x,y)$ holds for at most one value of y. For example, one can take the Halting predicate

$$A(x,t) = 1 \text{ iff } x \text{ is an encoding of a TM that halts in time exactly } t.$$

Algorithms and Computation in Mathematics,

As we have seen in §1.1.1, the predicate $A(x,y)$ is computable, but the predicate $B(x) = \exists t\ A(x,t)$ is not: it is precisely equivalent to the non-computable Halting Problem.

Let

$$S_A = \overline{\mathbb{U} - \bigcup_{A(x,y)=1} W\left(\frac{2\pi}{x}, \frac{1}{10x^2y}\right)}.$$

A sample set S_A is shown in Figure 6.1(b), and its boundary is shown in Figure 6.1(c). We first observe:

Proposition 6.1 *The boundary ∂S_A is not computable.*

Proof. Consider the part of S_A around the tip of the wedges $W(x,\bullet)$, which is located at point $p_x = \left(\frac{1}{2}\cos\frac{2\pi}{x}, \frac{1}{2}\sin\frac{2\pi}{x}\right)$. If $B(x) = 1$, then $p_x \in \partial S_A$. If $B(x) = 0$, then the ball $B\left(p_x, \frac{1}{10x^2}\right)$ is disjoint from ∂S_A. Thus if ∂S_A were computable, we would be able to compute $B(x)$, which is a contradiction. $\square$

The reason why ∂S_A is not computable is that the wedges that go into S_A, however thin, still affect the picture of ∂S_A dramatically in the sense of dist_H.

In contrast, very thin wedges are almost invisible in the picture of S_A. In fact, if we are interested in computing S_A with precision 2^{-n} we may safely ignore wedges that have width smaller than $\frac{1}{m} = 2^{-n-1}$. Hence to get such a picture we will only need to evaluate $A(x,y)$ for values of x and y such that $x^2y \le m$, and there are $< m^2$ such pairs. Thus, we have:

Proposition 6.2 *The set S_A is computable.*

The situation is analogous for the examples of non-computable Julia sets we have constructed. The non-computability of J_θ is due to narrow fjords of the Siegel disk Δ_θ. Accurately computing these fjords is equivalent to estimating the non-computable number $r(\theta)$, which is impossible. On the other hand, for an approximate picture of K_θ most of these fjords can be ignored.

6.2 Locally connected quadratic Julia sets

6.2.1 Local connectedness of sets in $\mathbb{C}$

Recall that a topological space X is locally connected if for each point $x \in X$ there exists a sequence of neighborhoods $U_i(x) \ni x$ such that:

(1) $U_i(x)$ is open and connected in X;
(2) $\cap U_i(x) = \{x\}$.

We remark that the condition (1) can be weakened:

(1a) $U_i(x)$ connected in X and contains an open neighborhood around x.

The main significance of local connectedness in the study of quadratic Julia sets comes from the following construction. Consider a quadratic polynomial $f_c(z) = z^2 + c$ with a connected Julia set. The Riemann mapping

$$\Phi : \hat{\mathbb{C}} \setminus K_c \to \hat{\mathbb{C}} \setminus \overline{\mathbb{U}}$$

is uniquely determined by the normalization $\Phi(\infty) = \infty$ and $\Phi(z) \sim z$ for $z \to \infty$. It then coincides with the *Böttcher coordinate* of $f_c(z)$ at infinity:

$$\Phi(f_c(z)) = (\Phi(z))^2. \tag{6.2.1}$$

As the map $z \mapsto z^2$ preserves the polar coordinate grid on $\hat{\mathbb{C}} \setminus \overline{\mathbb{U}}$, the equation (6.2.1) implies that the preimages of polar coordinate lines under Φ form an invariant grid for f_c. In particular, each radial curve

$$R_\theta \equiv \Phi^{-1}(\{re^{2\pi i\theta} \mid r \in (1,\infty)\})$$

is mapped onto the curve $R_{\theta'}$ by f_c, with $\theta' \equiv 2\theta \bmod \mathbb{Z}$. These curves are known as the *external rays* of J_c. For a fixed angle θ, as $r \to 1+$, the points $re^{2\pi i\theta}$ approach the Julia set J_c. We say that a ray R_θ *lands* at a point $z \in J_c$ if

$$\lim_{r \to 1+} \Phi^{-1}(re^{2\pi i\theta}) = z.$$

In this case, the point z is *accessible from infinity*.

The *equipotential* curve E_r for $r > 1$ is the preimage

$$E_r \equiv \Phi^{-1}(\{re^{2\pi i\theta} \mid \theta \in \mathbb{T}\}).$$

It is mapped to E_{r^2} by f_c.

It is well-known that a connected Julia set may fail to be locally connected. In particular, the following theorem was proved by Douady and Sullivan [Sul83], and independently by Lyubich [Lyu86]:

Proposition 6.3 *If a polynomial f_c has a periodic point of Cremer type, then its Julia set is not locally connected. Moreover, if f_c has a cycle of Siegel disks, and J_c is locally connected, then necessarily the critical point 0 of f_c is contained in the boundary of one of the periodic Siegel disks.*

In the case when the Julia set J_c is locally connected, a key to its topological structure is given by the Theorem of Carathéodory. Recall that a set $K \subset \mathbb{C}$ is *full* if its complement is connected in $\mathbb{C}$:

Carathéodory's Theorem. *For a connected compact and full set $K \subset \mathbb{C}$ denote by Φ the Riemann mapping*

$$\Phi : \hat{\mathbb{C}} \setminus K \to \hat{\mathbb{C}} \setminus \overline{\mathbb{U}} \text{ with } \Phi(\infty) = \infty \text{ and } \Phi'(\infty) = 1.$$

Then the following conditions are equivalent:

- *the set K is locally connected;*
- *the set $J = \partial K$ is locally connected;*
- *the inverse mapping Φ^{-1} extends continuously to a map $S^1 \to J$;*
- *every radial ray $\Phi^{-1}(\{re^{2\pi i\theta} \mid r > 1\})$ lands at a point of J.*

As an immediate corollary we have the following:

Corollary 6.4 *Assume that the Julia set of f_c is connected and locally connected. Then the inverse Böttcher map Φ^{-1} continuously extends to a surjection $\psi : S^1 \to J_c$ which is a semi-conjugacy*

$$\psi(z^2) = f_c(\psi(z)).$$

The parametrization

$$\gamma_c : \theta \mapsto z = \exp(2\pi i\theta) \mapsto \psi(z) \in J_c$$

is known as the *Carathéodory loop* of J_c.

6.2.2 Local connectedness of J_c for a hyperbolic parameter

As our first example, consider a hyperbolic parameter $c \in \mathscr{M}$:

Proposition 6.5 *The Julia set of a hyperbolic quadratic polynomial in $\mathscr{M}$ is locally connected.*

As an illustration, consider the example of Douady's rabbit. Denote by $p \in J_c$ the repelling fixed point where the ears are attached. It is known that every such point is accessible from infinity; in this case by three external rays $R_{1/7}$, $R_{2/7}$, $R_{4/7}$ (cf. Figure 6.2). It follows that removing p separates J_c into three connected sets J_i, $i = 1,2,3$. Hyperbolicity of the polynomial implies that the sizes of the connected components $f_c^{-k}(J_i)$ shrink geometrically with k. These components can now be used to define a basis of connected neighborhoods at every point $z \in J_c$.

In fact, in the case of a hyperbolic quadratic polynomial, an explicit topological model of the dynamical system $f_c : J_c \to J_c$ can be defined following Thurston and Douady (see [Thu, Dou93]).

Let B be the connected component of the immediate basin of the finite attracting orbit of J_c which contains the critical value c in its interior. Consider the periodic point $p \in \partial B$ with the lowest period. Such a point is always unique. It is accessible from ∞ by two or more external rays. Select from them the pair of rays which separates B from the others, and let $\theta_* \in [0,1)$ be the smaller of their arguments.

Partition the unit circle S^1 into two half-open arcs:

$$A_0 = \left(\frac{\theta_*}{2}, \frac{\theta_*+1}{2}\right], \; A_1 = \left(\frac{\theta_*+1}{2}, \frac{\theta_*+2}{2}\right].$$

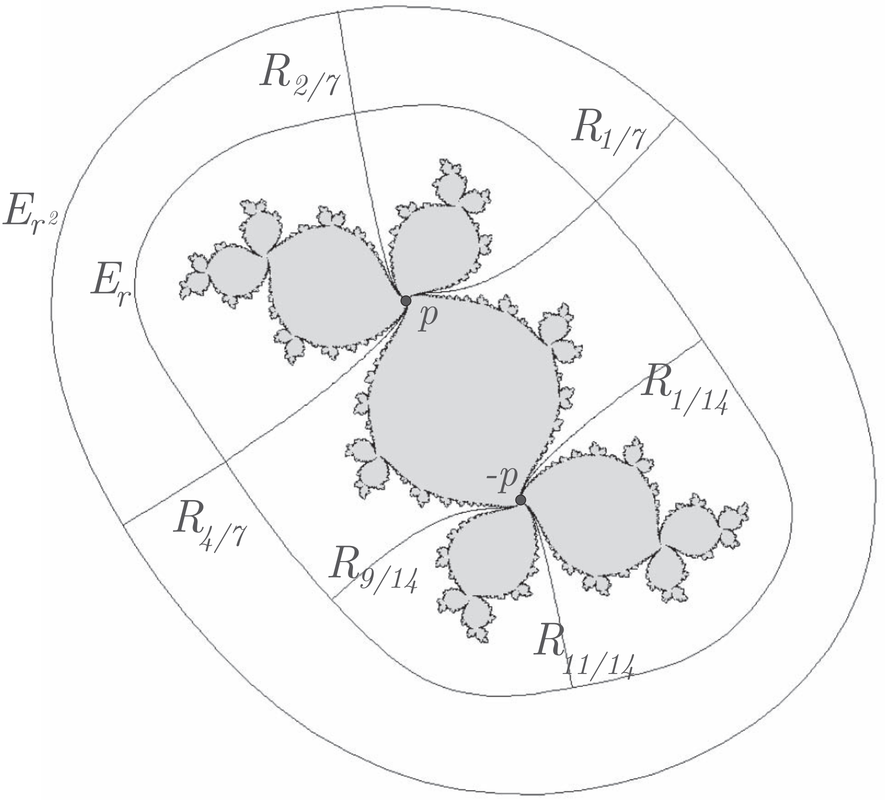


Fig. 6.2 The Julia set of a rabbit. The external rays $R_{1/7}, R_{2/7}, R_{4/7}$ landing at a fixed point p are indicated, as well as their preimages $R_{1/14}, R_{9/14}, R_{11/14}$ which land at $-p$. An equipotential curve E_r is also drawn, together with its image E_{r^2}. The figure is produced using the software package *iDynamics* for Macintosh, written by M. Shishikura.

Define the *itinerary* of a point $x = \exp(2\pi it) \in S^1$ as a sequence of 0's and 1's in which the n-th symbol σ_n is chosen so that

$$2^n t \in A_{\sigma_n}.$$

Define an equivalence relation $\sim_c$ on S^1 which identifies those points whose itineraries coincide.

Proposition 6.6 *Under these definitions, $s \sim_c t$ if and only if $\gamma_c(s) = \gamma_c(t)$.*

It is not difficult to see that every equivalence class of $\sim_c$ is finite. Moreover, for every such class $\bar{s} = \{s_1, \ldots, s_k\}$ denote by $C(\bar{s})$ its convex hull in $\overline{\mathbb{U}}$. This is either a finite-sided polygon, a chord, or a single point in S^1. For different equivalence classes, these convex hulls turn out to be disjoint. A natural way to extend the relation $\sim_c$ to $\mathbb{U}$ is by identifying the points which fall into the same $C(\bar{s})$.

The quotient $\overline{\mathbb{U}}/_{\sim_c}$ is known as the *pinched disk model* of K_c:

$$\overline{\mathbb{U}}/_{\sim_c} \simeq K_c.$$

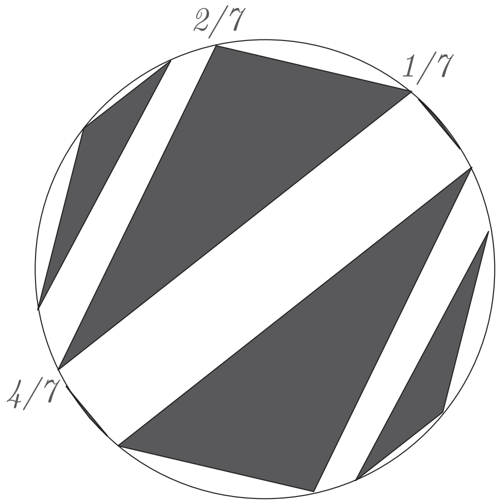


Fig. 6.3 The equivalence relation $\sim_c$ in $\mathbb{U}$ for a rabbit. The large triangle with the vertices $1/7$, $2/7$, $4/7$ corresponds to the fixed point p. Note that the three vertices form a set invariant under angle-doubling. The symmetric large triangle corresponds to $-p$.

By construction, the equivalence relation $\sim_c$ on S^1 is equivariant with respect to the angle-doubling map $f_0(z) = z^2$. Hence f_0 projects to a well-defined mapping F of the quotient $S^1/_{\sim_c} \simeq J_c$. The pair $(S^1/_{\sim_c}, F)$ forms a topological model of the restriction of the dynamical system f_c to J_c.

6.2.3 Locally connected Siegel Julia sets

As was first shown by Herman [Her85], there exist quadratic polynomials in the family $P_\theta(z) = z^2 + e^{2\pi i\theta}z$ with a Siegel disk Δ_θ at the origin, such that the critical point

$$p_\theta = -e^{2\pi i\theta}/2 \notin \partial\Delta_\theta.$$

By Proposition 6.3 in this case J_c is not locally connected.

In the case when $c \in \partial\Delta_\theta$, the boundary of Δ_θ cannot be a smooth curve. In recent papers of Buff-Chéritat [BC02] and Avila-Buff-Chéritat [ABC04], it is shown that in some cases, the boundary

$\partial \Delta_\theta$ of the Siegel disk can itself have smoothness just a hair breadth short of analytic.

There exist, however, topologically well-behaved examples with Siegel disks. Their existence was first demonstrated by Petersen [Pet96]. A different proof was also given by the second author in [Yam99]. Recall that an irrational number $\theta \in (0,1)$ is of bounded type if there exists a finite upper bound on the terms in its continued fraction expansion.

Theorem 6.7 ([Pet96]) *If θ is an irrational number of bounded type, then the Julia set J_θ is locally connected.*

The construction of Petersen and Zakeri [PZ04] later extended this result to a class of rotation numbers θ having full measure in S^1:

Theorem 6.8 ([PZ04]) *Suppose that θ is an irrational angle which satisfies the condition (5.6.1). Then J_θ is locally connected.*

Assume now that P_θ has a Siegel disk with the critical point $p_\theta = -e^{2\pi i\theta}/2$ in the boundary. Assume further that this point is accessible from infinity. In this case, $J_\theta \setminus \{p_\theta\}$ has two connected components; we denote by L_0 the one which does not contain Δ_θ. A *limb of generation n* is a component of $P_\theta^{-n}(L_0)$.

There exist various natural ways of labeling limbs of generation n. For instance, let R_1 and R_2 denote the two external rays which land at p_θ, and set

$$\Gamma = R_1 \cup R_2 \cup \{p_\theta\}.$$

Then we have two well-defined branches of the inverse map P_θ^{-1} mapping $\mathbb{C} \setminus P_\theta(\Gamma)$ to one of the components of $\mathbb{C} \setminus \Gamma$. Let ψ_0 denote the inverse branch which fixes Δ_θ, and ψ_1 the other one. We can then distinguish the limbs of the same generation by the order in which the two inverse branches were applied, so for $\bar{\sigma} \in \{0,1\}^n$ we have

$$L_{\bar{\sigma}} = \psi_{\sigma_n} \circ \cdots \circ \psi_{\sigma_1}(L_0).$$

Theorem 6.9 *The Julia set J_θ is locally connected if and only if the following three properties hold:*

(I) $\partial \Delta_\theta$ is a Jordan curve, and contains p_θ;
(II) the point p_θ is accessible from infinity;
(III) there exists a positive function $s : \mathbb{N} \to \mathbb{R}$ with $s(n) \underset{n\to\infty}{\longrightarrow} 0$ such that the diameter of each limb of generation n is bounded from above by $s(n)$.

The necessity of the condition (III) is not difficult to see. If there existed a non-trivial accumulation set of an infinite sequence of limbs (a "ghost limb") then all its points

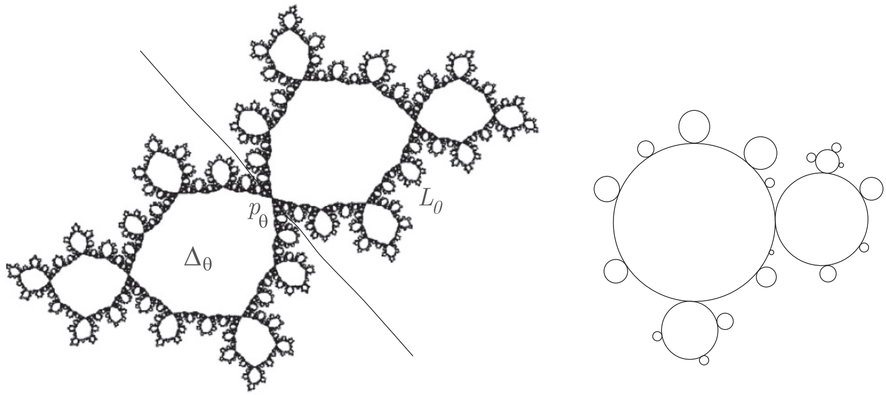


Fig. 6.4 On the left is the filled Julia set of the quadratic polynomial P_θ with $\theta = \cfrac{1}{1+\cfrac{1}{1+\cdots}}$. The critical point p_θ is on the boundary of the Siegel disk; the two external rays landing at p_θ and the initial limb L_0 are also indicated. On the right is an illustration to the topological model of J_θ.

would have to correspond to a single external ray R_θ, in violation of Carathéodory's Theorem.

As for the sufficiency of conditions (I)-(III), the limbs themselves can be used to construct a basis of connected neighborhoods. For more details, see e.g. [Yam99].

Note that Carathéodory's Theorem implies that, if J_θ is locally connected, then the conformal linearizing coordinate

$$\phi_\theta : \mathbb{U} \to \Delta_\theta$$

extends continuously to the boundary. Hence the restriction

$$P_\theta : \partial\Delta_\theta \to \partial\Delta_\theta$$

is conjugated by a homeomorphic change of coordinates $\phi_\theta : S^1 \to \partial\Delta_\theta$ to an irrational rotation of the circle. As $p_\theta \in \partial\Delta_\theta$, we obtain the following:

Proposition 6.10 *If J_θ is locally connected, then*

$$\partial\Delta_\theta = \overline{\mathrm{Postcrit}(P_\theta)}.$$

Recall that, by Proposition 4.10 in the case when θ is of bounded type, the curve $\partial\Delta_\theta$ is a quasicircle.

If J_θ is locally connected, then a topological model for the dynamics of $P_\theta : J_\theta \to J_\theta$ can be constructed similarly to what was done above for the hyperbolic case. However, if we are interested in constructing a topological model of J_θ *without the dynamics*, the exercise becomes rather trivial. We can, for instance, replace the Siegel disk itself, as well as its every preimage, with a round circle. Each of the circles has a countable set of circles attached to its boundary, at a dense set of points. Putting them together has to be done so that there are no intersections not only of the circles themselves, but of the closures of infinite chains of circles.

Here is one explicit way to do this: at each of the points of $C = S^1$ with a rational angle p/q we adjoin a small round circle $C_{p/q}$, with radius $\mathrm{rad}(C_{p/q}) = 0.1q^{-3}$. Parametrizing $C_{p/q}$ by the angular coordinate with angle 0 corresponding to the point $C_{p/q} \cap S^1$, we in turn attach a round disk $C_{p/q,s/t}$ at each rational angle s/t. The radius of $C_{p/q,s/t}$ will be equal to $0.1t^{-3} \times \mathrm{rad}(C_{p/q})$. The process is illustrated in Figure 6.4. Continuing this procedure indefinitely, and taking the closure, we obtain a set S with the property:

Proposition 6.11 *Any locally connected J_θ is homeomorphic to S.*

We leave the formal verification of this to the reader as a straightforward exercise.

As we will see below, the triviality of the topological model in this case does not guarantee that the computational properties of J_θ will be simple.

6.3 Local connectedness versus computability of J_θ

6.3.1 Non locally connected examples

Let us first note the following, rather elementary, observation:

Proposition 6.12 *Suppose that $\partial\Delta_\theta$ contains the critical point p_θ of the quadratic map P_θ. Then $\partial\Delta_\theta$ cannot be a smooth curve.*

Proof. Smoothness of $\partial\Delta_\theta$ at $P_\theta(p_\theta)$ would imply that $\partial\Delta_\theta$ has a cusp at p_θ, contradicting its smoothness at p_θ. □

Below we will demonstrate the following refinement of the "if" part of Theorem 5.16:

Theorem 6.13 *For each right-computable $r \in (0, r_{sup})$ there exists a computable θ such that $r = r(\theta)$ and, moreover, $\partial\Delta_\theta$ is a C^∞-smooth curve.*

In view of Proposition 6.3 and Proposition 6.12, this implies:

Corollary 6.14 *There exists a computable parameter θ such that J_θ is not computable, and not locally connected.*

We first showed the *existence* of a non-computable non locally-connected J_θ in [BY06]. Our argument used a construction of Siegel disks with C^∞ boundaries given by Buff and Chéritat [BC02]. The proof of Theorem 6.13 will follow the streamlined argument of Avila, Buff, and Chéritat [ABC04].

Let θ be a Brjuno number. It will be convenient for us to consider the conformal mapping

$$h_\theta : B(0, r(\theta)) \to \Delta_\theta, \text{ with } h_\theta(0) = 0, \text{ and } h'_\theta(0) = 1.$$

Of course,

$$h_\theta(z) = \phi_{(\Delta_\theta, 0)}(z/r(\theta)),$$

and thus

$$(h_\theta)^{-1} \circ P_\theta \circ h_\theta(z) = e^{2\pi i\theta} z. \tag{6.3.1}$$

As a preliminary step, let us formulate a standard fact:

Proposition 6.15 *Let θ_n be a sequence of Brjuno numbers converging to a Brjuno number θ, and such that*

$$r(\theta_n) \geq r.$$

Then

$$h_{\theta_n} \to h_\theta, \textit{ uniformly on compact subsets of } B(0, r).$$

Proof. By the Koebe Distortion Theorem, the sequence h_{θ_n} is equicontinuous on compact subsets of $B(0, r)$. Hence, from any subsequence we can extract a converging one. It remains to show that, if

$$h_{\theta_{n_k}} \to h$$

for some $\{n_k\}$, then $h = h_\theta$. Passing to the limit in the linearization equation (6.3.1), we see that h is a linearizing coordinate for P_θ on $B(0, r)$. The uniqueness part of the Riemann Mapping Theorem implies that h coincides with h_θ on $B(0, r)$. □

Proof (Theorem 6.13). We will modify the proof of the "if" part of Theorem 5.16 by adding a condition (7) to the algorithm of constructing θ with $r = r(\theta)$ (page 99):

(7) for each $k \geq m \geq 0$ the difference between the m-th derivatives

$$|h^{(m)}_{\gamma_k}(x) - h^{(m)}_{\gamma_{k+1}}(x)|$$

is bounded by 2^{-k} for all x with $|x| \leq r$.

To ensure that this property holds for each k, we modify the argument slightly. We construct a sequence

$$\gamma^l_k = [I^l_k, 1, 1, 1, \ldots] \longrightarrow \gamma_{k-1},$$

where each I^l_k extends I_k, and

$$r(\gamma^l_k) \to r_{s+k} + 2^{-(k+1)}\varepsilon.$$

Now by Proposition 6.15, the maps $h_{\gamma_k^j} \to h_{\gamma_{k-1}}$ uniformly on $B(0,r)$. By the Koebe Distortion Theorem, these mappings are equicontinuous on $B(0,r)$ with a constructive bound on the modulus of continuity. Hence we can algorithmically find $\gamma_k \equiv \gamma_k^j$, for which (7) holds, along with (4)–(6).

With this change in the argument, C^∞-smoothness of

$$\partial\Delta_\theta = h_\theta(\{|x| = r\})$$

is assured by the uniform in k convergence of the derivatives $\{h_{\gamma_k}^{(m)}\}$ for $m \geq 0$.

□

6.3.2 Non-computable locally connected J_θ

It would be comforting to think that high computational hardness of J_θ *always* implies that the Julia set has a bad topology. In view of Proposition 6.11, locally connected Siegel Julia sets are particularly simple from a topological point of view. It is somewhat surprising then, that we can show the following:

Theorem 6.16 *There exist values of θ for which J_θ is locally connected and not computable by any Turing Machine with oracle access to θ.*

Thus, a topological model for such J_θ is *easy* to draw, but the true picture of J_θ is impossible to draw. The method of the proof, combined with techniques of [BBY06], also yields locally connected Julia sets of arbitrarily high computational complexity, giving the following strengthening of Theorem 5.24:

Theorem 6.17 *For every computable function $h : \mathbb{N} \to \mathbb{N}$ there exists a computable value of θ such that the computational complexity of J_θ is greater than h, and J_θ is locally connected.*

Sets without simple topological models

Obviously, there are many other examples of computationally hard sets which have easily computable topological models. The simplest one is, of course, a point $\{x\} \subset \mathbb{R}$. The set $\{x\}$ is not computable whenever the number x is non-computable. On the other hand, regardless of the number x the computable set $\{0\} \subset \mathbb{R}$ is a topological model for $\{x\}$.

It is important to note, however, that it is not true that *every* set in $\mathbb{R}^k$, no matter how computationally complex, has a computable topological model. The phenomenon described in Theorem 6.17 is specific to the examples of Julia sets we construct.

Consider, for instance, the following family of compact subsets of $\mathbb{R}$. For every sequence $S = \{a_1, a_2, \ldots\}$ of 0's and 1's define

$$A_S = \bigcup_{n \text{ for which } a_n=0} \left[\frac{1}{n} - \frac{1}{n(n+2)}, \frac{1}{n}\right] \bigcup_{n \text{ for which } a_n=1} \left\{\frac{1}{n}\right\} \bigcup \{0\}.$$

Thus the set A_S consists of "dashes" and "dots", where the former mark 0's and the latter 1's in the sequence S. It is not hard to see that

Proposition 6.18 *If S_1 and S_2 are two different sequences, then A_{S_1} and A_{S_2} are not homeomorphic.*

On the other hand, there exist countably many Turing Machines, and hence only countably many of possible sets A_S may have computable topological models. This example easily generalizes to $\mathbb{R}^k$ for any $k \in \mathbb{N}$.

Existence of non-computable Julia sets following [BY06]

Theorem 6.16 does not say that the θ in its statement can be computed explicitly, and thus does not directly generalize Corollary 5.17. It is, however, a direct generalization of the following theorem, proved in [BY06].

Theorem 6.19 *There exist values of θ for which J_θ is not computable by any Turing Machine with oracle access to θ.*

We will outline the proof of Theorem 6.19 below.

There are countably many oracle Turing Machines. Let us write them in a sequence

$$M_1^\phi, M_2^\phi, \ldots$$

The argument proceeds by induction, in which at the n-th step we maintain a growing segment of a continued fraction

$$I_n = [a_1, a_2, \ldots, a_{K_n}],$$

and a shrinking interval of values $[l_n, r_n]$ of length $\ell_n = r_n - l_n$, such that the following two properties hold:

- $r_n = r(\gamma_n),$ where $\gamma_n = [I_n, 1, 1, \ldots],$
- for any
$$\beta = [I_n, t_{K_n+1}, t_{K_n+2}, \ldots] \text{ with } r(\beta) \in [l_n, r_n]$$
the machine M_n^ϕ fails to compute J_β.

To perform the step of the induction, let us select $\ell_{n+1} = \ell_n/20$, and try to "fool" the machine M_{n+1}^ϕ. Two possibilities exist. In the first case, M_{n+1}^ϕ does not compute J_β for *any* $\beta = [I_n, \ldots]$ with $r(\beta) \in [r_n - \ell_{n+1}, r_n]$. Then we just choose

$$[l_{n+1}, r_{n+1}] = [r_n - \ell_{n+1}, r_n], \;\; I_{n+1} = [I_n, 1]$$

and continue.

In the other case, there is a value

$$\beta = [I_n, t_{K_n+1}, t_{K_n+2}, \ldots] \text{ with } r(\beta) \in [r_n - \ell_{n+1}, r_n],$$

such that M^ϕ_{n+1} correctly computes the set J_β. In particular, we can use M^ϕ_{n+1} with an oracle access to β to evaluate $r(\beta)$ with precision ℓ_{n+1}. In the process, M^ϕ_{n+1} learns only some finite number $K_n + L_{n+1}$ digits in the continued fraction expansion of β. Thus running M^ϕ_{n+1} on any $\beta' = [I', \ldots]$ with

$$I' = [I_n, t_{K_n+1}, t_{K_n+2}, \ldots, t_{K_n+L_{n+1}}]$$

will result in an output with conformal radius within $[r_n - 2\ell_{n+1}, r_n + \ell_{n+1}]$. We will "fool" the $(n+1)$-st machine, by making a change in finitely many digits of β' beyond the $K_n + L_{n+1}$-th position, to reduce the conformal radius of the Siegel disk between $r_n - 8\ell_{n+1}$ and $r_n - 7\ell_{n+1}$. Since our machine only reads the first $K_n + L_{n+1}$ digits of the continued fraction, it will not suspect the change. We will then define the new segment I_{n+1} to include all digits including the changed one, set

$$[l_{n+1}, r_{n+1}] = [r(\gamma_{n+1}) - \ell_{n+1}, r(\gamma_{n+1})],$$

and the induction step will be complete. The drop in the value of the conformal radius is made possible by the technical lemmas from §5.2.

It is not difficult to carry out the above induction to ensure that

$$\lim r_n = r(\lim \gamma_n).$$

We then set $\gamma = \lim \gamma_n$. For each n, the continued fraction expansion of γ starts with I_n and $r(\gamma) \in [l_n, r_n]$, and thus by the construction M^ϕ_n will fail to compute J_γ. Since this is true for every n, the set J_γ is non-computable.

The difficulty

Before we proceed any further, let us describe informally the problem we will need to tackle. By Theorem 6.7, each of the Julia sets J_{γ_n} is locally connected because γ_n is of bounded type.

What would it take to show that the limiting Julia set also satisfies the conditions of Theorem 6.9? For instance, how could we ensure that the first condition, which asserts that $\partial\Delta_\theta$ is a Jordan curve passing through the critical point, holds?

It would be sufficient to show that the parametrizations

$$\phi_{\gamma_n} : S^1 \to \partial\Delta_{\gamma_n}$$

converge uniformly to a homeomorphism ϕ. A resulting P_θ-invariant Jordan curve $\phi(S^1)$ would pass through the critical point, and thus would coincide with the boundary of Δ_θ.

It is a much taller order to show that a perturbation $\gamma_n \mapsto \gamma_{n+1}$ can be carried out so that not only $|r_n - r_{n+1}|$ is small, but also the geometric shapes of the boundaries of the two Siegel disks are close. In view of Proposition 6.10 the question becomes:

How do we carry out a perturbation $\gamma_n \to \gamma_{n+1}$ *so that* $\mathrm{Postcrit}(P_{\gamma_{n+1}})$ *remains near* $\mathrm{Postcrit}(P_{\gamma_n})$*?*

6.3.3 *Cylinder renormalization and the control of the postcritical set*

Cylinder renormalization is the tool which we will use to gain control of the postcritical set of P_{θ_n} in the above discussion. It was introduced by the second author in [Yam02], and applied to maps with Siegel disks in [Yam08]. We refer the reader to these two works for a more detailed description.

To define the procedure, we start with an analytic map f defined in a neighborhood W of the origin, and of the form

$$f(z) = e^{2\pi i\theta} z + o(z),$$

where θ is some Brjuno number. Recall that $\{p_n/q_n\}$ denote its rational convergents. Fix some $n \geq 0$. Assume that there exists a simple arc $l \subset W$ which connects a fixed point a of the iterate f^{q_n} to 0, and has the property that $f^{q_n}(l)$ is again a simple arc whose only intersection with l is at the two endpoints. Let C_f be the topological disk in $\mathbb{C}\setminus\{0\}$ bounded by l and $f^{q_n}(l)$. We say that C_f is a *fundamental crescent* if the inverse branch $f^{-q_n}|_{C_f}$ mapping $f^{q_n}(l)$ to l is defined and univalent, and the quotient of

$$\overline{C_f \cup f^{-q_n}(C_f)}\setminus\{0,a\}$$

by the iterate f^{q_n} is conformally isomorphic to $\mathbb{C}/\mathbb{Z}$.

For a point z in the fundamental crescent, consider the *first return map* $R_f(z)$ given by the smallest iterate $f^i(z)$ which is again contained in C_f, assuming such an i exists. It will, of course, exist, and will be locally constant for all z in the intersection of C_f with the Siegel disk Δ_f.

Let us now select a conformal isomorphism

$$\kappa : \left(\overline{C_f \cup f^{-q_n}(C_f)}\setminus\{0,a\}\right)/_{f^{q_n}} \xrightarrow{\simeq} \mathbb{C}/\mathbb{Z},$$

which sends the puncture at $\{0\}$ to the "upper" end $+i\cdot\infty$ of $\mathbb{C}/\mathbb{Z}$. Its composition with the exponential map $\chi(z) = \exp(2\pi i\kappa(z))$ maps the quotient of the crescent to the complex plane punctured at the origin. Consider the map

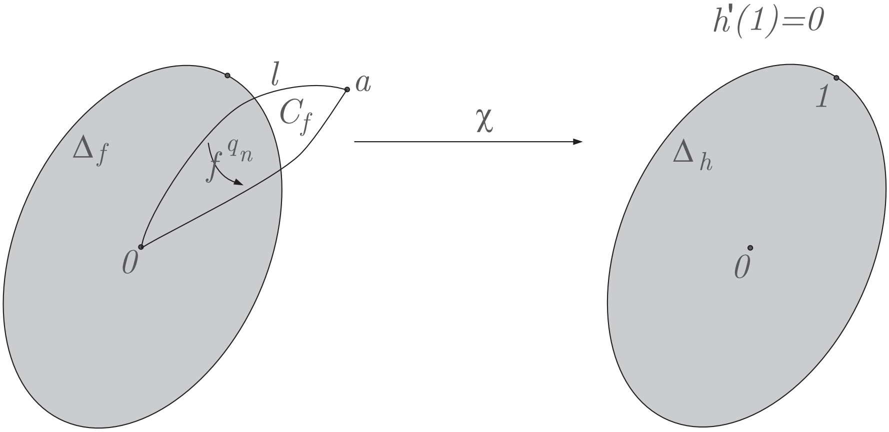


Fig. 6.5 Schematics of cylinder renormalization.

$$h = \chi \circ R_f \circ \chi^{-1}$$

It is not difficult to see that it is an analytic function defined in a neighborhood of the origin. Moreover, filling in the removable singularity at 0, we have:

$$h = \exp(2\pi i\theta')z + o(z), \text{ with } \theta' = G^{n+1}(\theta),$$

where $G(\theta) = \left\{\frac{1}{\theta}\right\}$ is the Gauss map. How well-defined is h? First, and most crucially, Liouville's Theorem implies that the only flexibility we have in the choice of χ is in post-composing it with a homothety around 0. A different choice of C_f could *a priori* produce a different h. However,

Proposition 6.20 *Every other fundamental crescent C'_f with the same endpoints as C_f, and such that $C'_f \cup C_f$ is a topological disk, produces the same renormalized map h (defined up to a change of coordinates by a homothety).*

Now, let us suppose that θ is of bounded type, and the Siegel disk Δ_f is contained in the domain W of f. Further, let the boundary of Δ_f contain a unique critical point of f. Then h is also going to have a single critical point on the boundary of its Siegel disk. Let us uniquely specify χ by putting this point at 1. We then call the map h a *cylinder renormalization* of f with period q_n.

The boundary of the Siegel disk of h is obtained by a conformal "blow-up" of an arc of the boundary of Δ_f. The cylinder renormalization acts as a zoom-in into the postcritical set.

Now let us specialize to the case of quadratic polynomials P_θ:

Theorem 6.21 *Let θ be of a bounded type. Let P_θ be as above. There exists a sequence g_n, $n \in \mathbb{N}$, of cylinder renormalizations of P_θ with the following properties.*

(I) For each n, the map g_n is a cylinder renormalization of P_θ with period q_n. Thus g_n has a Siegel disk with rotation number $G^{n+1}(\theta)$ centered at the origin, whose boundary is a quasicircle, containing the critical point 1.

(II) Denoting C_n the fundamental crescent of the respective renormalization, we have

$$d_n = \sup_{z \in C_n} \operatorname{dist}(z, \Delta_\theta) \to 0.$$

Moreover, d_n is commensurable with $C_n \cap \partial \Delta_\theta$, and hence

$$d_n < Ab^{-n} \text{ for some } A > 0, \text{ and } b > 1.$$

(III) Finally, there exists $k \in \mathbb{N}$ such that, for all n_1 and for $n_2 \geq n_1 + k$, the map g_{n_2} is a cylinder renormalization of g_{n_1}.

What can we say about the sequence of the cylinder renormalizations thus obtained? A recent result of Inou and Shishikura [IS06] implies that under an additional assumption on θ all of these analytic maps belong to a compact family:

Theorem 6.22 *There exist $N_0 \in \mathbb{N}$, a pair of topological disks $\widetilde{W} \Supset W \ni \{0,1\}$, an open set $\mathscr{V}$ in the Banach space of analytic maps in W with the sup-norm, and a compact subset $\mathscr{Y} \Subset \mathscr{V}$ such that the following is true.*

- *Let $\theta = [a_1, a_2, \ldots] \in (0,1) \setminus \mathbb{Q}$ with $a_i \geq N_0$. For every $f \in \mathscr{V}$ with $f'(0) = e^{2\pi i \theta}$ we have the following. The map f is cylinder renormalizable with period $1 = q_0$, and the corresponding cylinder renormalization*

  $$g(z) = \exp(2\pi i G(\theta)) z + o(z) \in \mathscr{Y}.$$

  Moreover, g analytically extends to the larger domain $\widetilde{W}$.
- *Further, consider the quadratic polynomial $f = P_\theta(z)$. Set g_n to be the sequence of cylinder renormalizations of f as in Theorem 6.21. Then there exists $j \in \mathbb{N}$ such that $g_j|_W \in \mathscr{Y}$.*

As an easy corollary, note that:

Corollary 6.23 *Let $g(z)$ and W be as in the above theorem. Then the critical orbit*

$$\bigcup_{n \geq 0} g^n(1) \subset W.$$

Proof. Indeed, the theorem implies that there exists an infinite sequence of cylinder renormalizations of the restriction $g|_W$. Hence, iterates $(g|_W)^n(1)$ are defined for arbitrarily large values of n.

∎

Much more can be said about the sequence of cylinder renormalizations of a quadratic map, than just compactness. Let us define some function spaces. For a topological disk $W \subset \mathbb{C}$ containing 0 and 1 we will denote by $\mathbf{A}_W$ the Banach space of bounded analytic functions in W equipped with the sup norm. Let us denote by $\mathbf{C}_W$ the Banach subspace of $\mathbf{A}_W$ consisting of analytic mappings $h : W \to \mathbb{C}$ such that $h(0) = 0$ and $h'(1) = 0$.
Suppose that $f \in \mathbf{C}_W$ is cylinder renormalizable with some period k. Let C_f be the corresponding fundamental crescent. Assume further that the first return map R_f has a critical point $\zeta \in C_f$ which corresponds to an orbit of f passing through 1. Let h be obtained from R_f as before, by uniformizing the quotient of C_f, with the normalization, sending ζ to 1. Assume further that $h \in \mathbf{C}_V$ for some V.

Proposition 6.24 *Suppose that $f \in \mathbf{C}_W$ is cylinder renormalizable, and that its renormalization h_f is contained in $\mathbf{C}_V$. Then the following holds.*

- *There exists an open neighborhood $U(f) \subset \mathbf{C}_W$ such that every map $g \in \mathbf{C}_W$ is cylinder renormalizable, with a fundamental crescent C_g which can be chosen to move continuously with g.*
- *Moreover, the dependence $g \mapsto h_g$ of the cylinder renormalization on the map g is an analytic mapping $\mathbf{C}_W \to \mathbf{C}_V$.*

We have thus defined an analytic *cylinder renormalization operator* $\mathscr{R}_{\mathrm{cyl}}$ from an open neighborhood in $\mathbf{C}_W$ to $\mathbf{C}_V$.

Hyperbolicity of Renormalization ([Yam08]). *There exists a domain $U \supset \{0,1\}$ such that the following holds. Every periodic continued fraction*

$$\theta = [a_1, a_2, \ldots, a_n, a_1, a_2, \ldots]$$

corresponds to a periodic point $\hat{P}_\theta \in \mathbf{C}_U$ of $\mathscr{R}_{cyl}$ with a Siegel disk with the same rotation number. The operator $\mathscr{L} = D\mathscr{R}^n_{cyl}|_{\hat{P}_\theta}$ is compact.
Further, let θ_1 be such that $G^k(\theta_1) = \theta$ for some k, and as before write $P_{\theta_1}(z) = e^{2\pi i\theta_1} z + z^2$. Then

$$\mathscr{R}^{k+mn}_{cyl} P_{\theta_1} \to \hat{P}_\theta$$

in the space $\mathbf{C}_U$ geometrically in m.
Moreover, suppose that $\min a_i \geq N_0$. Then $\mathscr{L}$ is hyperbolic with one-dimensional unstable direction.

6.4 Non-computable locally connected Julia sets

Let N_0 be as in Theorem 6.22, and fix $N > N_0$. An *admissible* irrational number $\theta = [a_1, a_2, \ldots] \in \mathbb{T}$ has all of the coefficients $a_i \geq N_0$, and $a_j = N$ for all sufficiently large values of j.

The changes in the supporting lemmas of §5.4 needed to replace the digit 1 by N in the tails of the continued fractions θ_i are trivial and will be left to the reader.

One consequence of the renormalization picture we have described above is the following strengthening of Proposition 4.10:

Proposition 6.25 *There exists $B = B(N)$ such that the following holds. Suppose that θ is an admissible number. Then there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ the boundary of the Siegel disk $\partial\Delta_{g_k}$ is a B-quasicircle, where g_k is the cylinder renormalization of P_θ from Theorem 6.21,*

In fact, denoting by $\hat{f}$ the fixed point of $\mathscr{R}_{cyl}$ with rotation number $[N, N, N, \ldots]$ whose existence is postulated in the Theorem on Hyperbolicity of Renormalization, we see that the boundary of the Siegel disk of P_θ at small scales looks like that of $\hat{f}$.

Definition 6.4.1 *Now let α be a Brjuno number such that J_α is locally connected. We will say that J_β is an admissible 2^{-n}-perturbation of J_α if the following properties hold.*

1. *The Julia set J_β is locally connected.*
2. $$\mathrm{dist}_H(\partial\Delta_\alpha, \partial\Delta_\beta) < 2^{-n}.$$
3. *For each n and each $\bar{\sigma} \in \{0,1\}^n$, denoting by $L^\alpha_{\bar{\sigma}}$ and $L^\beta_{\bar{\sigma}}$ the limbs of J_α and J_β respectively, we have*
$$|\,\mathrm{diam}(L^\alpha_{\bar{\sigma}}) - \mathrm{diam}(L^\beta_{\bar{\sigma}})| < 2^{-n}.$$
4. *Consider the Riemann mapping*
$$\Psi_\alpha : \mathbb{U} \to \Delta_\alpha \text{ normalized by } \Psi_\alpha(0) = 0, \Psi'_\alpha(0) = 1,$$
and a similarly defined Ψ_β. Then
$$\sup_{z \in \overline{\mathbb{U}}} |\Psi_\alpha - \Psi_\beta| < 2^{-n}.$$
5. *Similarly, let*
$$\Phi_\alpha : \hat{\mathbb{C}} \setminus \overline{\mathbb{U}} \to \hat{\mathbb{C}} \setminus K_\alpha$$

be the Böttcher map of P_α, and Φ_β the Böttcher map of P_β. Let $\gamma_\alpha^\pm$ be the angles of the two external rays of J_α, which land at the critical point p_α, and similarly for $\gamma_\beta^\pm$. Then

$$||\Phi_\beta(te^{2\pi i\gamma_\beta^\pm}) - \Phi_\alpha(te^{2\pi i\gamma_\alpha^\pm})|| < 2^{-n}$$

in the spherical norm for $t \in [1,\infty)$. In particular,

$$|p_\alpha - p_\beta| < 2^{-n}.$$

We now formulate the following key consequence of the result of Inou and Shishikura (cf. the discussion in [BC05]):

Proposition 6.26 *Consider an admissible number*

$$\alpha = [I_\alpha, N, N, N, \ldots],$$

where I_α is some initial segment of the continued fraction. For every $\varepsilon > 0$ there exist $\delta > 0$ and $M \in \mathbb{N}$ such that the following holds. Let β be a perturbation of α of the form

$$\beta = [I_\alpha, \underbrace{N, N, \ldots, N}_{m}, A_1, A_2, \ldots, A_k, N, N, N, \ldots], \text{ where } m \geq M \text{ and } A_i \geq N,$$

and such that

$$|r_\alpha - r_\beta| < \delta.$$

Then

$$\mathrm{dist}_H(\partial\Delta_\alpha, \partial\Delta_\beta) < \varepsilon.$$

Sketch of proof of Proposition 6.26. The boundary of Δ_α is obtained by taking the closure of the critical orbit $\{P_\alpha^n(1)\}$. By simple considerations of continuity, there exists $k_0 \in \mathbb{N}$ such that, for every $m \geq k_0$,

$$\partial\Delta_\alpha \subset B(\partial\Delta_\beta, \varepsilon).$$

Let τ be any positive number smaller than ε. For the map P_α select C_n as in Theorem 6.21, (II). Consider the arc

$$\ell_n = \overline{\partial\Delta_\alpha \cap C_n}$$

of the boundary of the Siegel disk trapped inside the fundamental crescent. By the inverse branch $(P_\alpha)^{-1}$, fixing the Siegel disk, the arc is rotated around the boundary. An inspection shows that

$$\left(\bigcup_{j=0}^{q_n} (P_\alpha)^{-j}(\ell_{n-1})\right) \bigcup \left(\bigcup_{j=0}^{q_{n-1}} (P_\alpha)^{-j}(\ell_n)\right) \supset \partial\Delta_\alpha.$$

Denote by $W_n \subset C_n$ the lift of the domain W from Theorem 6.22. Note that by Corollary 6.23,

$$\ell_n \subset \overline{W_n}.$$

By Theorem 6.21 (II), for any $\nu > 0$, there exists $k_1 \in \mathbb{N}$ such that

$$W_n \Subset B(\Delta_\alpha, \nu)$$

for $n \geq k_1$. An application of the Koebe Distortion Theorem to pull-backs

$$Y_n \equiv \left(\bigcup_{j=0}^{q_n} (P_\alpha)^{-j}(W_{n-1}) \right) \bigcup \left(\bigcup_{j=0}^{q_{n-1}} (P_\alpha)^{-j}(W_n) \right)$$

implies the existence of $k_2 \in \mathbb{N}$ such that, for $n \geq k_2$,

$$Y_n \subset B(\Delta_\alpha, \tau/2).$$

Set

$$k = k_2 + |I_\alpha|.$$

Now let C'_n, W'_n, and Y'_n denote the corresponding objects for P_β. By considerations of continuity, C'_n, C'_{n-1} are small perturbations of C_n, C_{n-1}, provided that

$$m \gg n \geq k.$$

By Corollary 6.23, we have

$$\overline{\Delta_\beta} \cap C'_n \subset W'_n.$$

Select m_τ large enough so that for $m > m_\tau$ the previous inclusions hold, and

$$Y'_n \subset B(Y_n, \tau/2).$$

Then

$$\overline{\Delta_\beta} \subset B(\Delta_\alpha, \tau).$$

Thus by moving the perturbation far enough to the right in the continued fraction of α, we can guarantee that $\partial\Delta_\beta$ does not extend outside a small neighborhood of Δ_α.

It remains to ensure that $\partial\Delta_\beta$ does not have decorations which grow deep into Δ_α. The easiest way to see this is to note that, by Proposition 6.25, $\partial\Delta_\alpha$ is a B-quasicircle for some $B \in \mathbb{N}$. Hence, for every $\delta > 0$, there exists $0 < \tau < \varepsilon/2$ such that, setting

$$U_\tau = B(\Delta_\alpha, \tau),$$

we have

$$r(U_\tau, 0) - r_\alpha < \delta, \text{ so that } r(U_\tau, 0) - r_\beta < 2\delta.$$

By Proposition 5.8 applied to uniformization of U_τ, for δ small enough, the above inequality implies that

$$\partial\Delta_\beta \subset B(\partial U_\tau, \varepsilon).$$

Taking these δ and τ, and $m > m_\tau$ we have

$$\partial\Delta_\beta \subset B(\Delta_\alpha,\varepsilon)\cap B(\partial U_\tau,\varepsilon)\subset B(\partial\Delta_\alpha,\varepsilon).$$

$\square$

We now state:

Proposition 6.27 *Consider an admissible number*

$$\alpha = [I_\alpha, N, N, N, \ldots].$$

For every $n\in\mathbb{N}$ there exist $\delta=\delta(\alpha,n)>0$ and $M=M(\alpha,n)\in\mathbb{N}$ such that the following holds. Let β be a perturbation of α of the form

$$\beta = [I_\alpha,\underbrace{N,N,\ldots,N}_{m},A_1,\ldots,A_k,N,N,N,\ldots],\ \textit{where } m\geq M \textit{ and } A_i\geq N,$$

and such that

$$|r_\alpha - r_\beta| < \delta.$$

Then J_β is an admissible 2^{-n}-perturbation of J_α.

The property (1) of an admissible perturbation follows by Petersen's theorem [Pet96]. Property (2) is proved in Proposition 6.26, and the stronger property (4) follows by similar considerations, together with Carathéodory's Theorem.
Control of the postcritical set of Δ_β allows us to show that all limbs of J_β of a sufficiently high generation are uniformly small. This technical and rather difficult exercise is carried out in [Yam99]. The property (3) follows from this, and simple considerations of continuity (limbs of a low generation do not move much under a small perturbation of the parameter). Finally, (5) follows from this, Proposition 6.26, and Carathéodory's Theorem.

The induction in the proof of Theorem 6.19 can now be modified using Proposition 6.27 so that each $J_{\gamma_{n+1}}$ is an admissible $2^{-(n+1)}$-perturbation of J_{γ_n}. We modify the induction statement to be:

- $r_n = r(\gamma_n)$, where $\gamma_n = [I_n, N, N, \ldots]$ is an admissible number,
- for any

$$\beta = [I_n, t_{K_n+1}, t_{K_n+2}, \ldots] \text{ with } r(\beta)\in[l_n, r_n] \text{ and } t_{K_n+i}\geq N \text{ for all } i$$

  the machine M_n^ϕ fails to compute J_β,
- J_{γ_n} is an admissible 2^{-n}-perturbation of $J_{\gamma_{n-1}}$.

To guarantee that $J_{\gamma_{n+1}}$ is an admissible $2^{-(n+1)}$-perturbation of J_{γ_n}, we first select $\delta = \delta(\gamma_n, n+1)$ and $M = M(\gamma_n, n+1)$ as in Proposition 6.27 and set

$$I'_n = [I_n, \underbrace{N, N, \ldots, N}_{M}] \text{ and } \ell_{n+1} = \delta/20.$$

We then choose I_{n+1} to be an extension of I'_n which satisfies the first two conditions of the induction statement. $J_{\gamma_{n+1}}$ is an admissible $2^{-(n+1)}$-perturbation of J_{γ_n} by Proposition 6.27, since by the construction

$$r(\gamma_{n+1}) \in [r(\gamma_n) - 20\ell_{n+1}, r(\gamma_n)],$$

and thus $|r(\gamma_n) - r(\gamma_{n+1})| < \delta$.

In view of Theorem 6.9, the limiting Julia set J_θ will be locally connected and non-computable, as required in Theorem 6.16.

Concluding Remarks

Further questions about quadratic Julia sets

The computability theory of quadratic Julia sets presented in this book is fairly complete. We know that non-computable examples exist only among maps with Siegel disks. The computability of the conformal radius r of the Siegel disk is equivalent to the computability of the Julia set. Finally, if we restrict ourselves to maps $P_\theta(z) = z^2 + e^{2\pi i\theta}z$ with a neutral fixed point at the origin, then computability of the parameter θ implies right-computability of r_θ. Conversely, for each right-computable r there is a computable θ with $r_\theta = r$.

When it comes to computational complexity, the picture becomes much murkier. All the examples of Julia sets with non-polynomial complexity in this book come from Siegel disks. Somewhat counter-intuitively, parabolic Julia sets have polynomial time complexity. Cremer Julia sets present the biggest mystery. No informative pictures of these sets have ever been produced, and yet we have shown that all of them are computable. It is worth repeating here the two questions we have posed in §5.3.2:

- *Is there any Cremer quadratic Julia set with a low complexity bound? More specifically, is there a practical algorithm to draw pictures of at least one Cremer Julia set?*
- *Does there exist any Cremer quadratic Julia set with a complexity higher than polynomial? Moreover, are there Cremer Julia sets of an arbitrarily high complexity?*

Much is known about quadratic maps with *weakly hyperbolic* dynamics, such as, for instance, Collet-Eckmann maps. Deep analytic results about them should translate into complexity bounds – this is an interesting direction of further study. At the other end of the spectrum are extremely non-hyperbolic examples of *infinitely renormalizable* maps. Some of them should be at least as bad as Cremer polynomials. Other examples, such as the celebrated Feigenbaum map, have been frequently

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simulated numerically. We expect them to be of a low complexity, perhaps even polynomial, and ask:

- *What is the computational complexity of the Feigenbaum quadratic map?*

Turning to topological properties, as we have shown in Theorem 6.16, there *exist* non-computable and locally connected Julia sets. It is natural to ask:

- *Can a parameter for a non-computable locally connected quadratic Julia set be produced constructively?*

In the opposite direction, the question remains open:

- *Can a Julia set with a low computational complexity be non locally-connected?*

This is, of course, related to the complexity lower bounds in the Cremer case.

Extending the results to other dynamical systems

One of the surprises of our study is how delicate the non-computability results are. To isolate the class of the parameters for which they hold true, we need a cutting edge analytical machinery. Moreover, from the sharpness of the results, it appears that any alternative route would require a machinery of a similar strength. The non-computable examples are very fragile, destroyed both by "filling in", and by adding small perturbations to the parameter. All this suggests that extending this study to attractors/repellers in other natural families of dynamical systems (such as polynomials in higher dimension, for example) will be both interesting, and very challenging.

There are, of course, many examples which can be directly reduced to the study of quadratic maps. Notably, using the Douady-Hubbard theory of quadratic-like maps (see p. 114), it is easy to produce non-computable Julia sets of polynomials of an arbitrary degree. Rational maps with Herman rings should also present non-computable cases, and finding such examples may be possible along similar lines.

It is worth noting that there exists an approach to creating dynamical systems which are difficult to simulate by embedding a copy of a Turing Machine into the dynamics. This translates simple questions about infinite orbits into intractable problems, such as the Halting Problem. Even seemingly innocuous examples, such as piecewise-linear maps of the square (see [Moo90]), are capable of simulating a universal computer. It is unlikely that our results can be derived in a similar manner. However, in other settings this approach should lead to non-computability statements for attractors. This is an interesting question to study, as "Turing completeness" has been shown to exist in many natural families of dynamical systems. It is an important problem (and an object of much speculation in the literature) whether computational hardness due to completeness can be observed in nature. It is possible that it is destroyed by small random perturbations, so even a very complex system becomes easy to predict statistically when a random noise is added.

Practical implications of non-computability results

The main result of this book seems to go against our established intuition. Computer simulation of dynamical systems has become ubiquitous. This is particularly true in complex dynamics, where computational experiments have motivated much of the development in the field. Numerical modeling of attractors/repellers ranges from Julia sets to attractors of truly staggering complexity, such as weather patterns. It is difficult to imagine that for a simple non-linear dynamical system in one complex dimension, whose coefficients can be computed efficiently (and very likely *easily*), the global repeller cannot be numerically simulated. We can take some comfort in the fact that for Julia sets these parameters are very rare. Still, they exist, and proving this connects Computational Complexity Theory to a beautiful chapter in modern Complex Dynamics.

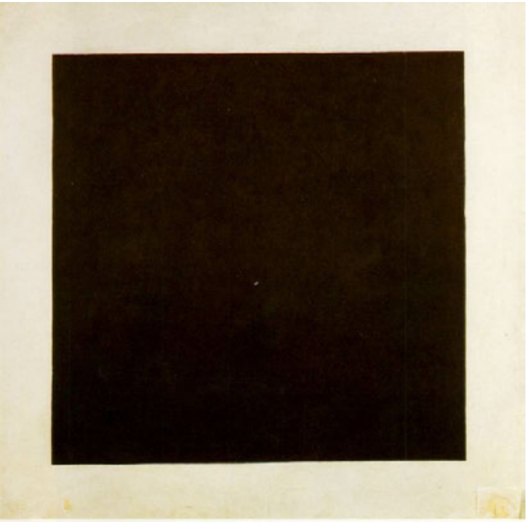

Fig. 6.6 Kasimir Malevich. *Black Square*. 1923–29, Oil on canvas. Russian Museum, St. Petersburg.

A striking practical consequence of the existence of non-computable J_c is that we will never see their pictures. On a 100×100 screen there are 2^{100^2} possible black and white pictures. Non-computability of J_c means that there exists no systematic way to distinguish a correct one among these. The likelihood of stumbling upon it even for a screen of this size is very slim indeed. It is not much of an exaggeration to say that we should not expect to do much better than a *true* one-pixel approximation of J_c – the Black Square.

A Brief Historical Note

Blum, Cucker, Shub, and Smale [BCSS98] were among the first to approach the computer-theoretic foundations of the numerical study of dynamics in a systematic way. While the model of computation they used was algebraic in nature, and not suitable to study fractal objects, they have also listed understanding of the computational hardness of Julia sets and the Mandelbrot set as one of their prime motivations.

Our work fits into the tradition of Computable Analysis, founded by Banach and Mazur [BM37] in 1937, only one year after the birth of Turing Machines and Post Machines. Computability of Euclidean sets has been discussed recently by Brattka and Weihrauch [BW99], Chou and Ko [CK95], and others. Probably, the first work addressing Julia sets in this context was the work of Zhong [Zho98] (also partly motivated by the discussion in [BCSS98]). Several other papers in a similar vein have appeared, notably the work of Hertling [Her05] on computability of the Mandelbrot set, and Rettinger and Weihrauch on complexity of some Julia sets [RW03], [Ret05].

Countless programs to visualize Julia sets have been written by practicing complex dynamicists. Perhaps the only systematic effort to bound their complexity had been the introduction of Distance Estimators by Milnor [Mil06] and Fisher [Fis89], whose ideas play an important role in our study.

Our main break-through came in the work [BY06] where we first showed that some quadratic Julia sets are not computable. This had taken us completely by surprise, and had motivated further study of which this book is a result. Some of the results presented here were obtained in collaboration with Ilia Binder [BBY07a], [BBY06]. Much of the material is new, and has never been published previously.

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